SIMULTANEOUS EXTRAGRADIENT ITERATIVE METHOD TO A SPLIT EQUALITY VARIATIONAL INEQUALITY PROBLEM AND A MULTIPLE-SETS SPLIT EQUALITY FIXED POINT PROBLEM FOR MULTI-VALUED DEMICONTRACTIVE MAPPINGS

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Abstract. This paper deals with a strong convergence theorem for a simultaneous extragradient iterative method to approximate a common solution to a split equality variational inequality problem and a multiple-sets split equality fixed point problem for two countable families of multi-valued demicontractive mappings in real Hilbert spaces. Further, we give a numerical example to justify the main result. The method and results presented in this paper extend and unify some recent known results in the literature.

Keywords: split equality variational inequality problem.; multiple-sets split equality fixed point problem; multi-valued demicontractive mapping; simultaneous extragradient iterative method; monotone mapping.

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1. **Introduction**

Let $H_1$, $H_2$ and $H_3$ be real Hilbert spaces, let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed and convex sets. We denote the inner products and induced norms of $H_1$, $H_2$ and $H_3$ by notations $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively.

The split feasibility problem (in short, S\textsubscript{p}FP) is to find:

(1) \[ x^* \in C \text{ such that } Ax^* \in Q, \]

where $A : H_1 \to H_2$ is a bounded linear operator. The S\textsubscript{p}FP(1) in finite dimensional Hilbert spaces was introduced by Censor and Elfving [5] for modeling inverse problem which arises from retrievals and in medical image reconstruction [4]. Since then various iterative methods have been proposed to solve S\textsubscript{p}FP(1); see for instance [1, 3, 12, 21].

Censor et al. [7] proposed the following multiple-sets split feasibility problem (in short, MSS\textsubscript{p}FP), which arises in applications such as intensity modulated radiation therapy [20]:

(2) \[ x^* \in \bigcap_{i=1}^{N} C_i \text{ such that } Ax^* \in \bigcap_{j=1}^{M} Q_j, \]

where $N$ and $M$ are positive integers, for each $i, j$, $C_i \subset H_1$ and $Q_j \subset H_2$ are nonempty, closed and convex sets.

A mapping $F_1 : H_1 \to H_1$ is said to be firmly quasi-nonexpansive if $\text{Fix}(F_1) \neq \emptyset$ and

(3) \[ \|F_1 x - x^*\|^2 \leq \|x - x^*\|^2 - \|x - F_1 x\|^2, \forall x^* \in \text{Fix}(F_1), x \in H_1, \]

where $\text{Fix}(F_1) := \{ x \in H_1 : x = F_1 x \}$, the set of fixed points of $F_1$.

A mapping $F_1 : C \to C$ is said to be $k$-demicontractive if $\text{Fix}(F_1) \neq \emptyset$ and there exists a constant $k \in (0, 1)$ such that

(4) \[ \|F_1 x - x^*\|^2 \leq \|x - x^*\|^2 + k \|x - F_1 x\|^2, \forall x^* \in \text{Fix}(F_1), x \in C. \]

Evidently, the class of demicontractive mappings properly includes the class of firmly quasi-nonexpansive mappings.

**Remark 1.1.** [8] For negative values of $k$ the class of demicontractive mappings is diminished to a great extent. Such class with negative value of $k$ was considered under the name of strongly
attracting mapping. In particular, the mapping $F_1$ which satisfies (4) with $k = -1$ is called pseudo-contractive. Note that a mapping $F_1$ satisfying (4) with $k = 1$ is usually called hemicontractive and used in connection with the strong convergence of the implicit Mann-type iteration method.

**Example 1.1.** [11] Let $f$ be a real function defined by $f(x) = -x^2 - x$; it can be seen that $f: [-2, 1] \rightarrow [-2, 1]$. This function is demicontractive on $[-2, 1]$ and continuous. It is not quasi-nonexpansive and is not pseudo-contractive on $[-2, 1]$.

Rest of the paper, unless specified, let $A: H_1 \rightarrow H_2$ and $B: H_2 \rightarrow H_3$ be bounded linear operators.

In 2013, Moudafi et al. [17] introduced and studied the following split equality fixed point problem (in short, $S_p$EFPP) which is a generalization of $S_p$FP (1): Find $x^* \in C$ and $y^* \in Q$ such that

**Definition 1.1.** Let $T_1: H_1 \Rightarrow CB(H_1)$ be a multi-valued mapping. $x^* \in H_1$ is said to be fixed point of $T_1$ if $x^* \in T_1 x^*$. We denote by $\text{Fix}(T_1)$, the set of fixed points of $T_1$ defined by

$$\text{Fix}(T_1) = \{ x \in H_1 : x \in T_1 x \}.$$

**Definition 1.2.** A multi-valued mapping $T_1: \mathcal{D}(T_1) \subset H_1 \Rightarrow CB(H_1)$ is said to be:

(i) nonexpansive if

$$D(T_1 x, T_1 y) \leq \| x - y \|, \forall x, y \in \mathcal{D}(T_1);$$
(ii) quasi-nonexpansive if \( \text{Fix}(T_1) \neq \emptyset \) and

\[
D(T_1 x, T_1 x^*) \leq \|x - x^*\|, \forall x^* \in \text{Fix}(T_1), x \in \mathcal{D}(T_1);
\]

(iii) \( k \)-demicontractive if \( \text{Fix}(T_1) \neq \emptyset \) and there exists a constant \( k \in (0, 1) \) such that

\[
(D(T_1 x, T_1 x^*))^2 \leq \|x - x^*\|^2 + k(D(x, T_1 x))^2, \forall x^* \in \text{Fix}(T_1), x \in \mathcal{D}(T_1),
\]

where \( \mathcal{D}(T_1) \) denotes the domain of \( T_1 \).

Evidently, the class of multi-valued demicontractive mappings properly includes the class of multi-valued quasi-nonexpansive mappings. The class of demicontractive mappings is important because several common types of operators arising in optimization problems belong to this class, see for example, Chidume and Maruster [8], Maruster and Popirlan [16] and references therein.

**Example 1.2.** Let \( H_1 = \mathbb{R} \), the set of all real numbers, \( T_1 : \mathbb{R} \to CB(\mathbb{R}) \) be defined by \( T_1(x) = \{-2x\}, \forall x \in \mathbb{R} \). We have that \( \text{Fix}(T_1) = \{0\} \) and \( T_1 \) is a multi-valued demicontractive mapping which is not quasi-nonexpansive. In fact, for each \( x \in \mathbb{R} \), we have

\[
(D(T_1 x, T_1 0))^2 = 4|x - 0|^2,
\]

which implies that \( T_1 \) is not quasi-nonexpansive. Also we have

\[
(D(T_1 x, T_1 0))^2 = |x - 0|^2 + \frac{1}{3}(d(x, T_1 x))^2.
\]

This implies that \( T_1 \) is demicontractive with \( k = \frac{1}{3} \).

In 2014, Wu et al. [23] introduced and studied the following multiple-sets split equality problem for finite families of multi-valued quasi-nonexpansive mappings:

\[
\text{Find } x^* \in \bigcap_{i=1}^{N} \text{Fix}(T_i) \text{ and } y^* \in \bigcap_{i=1}^{N} \text{Fix}(S_i) \text{ such that } Ax^* = By^*,
\]

where \( N \) is a positive integer, and \( \{T_i\}_{i=1}^{N} : H_1 \rightrightarrows CB(H_1), \{S_i\}_{i=1}^{N} : H_2 \rightrightarrows CB(H_2) \) are families of multi-valued quasi-nonexpansive mappings.
Very recently, Chidume [11] introduced and studied the following multiple-sets split equality fixed point problem (in short, MSS\(_p\)EFPP) for countable families of multi-valued demi-contractive mappings:

\[
\text{Find } x^* \in \bigcap_{i=1}^{\infty} \text{Fix}(T_i) \text{ and } y^* \in \bigcap_{i=1}^{\infty} \text{Fix}(S_i) \text{ such that } Ax^* = By^*,
\]

where \(\{T_i\}_{i=1}^{\infty} : H_1 \rightrightarrows \text{CB}(H_1)\) and \(\{S_i\}_{i=1}^{\infty} : H_2 \rightrightarrows \text{CB}(H_2)\) are countable families of multi-valued demicontractive mappings.

We consider the following split equality variational inequality problem (in short, S\(_p\)EVIP):

Find \(x^* \in C\) and \(y^* \in Q\) such that

\[
\langle f(x^*), x - x^* \rangle \geq 0, \forall x \in C
\]

(9)

\[
\langle g(y^*), y - y^* \rangle \geq 0, \forall y \in Q
\]

(10)

and \(Ax^* = By^*\),

where \(f : C \to H_1\) and \(g : Q \to H_2\) be single-valued mappings. When looked separately, (9) is called variational inequality problem (in short, VIP) and its solution set is denoted by \(\text{Sol}(\text{VIP}(9))\). The solution set of S\(_p\)EVIP(9)-(10) is denoted by \(\text{Sol}(\text{S}\(_p\)EVIP(9)-(10)) = \{(x^*, y^*) \in C \times Q : x^* \in \text{Sol}(\text{VIP}(9)), y^* \in \text{Sol}(\text{VIP}(10)) \text{ and } Ax^* = By^*\}\). S\(_p\)EVIP(9)-(10) generalizes split variational inequality problem (in short, S\(_p\)VIP) studied by Censor et al. [6].

In 1976, Korpelevich [13] introduced the following iterative method which is known as extragradient iterative method:

\[
\begin{align*}
    x_0 & \in C, \\
    y_n & = P_C(x_n - \lambda f x_n), \\
    x_{n+1} & = P_C(x_n - \lambda f y_n), \quad n \geq 0,
\end{align*}
\]

(11)

where \(\lambda > 0\) is a fixed number, \(f\) is a monotone and Lipschitz continuous mapping and \(P_C\) is the metric projection of \(H_1\) onto \(C\); and proved that if the \(\text{Sol}(\text{VIP}(9))\) is nonempty then, under some suitable conditions, the sequence \(\{x_n\}\) generated by algorithm (11) converges to a solution of variational inequality (9). Since then a number of generalizations of extragradient iterative
method has been studied for various important classes of problems, see for instance [15, 18, 22] and the relevant references therein.

Motivated by the ongoing research work in this direction, we propose and analyze a simultaneous extragradient iterative method to approximate a common solution to $S_p$EVIP(9)-(10) and $MSS_p$EFPP(8) for countable families of multi-valued demicontractive mappings in real Hilbert spaces. Further, we give a numerical example to justify the main result. The method and results presented in this paper extend and unify some recent known results in the literature; see for instance [6, 9, 11, 17, 24].

2. Preliminaries

We recall some definitions and results which are needed in sequel. Let $\to$ and $\rightharpoonup$ denote the strong and weak convergence, respectively and $\mathbb{N}$ denote the set of natural numbers.

For every point $x \in H_1$, there exists a unique nearest point in $C$ denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \forall y \in C.$$  

The mapping $P_C$ is called the metric projection of $H_1$ onto $C$. It is known that $P_C$ is nonexpansive and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \forall x \in H_1.$$  

Moreover, $P_C x$ is characterized by the fact that $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \forall y \in C$$  

which implies that

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \forall x \in H_1, y \in C.$$  

Definition 2.1. A mapping $f : C \to H_1$ is said to be:

(i) monotone, if

$$\langle fx - fy, x - y \rangle \geq 0, \forall x, y \in H_1;$$
(ii) \( \alpha \)-inverse strongly monotone, if there exists a constant \( \alpha > 0 \) such that
\[
\langle f(x) - f(y), x - y \rangle \geq \alpha \| f(x) - f(y) \|^2, \quad \forall x, y \in H_1;
\]

(iii) \( \beta \)-Lipschitz continuous, if there exists a constant \( \beta > 0 \) such that
\[
\| f(x) - f(y) \| \leq \beta \| x - y \|, \quad \forall x, y \in H_1.
\]

We note that if \( f \) is \( \alpha \)-inverse strongly monotone mapping, then \( f \) is monotone and \( \frac{1}{\alpha} \)-Lipschitz continuous but converse need not be true. For \( \alpha = 1 \), \( \alpha \)-inverse strongly monotone mapping \( f \) is called firmly nonexpansive mapping.

**Definition 2.2.** A mapping \( T_1 : H_1 \rightarrow H_1 \) is said to be:

(i) demiclosed at zero if for any sequence \( \{x_n\} \subset H_1 \), with \( x_n \rightharpoonup x^* \) and \( \| x_n - T_1x_n \| \rightarrow 0 \), we have \( x^* = T_1x^* \).

(ii) semicompact if for any bounded sequence \( \{x_n\} \subset H_1 \), with \( \| x_n - T_1x_n \| \rightarrow 0 \), there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( \{x_{n_k}\} \) converges strongly to some \( x^* \in H_1 \).

**Definition 2.3.** [2]. A multi-valued mapping \( T_1 : H_1 \rightrightarrows 2^{H_1} \) is said to be:

(i) monotone if
\[
\langle u - v, x - y \rangle \geq 0, \text{ whenever } u \in T_1(x), \ v \in T_1(y);
\]

(ii) maximal monotone if \( T_1 \) is monotone and the graph, \( \text{graph}(T_1) := \{(x,y) \in H_1 \times H_1 : y \in T_1(x)\} \), is not properly contained in the graph of any other monotone mapping.

It is well known that for each \( x \in H_1 \) and \( \lambda > 0 \) there is a unique \( z \in H_1 \) such that \( x \in (I + \lambda T_1)z \).

The mapping \( J^T_{\lambda} := (I + \lambda T_1)^{-1} \) is called the resolvent of \( T_1 \). It is a single-valued and firmly nonexpansive mapping defined on \( H_1 \).

**Definition 2.4.** A multi-valued mapping \( T_1 : H_1 \rightrightarrows CB(H_1) \) is said to be:

(i) demiclosed at zero if for any sequence \( \{x_n\} \subset H_1 \), with \( x_n \rightharpoonup x^* \) and \( d(x_n, T_1x_n) \rightarrow 0 \), we have \( x^* \in T_1x^* \).

(ii) hemicompact if for any bounded sequence \( \{x_n\} \subset H_1 \), with \( d(x_n, T_1x_n) \rightarrow 0 \), there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( \{x_{n_k}\} \) converges strongly to some \( x^* \in H_1 \).
Lemma 2.1. [11] Let \( C \) be a nonempty subset of a real Hilbert space \( H \) and let \( T_1 : C \rightrightarrows CB(C) \) be a multi-valued \( k \)-demiccontractive mapping. Let for every \( z \in \text{Fix}(T_1) \), \( T_1z = \{z\} \). Then there exists \( L > 0 \) such that

\[
D(T_1x, T_1z) \leq L\|x - z\|, \quad \forall x \in C, \ z \in \text{Fix}(T_1).
\]

Lemma 2.2. For all \( x, y \in H \), we have

(i) \( \|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle \);

(ii) \( 2\langle x, y \rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2 = \|x + y\|^2 - \|x\|^2 - \|y\|^2 \).

Lemma 2.3. [19] (Opial’s lemma) Let \( \{\mu_n\} \) be a sequence in Hilbert space \( H \), such that there exists a nonempty set \( W \subset H \) satisfying:

(i) For every \( \mu^* \in W \), \( \lim_{n \to \infty} \|\mu_n - \mu^*\| \) exists.

(ii) Any weak-cluster point of the sequence \( \{\mu_n\} \) belongs to \( W \);

Then there exists \( \mu^* \in W \) such that \( \{\mu_n\} \) weakly converges to \( \mu^* \).

Lemma 2.4. [10] Let \( \{x_i\}_{i=1}^m \) be a set in Hilbert space \( H \). For \( \{\alpha_i\}_{i=1}^m \subset (0, 1) \) such that \( \sum_{i=1}^m \alpha_i = 1 \). Then the following identity holds:

\[
\left\| \sum_{i=1}^m \alpha_i x_i \right\|^2 = \sum_{i=1}^m \alpha_i \|x_i\|^2 - \sum_{1 \leq i < j \leq m} \alpha_i \alpha_j \|x_i - x_j\|^2.
\]

Remark 2.1. It follows from Lemma 2 that the following identity holds:

\[
\left\| \sum_{i=1}^\infty \alpha_i x_i \right\|^2 = \sum_{i=1}^\infty \alpha_i \|x_i\|^2 - \sum_{i,j=1, i \neq j}^\infty \alpha_i \alpha_j \|x_i - x_j\|^2.
\]

for \( \sum_{i=1}^\infty \alpha_i = 1 \), provided that \( \{x_i\} \) is bounded.

3. Simultaneous Extragradient Iterative Algorithms

We propose the following simultaneous extragradient iterative algorithm to approximate a common solution of \( S_p\text{EVIP}(9) - (10) \) and \( MSS_p\text{EFPP}(8) \).
Algorithm 3.1. Let \((x_1, y_1) \in H_1 \times H_2\) be given. The iteration sequences \(\{x_n, y_n\}\) be generated by the schemes:

\[
\begin{align*}
p_n &= P_C(x_n - \gamma_n A^*(Ax_n - By_n)); \\
u_n &= P_C(I - \lambda_n f)p_n; \\
c_n &= P_C(p_n - \lambda_n f u_n); \\
x_{n+1} &= \alpha_0 c_n + \sum_{i=1}^{\infty} \alpha_i w_{i,n}, w_{i,n} \in T_ic_n; \\
q_n &= P_Q(y_n + \gamma_n B^*(Ax_n - By_n)); \\
v_n &= P_Q(I - \lambda_n g)q_n; \\
e_n &= P_Q(q_n - \lambda_n g v_n); \\
y_{n+1} &= \alpha_0 e_n + \sum_{i=1}^{\infty} \alpha_i z_{i,n}, z_{i,n} \in S_ie_n,
\end{align*}
\]

(15)

where \(\alpha_0 \in (k, 1), \alpha_i \in (0, 1)\), for each \(i \in \mathbb{N}\) such that \(\sum_{i=0}^{\infty} \alpha_i = 1\) and the step size \(\gamma_n\) is chosen in such a way that for some \(\varepsilon > 0\),

\[
\gamma_n \in (\varepsilon, \mu_n - \varepsilon), \ n \in \Lambda,
\]

(16)

otherwise \(\gamma_n = \gamma (\gamma \geq 0)\), where \(\mu_n := \frac{2\|Au_n - Bv_n\|^2}{\|A^*(Au_n - Bv_n)\|^2 + \|B^*(Au_n - Bv_n)\|^2}\) and the index set \(\Lambda = \{n : Ax_n - By_n \neq 0\}\).

Remark 3.1. \([24]\) It follows from condition (16) that \(\inf_{n \in \Lambda} \{\mu_n - \gamma_n\} > 0\). Since \(\|A^*(Au_n - Bv_n)\| \leq \|A^n\|\|Au_n - Bv_n\|\) and \(\|B^*(Au_n - Bv_n)\| \leq \|B^n\|\|Au_n - Bv_n\|\), we observe that \(\{\mu_n\}\) is bounded below by \(\frac{2}{\|A^n\| + \|B^n\|}\) and so \(\inf_{n \in \Lambda} \mu_n < +\infty\). Consequently \(\sup_{n \in \Lambda} \gamma_n < +\infty\) and hence \(\{\gamma_n\}\) is bounded.

For each \(i \in \mathbb{N}\), if \(T_i\) and \(S_i\) are single-valued demicontractive mappings then Algorithm 3.1 is reduced to the following simultaneous extragradient iterative algorithm to approximate a common solution of \(S_p\text{EVIP}(9)-(10)\) and \(S_p\text{EFPP} (5)\):
Algorithm 3.2. Let \((x_1, y_1) \in H_1 \times H_2\) be given. The iteration sequences \(\{(x_n, y_n)\}\) be generated by the schemes:

\[
\begin{align*}
  p_n &= P_C(x_n - \gamma_n A^*(Ax_n - By_n)); \\
  u_n &= P_C(I - \lambda_n f)p_n; \\
  c_n &= P_C(p_n - \lambda_n fu_n); \\
  x_{n+1} &= \alpha_0 c_n + \sum_{i=1}^{\infty} \alpha_i T_i c_n; \\
  q_n &= P_Q(y_n + \gamma_n B^*(Ax_n - By_n)); \\
  v_n &= P_Q(I - \lambda g)q_n; \\
  e_n &= P_Q(q_n - \lambda_n g v_n); \\
  y_{n+1} &= \alpha_0 e_n + \sum_{i=1}^{\infty} \alpha_i S_i e_n,
\end{align*}
\]

(17)

where \(\alpha_0 \in (k, 1), \alpha_i \in (0, 1)\), for each \(i \in \mathbb{N}\) such that \(\sum_{i=0}^{\infty} \alpha_i = 1\) and the step size \(\gamma_n\) is chosen in such a way that for some \(\epsilon > 0\),

\[
\gamma_n \in (\epsilon, \mu_n - \epsilon), \quad n \in \Lambda,
\]

(18)

otherwise \(\gamma_n = \gamma (\gamma \geq 0)\), where \(\mu_n := \frac{2\|Au_n - Bv_n\|^2}{\|A^*(Au_n - Bv_n)\|^2 + \|B^*(Au_n - Bv_n)\|^2}\) and the index set \(\Lambda = \{n : Ax_n - By_n \neq 0\}\).

4. Main results

We prove a strong convergence theorem to approximate a common solution to \(S_p\text{EVIP}(9)-(10)\) and \(\text{MS}_p\text{EFPP}(8)\) for countable families of multi-valued demicontractive mappings by selecting the step size in such a way that the implementation of the algorithm does not require the calculation or estimation of the operator norms.

Theorem 4.1. Let \(H_1, H_2\) and \(H_3\) be real Hilbert spaces and \(C \subset H_1\), \(Q \subset H_2\) be nonempty, closed and convex sets. Let \(A : H_1 \to H_3\), \(B : H_2 \to H_3\) be bounded linear operators with their adjoint operators \(A^*\) and \(B^*\), respectively. Let \(f : C \to H_1\) be monotone and \(\alpha\)-Lipschitz continuous mapping and let \(g : Q \to H_2\) be monotone and \(\beta\)-Lipschitz continuous mapping. Let \(\{T_i\}_{i=1}^{\infty} : H_1 \rightrightarrows CB(H_1)\) and \(\{S_i\}_{i=1}^{\infty} : H_2 \rightrightarrows CB(H_2)\) be families of multi-valued demicontractive mappings with demicontractive constants \(k_i\) and \(s_i\), respectively and let \(k_1 = \sup_{i \geq 1} \{k_i\} \in (0, 1)\)
and $k_2 = \sup\{s_i\} \in (0, 1)$. For each $i \in \mathbb{N}$, let $T_i$ and $S_i$ be demiclosed at 0. Assume that for $x \in \bigcap_{i=1}^{\infty} \text{Fix}(T_i)$, $T_i x = \{x\}$ and for $y \in \bigcap_{i=1}^{\infty} \text{Fix}(S_i)$, $S_i y = \{y\}$, for each $i \in \mathbb{N}$. Assume that

$$\Gamma := \text{Sol}(S_p\text{EVIP}(9) - (10)) \bigcap \left( \bigcap_{i=1}^{\infty} \text{Fix}(T_i) \times \bigcap_{i=1}^{\infty} \text{Fix}(S_i) \right) \neq \emptyset.$$ If the sequence $\{\lambda_n\}_{n \in \mathbb{N}} \subset [a, b]$, for some $a$ and $b$ with $0 < a < b < \frac{1}{\min\{\alpha, \beta\}}$ and $k \in (0, 1)$ where $k = \max\{k_1, k_2\}$, then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1 converges weakly to $(\bar{x}, \bar{y}) \in \Gamma$. In addition, if for each $i \in \mathbb{N}$, $T_i$ and $S_i$ are hemicompact, then $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y}) \in \Gamma$.

**Proof.** Let $(x, y) \in \Gamma$, i.e., for each $i \in \mathbb{N}$, $x \in T_i(x)$, $y \in S_i(y)$, $x \in \text{Sol}(\text{VIP}(9))$, $y \in \text{Sol}(\text{VIP}(10))$ and $Ax = By$. First, we prove that $\{w_{i,n}\}_{i=0}^{\infty}$ is bounded. Indeed, it follow from Lemma 2.1 that

$$\|w_{i,n} - x\| \leq D(T_i c_n, T_i x) \leq \frac{1 + \sqrt{k_1}}{1 - \sqrt{k_1}} \|c_n - x\| := M_n.$$ This implies that $\{w_{i,n}\}_{i=0}^{\infty}$ is bounded. Similarly, we obtain that $\{z_{i,n}\}_{i=0}^{\infty}$ is bounded.

We estimate

$$
\|p_n - x\|^2 = \|P_C(x_n - \gamma_n A^*(Ax_n - By_n)) - x\|^2 \\
\leq \|x_n - \gamma_n A^*(Ax_n - By_n) - x\|^2 \\
\leq \|x_n - x\|^2 - 2\gamma_n \langle x_n - x, A^*(Ax_n - By_n) \rangle + \gamma_n^2 \|A^*(Ax_n - By_n)\|^2 \\
(19) \leq \|x_n - x\|^2 - 2\gamma_n \langle Ax_n - Ax, Ax_n - By_n \rangle + \gamma_n^2 \|A^*(Ax_n - By_n)\|^2 \\
(20) \leq \|x_n - x\|^2 + 2\gamma_n \|Ax_n - Ax\| \|Ax_n - By_n\| + \gamma_n^2 \|A^*(Ax_n - By_n)\|^2.
$$

Using Lemma 2.2 (ii) in (19), we get

$$\|p_n - x\|^2 \leq \|x_n - x\|^2 - \gamma_n \|Ax_n - Ax\|^2 - \gamma_n \|Ax_n - By_n\|^2 + \gamma_n \|By_n - Ax\|^2 \\
+ \gamma_n^2 \|A^*(Ax_n - By_n)\|^2. \quad (21)$$

Similarly, we obtain

$$\|q_n - y\|^2 \leq \|y_n - y\|^2 - \gamma_n \|By_n - By\|^2 - \gamma_n \|Ax_n - By_n\|^2 + \gamma_n \|Ax_n - By\|^2 \\
+ \gamma_n^2 \|B^*(Ax_n - By_n)\|^2. \quad (22)$$
Adding (21) and (22), and using the fact that $Ax = By$, we get

$$
\|p_n - x\|^2 + \|q_n - y\|^2 \leq \|x_n - x\|^2 + \|y_n - y\|^2 - \gamma_n [2\|Ax_n - By_n\|^2 - \gamma_n (\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2)].
$$

(23)

From condition (16) on $\gamma$, we obtain from (23) that

$$
\|p_n - x\|^2 + \|q_n - y\|^2 \leq \|x_n - x\|^2 + \|y_n - y\|^2.
$$

(24)

Since $c_n = P_C (p_n - \lambda_n f u_n)$, it follows from (14) that

$$
\|c_n - x\|^2 = \|P_C (p_n - \lambda_n f u_n) - x\|^2
\leq \|p_n - \lambda_n f u_n - x\|^2 - \|p_n - \lambda_n f u_n - c_n\|^2
\leq \|p_n - x\|^2 - \|p_n - c_n\|^2 + 2\lambda_n \langle f u_n, x - c_n \rangle
\leq \|p_n - x\|^2 - \|p_n - c_n\|^2 + 2\lambda_n [\langle f u_n - fx, x - u_n \rangle + \langle fx, x - u_n \rangle + \langle f u_n, u_n - c_n \rangle]
$$

(25)

Since $f$ is monotone and the fact that $x \in \text{Sol}(\text{VIP}(9))$, we obtain from (25) that

$$
\|c_n - x\|^2 \leq \|p_n - x\|^2 - \|p_n - c_n\|^2 + 2\lambda_n \langle f u_n, u_n - c_n \rangle
\leq \|p_n - x\|^2 - \|p_n - u_n\|^2 - \|u_n - c_n\|^2 + 2\lambda_n \langle p_n - \lambda_n f u_n - u_n, c_n - u_n \rangle
$$

(26)

From (13), we have

$$
\langle p_n - \lambda_n f u_n - u_n, c_n - u_n \rangle = \langle p_n - \lambda_n f p_n - u_n, c_n - u_n \rangle + \lambda_n \langle f p_n - f u_n, c_n - u_n \rangle
\leq \lambda_n \langle f p_n - f u_n, c_n - u_n \rangle
$$

(27)

Since $f$ is $\alpha$-Lipschitz-continuous, we obtain

$$
2 \langle p_n - \lambda_n f u_n - u_n, c_n - u_n \rangle \leq 2 \lambda_n \|f p_n - f u_n\| \|c_n - u_n\|
\leq 2 \lambda_n \alpha \|p_n - u_n\| \|c_n - u_n\|
\leq (\lambda_n \alpha)^2 \|p_n - u_n\|^2 + \|c_n - u_n\|^2.
$$

(28)

(29)
Hence, from (26) and (29), we obtain

\begin{equation}
\|c_n - x\|^2 \leq \|p_n - x\|^2 - (1 - (\lambda_n \alpha)^2) \|p_n - u_n\|^2
\end{equation}

Similarly, we obtain

\begin{equation}
\|e_n - y\|^2 \leq \|q_n - y\|^2 - (1 - (\lambda_n \beta)^2) \|q_n - v_n\|^2
\end{equation}

On adding (30) and (31), we get

\begin{equation}
\|c_n - x\|^2 + \|e_n - y\|^2 \leq \|p_n - x\|^2 + \|q_n - y\|^2 - (1 - (\lambda_n \eta)^2)(\|p_n - u_n\|^2 + \|q_n - v_n\|^2),
\end{equation}

where \( \eta = \min\{\alpha, \beta\} \). Next, we estimate

\[
\|x_{n+1} - x\|^2 = \left\| \alpha_0 c_n + \sum_{i=1}^{\infty} \alpha_i w_{i,n} - x \right\|^2 \\
= \left\| \alpha_0 (c_n - x) + \sum_{i=1}^{\infty} \alpha_i (w_{i,n} - x) \right\|^2 \\
= \left\| \alpha_0 (c_n - x) + (1 - \alpha_0) \sum_{i=1}^{\infty} \frac{\alpha_i}{1 - \alpha_0} (w_{i,n} - x) \right\|^2 \\
= \alpha_0 \|c_n - x\|^2 + (1 - \alpha_0) \left( \left\| \sum_{i=1}^{\infty} \frac{\alpha_i}{1 - \alpha_0} (w_{i,n} - x) \right\|^2 \\
- \alpha_0 (1 - \alpha_0) \left\| \sum_{i=1}^{\infty} \frac{\alpha_i}{1 - \alpha_0} (w_{i,n} - c_n) \right\|^2 \right) \\
= \alpha_0 \|c_n - x\|^2 + (1 - \alpha_0) \left[ \left\| \sum_{i=1}^{\infty} \frac{\alpha_i}{1 - \alpha_0} w_{i,n} - x \right\|^2 \\
- \sum_{i,j=1, i \neq j}^{\infty} \frac{\alpha_i \alpha_j}{1 - \alpha_0} \|w_{i,n} - w_{j,n}\|^2 \right] - \alpha_0 (1 - \alpha_0) \sum_{i=1}^{\infty} \frac{\alpha_i}{1 - \alpha_0} \|w_{i,n} - c_n\|^2
\]
\[
\begin{align*}
= \alpha_0 \|c_n - x\|^2 + \sum_{i=1}^{\infty} \alpha_i \|w_{i,n} - x\|^2 \\
- \sum_{i=1}^{\infty} \alpha_0 \alpha_i \|c_n - w_{i,n}\|^2 - \sum_{i,j=1, i \neq j}^{\infty} \alpha_i \alpha_j \|w_{i,n} - w_{j,n}\|^2 \\
\leq \alpha_0 \|c_n - x\|^2 + \sum_{i=1}^{\infty} \alpha_i D^2(T_i c_n, T_i x) - \sum_{i=1}^{\infty} \alpha_0 \alpha_i d^2(c_n, T_i c_n) \\
\leq \alpha_0 \|c_n - x\|^2 + \sum_{i=1}^{\infty} \alpha_i \|c_n - x\|^2 \\
+ \sum_{i=1}^{\infty} \alpha_i k_i d^2(c_n, T_i c_n) - \sum_{i=1}^{\infty} \alpha_0 \alpha_i d^2(c_n, T_i c_n) \\
\leq \|c_n - x\|^2 - \sum_{i=1}^{\infty} \alpha_i (\alpha_0 - k_i) d^2(c_n, T_i c_n)
\end{align*}
\]

Similarly, we obtain

\[
\|y_{n+1} - y\|^2 \leq \|e_n - y\|^2 - (\alpha_0 - k_1) \sum_{i=1}^{\infty} \alpha_i d^2(e_n, S_i e_n).
\]

On adding the inequalities (34), (35) and using \(k = \max\{k_1, k_2\}\), we get

\[
\|x_{n+1} - x\|^2 + \|y_{n+1} - y\|^2 \leq \|x_n - x\|^2 + \|e_n - y\|^2 - (\alpha_0 - k) \left( \sum_{i=1}^{\infty} \alpha_i d^2(e_n, S_i e_n) + \sum_{i=1}^{\infty} \alpha_i d^2(e_n, S_i e_n) \right).
\]

On using (23) and (32) in (36), we obtain

\[
\|x_{n+1} - x\|^2 + \|y_{n+1} - y\|^2 \leq \|x_n - x\|^2 + \|y_n - y\|^2 - (1 - (\lambda_n \eta)^2) (\|p_n - u_n\|^2 + \|q_n - v_n\|^2) \\
- \gamma_n [2\|Ax_n - By_n\|^2 - \gamma_n (\|A^* (Ax_n - By_n)\|^2 + \|B^* (Ax_n - By_n)\|^2)] \\
- (\alpha_0 - k) \left( \sum_{i=1}^{\infty} \alpha_i d^2(c_n, T_i c_n) + \sum_{i=1}^{\infty} \alpha_i d^2(e_n, S_i e_n) \right).
\]
Now, setting $\rho_n(x,y) := \|x_n - x\|^2 + \|y_n - y\|^2$ in (37), we obtain

\[
\rho_{n+1}(x,y) \leq \rho_n(x,y) - (1 - (\lambda_n \eta)^2)(\|p_n - u_n\|^2 + \|q_n - v_n\|^2)
- \gamma_n[2\|Ax_n - By_n\|^2 - \gamma_n(\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2)]
\]

\[
= -(\alpha_0 - k)\left(\sum_{i=1}^{\infty} c_i d^2(c_n, T_i c_n) + \sum_{i=1}^{\infty} d^2(e_n, S_i e_n)\right).
\]

(38)

Since $\lambda_n < \frac{1}{n}$ and $\alpha_0 \in (k, 1)$ then $(\alpha_0 - k) > 0$ and hence it follows from condition (16) on $\gamma_n$ that

\[
\rho_{n+1}(x,y) \leq \rho_n(x,y).
\]

This implies that the sequence $\{\rho_n(x,y)\}$ is non-increasing and bounded below and hence it converges to $\rho(x,y)$ (say). Thus, condition (i) of Lemma 2.3 is satisfied with $\mu_n = (x_n, y_n)$, $\mu^* = (x, y)$ and $W := \Gamma \subset H = H_1 \times H_2$ with norm $\|(x,y)\| = (\|x\|^2 + \|y\|^2)^{\frac{1}{2}}$.

Since $\|x_n - x\|^2 \leq \rho_n(x,y)$, $\|y_n - y\|^2 \leq \rho_n(x,y)$ and $\lim_{n \to \infty} \rho_n(x,y)$ exists, we observe that $\{x_n\}$ and $\{y_n\}$ are bounded and $\limsup_{n \to \infty} \|x_n - x\|$ and $\limsup_{n \to \infty} \|y_n - y\|$ exist. Also, $\limsup_{n \to \infty} \|Ax_n - Ax\|$ and $\limsup_{n \to \infty} \|By_n - By\|$ exist. Further, from (24) and (32), we easily observe that the sequences $\{c_n\}, \{e_n\}, \{p_n\}$ and $\{q_n\}$ are bounded.

Now, since $\{\gamma_n\}$ is bounded, $(1 - (\lambda_n \eta)^2) > 0$ and $(\alpha_0 - k) > 0$ then it follows from the convergence of the sequence $\{\rho_n(x,y)\}$ and (38) that

\[
\lim_{n \to \infty} (\|p_n - u_n\|^2 + \|q_n - v_n\|^2) = 0,
\]

(39)

\[
\lim_{n \to \infty} (d^2(c_n, T_i c_n) + d^2(e_n, S_i e_n)) = 0, \quad \text{for each } i \in \mathbb{N},
\]

(40)

and

\[
\lim_{n \to \infty} (\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2) = 0.
\]

(41)

Note that $Ax_n - By_n = 0$, if $n \notin \Lambda$. Hence, we obtain

\[
\lim_{n \to \infty} \|p_n - u_n\| = \lim_{n \to \infty} \|q_n - v_n\| = 0,
\]

(42)

\[
\lim_{n \to \infty} d(c_n, T_i c_n) = 0, \quad \text{for each } i \in \mathbb{N},
\]

(43)
\[ \lim_{n \to \infty} d(e_n, S_ie_n) = 0, \text{ for each } i \in \mathbb{N}, \]

and

\[ \lim_{n \to \infty} \|Ax_n - By_n\| = 0. \tag{45} \]

Let \( \bar{x}, \bar{y} \) be weak cluster points of the bounded sequences \( \{x_n\}, \{y_n\} \), respectively. It follows from Lemma 2.2 (i) that

\[ \|x_{n+1} - x_n\|^2 = \|x_{n+1} - x - x_n + x\|^2 \]
\[ = \|x_{n+1} - x\|^2 - \|x_n - x\|^2 - 2 \langle x_{n+1} - x_n, x_n - x \rangle \]
\[ = \|x_{n+1} - x\|^2 - \|x_n - x\|^2 - 2 \langle x_{n+1} - \bar{x}, x_n - x \rangle + 2 \langle x_n - \bar{x}, x_n - x \rangle. \tag{46} \]

Since \( \limsup_{n \to \infty} \|x_n - x\| \) exists, it follows from (46) that

\[ \limsup_{n \to \infty} \|x_{n+1} - x_n\| = 0. \]

Consequently,

\[ \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{47} \]

Similarly, we obtain

\[ \lim_{n \to \infty} \|y_{n+1} - y_n\| = 0. \tag{48} \]

Since \( P_C \) is firmly nonexpansive, we have

\[ \|p_n - x\|^2 = \|P_C(x_n - \gamma_n A^*(Ax_n - By_n)) - x\|^2 \]
\[ \leq \langle p_n - x, x_n - \gamma_n A^*(Ax_n - By_n) - x \rangle \]
\[ = \frac{1}{2} \left\{ \|p_n - x\|^2 + \|x_n - \gamma_n A^*(Ax_n - By_n) - x\|^2 \right. \]
\[ - \|p_n - x_n + \gamma_n A^*(Ax_n - By_n)\|^2 \right\}. \]
which implies that
\[
\| p_n - x \|^2 \leq \| x_n - x \|^2 - 2\gamma_n \langle x_n - x, A^* (Ax_n - By_n) \rangle + \gamma_n^2 \| A^* (Ax_n - By_n) \|^2 - 2\gamma_n \langle p_n - x_n, A^* (Ax_n - By_n) \rangle \\
\leq \| x_n - x \|^2 + 2\gamma_n \| Ax_n - Ax \| \| Ax_n - By_n \| - \| p_n - x_n \|^2
\]
(49)
\[
+ 2\gamma_n \| Ap_n - Ax_n \| \| Ax_n - By_n \|.
\]

Since (33) can also be written as
\[
\| x_{n+1} - x \|^2 \leq \| c_n - x \|^2 + \sum_{i=1}^{\infty} \alpha_i k_i d^2 (c_n, T_i c_n).
\]
(50)
Using (30) and (49) in (50), we get
\[
\| x_{n+1} - x \|^2 \leq \| x_n - x \|^2 + 2\gamma_n \| Ax_n - Ax \| \| Ax_n - By_n \| - \| p_n - x_n \|^2 \\
+ 2\gamma_n \| Ap_n - Ax_n \| \| Ax_n - By_n \| + \sum_{i=1}^{\infty} \alpha_i k_i d^2 (c_n, T_i c_n), \\
-(1 - (\lambda_n \alpha)^2) \| p_n - u_n \|^2,
\]
which, in turn, implies that
\[
\| p_n - x_n \|^2 \leq (\| x_n - x \| + \| x_{n+1} - x \|) \| x_{n+1} - x_n \| + 2\gamma_n \| Ax_n - Ax \| \| Ax_n - By_n \| \\
+ 2\gamma_n \| Ap_n - Ax_n \| \| Ax_n - By_n \| + \sum_{i=1}^{\infty} \alpha_i k_i d^2 (c_n, T_i c_n).
\]
(51)

Since \{p_n\}, \{x_n\} are bounded and A is a bounded linear operator, then \{Ap_n - Ax_n\} is bounded. Now, using (45), (43) and (47) in (51), we have
\[
\lim_{n \to \infty} \| p_n - x_n \| = 0.
\]
(52)
Since
\[
\| u_n - x_n \| \leq \| u_n - p_n \| + \| p_n - x_n \|,
\]
(53)
using (42), (52) in (53), we have
\[
\lim_{n \to \infty} \| u_n - x_n \| = 0.
\]
(54)
Similarly, we obtain

\[(55) \lim_{n \to \infty} \| q_n - y_n \| = 0, \]

\[(56) \lim_{n \to \infty} \| v_n - y_n \| = 0. \]

It follows from (21), (26), (28) and (50) that
\[
\| x_n + 1 - x \|^2 \leq \| x_n - x \|^2 + 2\gamma_n \| Ax_n - Ax \| \| Ax_n - By_n \| - \| p_n - x_n \|^2 \\
+ 2\lambda_n \alpha \| p_n - u_n \| \| c_n - u_n \| + \sum_{i=1}^{\infty} \alpha_i k_i d^2 (c_n, T_i c_n),
\]
which, in turn, implies that

\[
\| u_n - c_n \|^2 \leq (\| x_n - x \| + \| x_n + 1 - x \|) \| x_n + 1 - x_n \| + 2\gamma_n \| Ax_n - Ax \| \| Ax_n - By_n \| \\
+ 2\gamma_n \| Ap_n - Ax_n \| \| Ax_n - By_n \| + \sum_{i=1}^{\infty} \alpha_i k_i d^2 (c_n, T_i c_n) \\
+ 2\lambda_n \alpha \| p_n - u_n \| \| c_n - u_n \|.
\]

(57)

Since \( \{c_n\}, \{u_n\} \) are bounded, using (45), (43) and (47) in (57), we have that

\[(58) \lim_{n \to \infty} \| u_n - c_n \| = 0. \]

Since

\[(59) \| c_n - x_n \| \leq \| c_n - u_n \| + \| u_n - x_n \|, \]

using (58), (54) in (59), we have

\[(60) \lim_{n \to \infty} \| c_n - x_n \| = 0. \]

Similarly, we obtain

\[(61) \lim_{n \to \infty} \| v_n - e_n \| = 0, \]

\[(62) \lim_{n \to \infty} \| e_n - y_n \| = 0. \]

Since every Hilbert space satisfies Opial’s condition, Opial’s condition guarantees that the weakly subsequential limit of \( \{x_n\} \) and \( \{y_n\} \) is unique. Since \( \{x_n\} \) is bounded, there exists
a subsequence \( \{x_{n_i}\} \) of \( \{x_n\} \) such that \( x_{n_i} \to \bar{x} \) and hence it follows from (60) that there is
a subsequence \( \{c_{n_i}\} \) of \( \{c_n\} \) such that \( c_{n_i} \to \bar{c} \). Further, demiclosedness of \( T_i \) at 0 for each
\( i \in \mathbb{N} \) and (43) imply that \( \bar{x} \in T_i \bar{x} \) for each \( i \in \mathbb{N} \). Hence \( \bar{x} \in \bigcap_{i=1}^{\infty} \text{Fix}(T_i) \). Also, it follows from
boundedness of \( \{y_n\} \) and (62) that there exist subsequences \( \{y_{n_i}\} \) of \( \{y_n\} \) and \( \{e_{n_i}\} \) of \( \{e_n\} \)
such that \( y_{n_i} \to \bar{y} \) and \( e_{n_i} \to \bar{y} \) and hence demiclosedness of \( S_i \) at 0 along with (44) yield that
\( \bar{y} \in S_i \bar{y} \) for each \( i \in \mathbb{N} \). Thus \( \bar{y} \in \bigcap_{i=1}^{\infty} \text{Fix}(S_i) \).

Now, we show that \( \bar{x} \in \text{Sol}(\text{VIP}(9)) \). Since \( \lim_{n \to \infty} \|p_n - u_n\| = 0 \) and \( \lim_{n \to \infty} \|p_n - x_n\| = 0 \), there
exist subsequences \( \{p_{n_i}\} \) and \( \{u_{n_i}\} \) of \( \{p_n\} \) and \( \{u_n\} \), respectively such that \( p_{n_i} \to \bar{x} \) and \( u_{n_i} \to \bar{x} \).
Let
\[
T_v = \begin{cases} 
fv + N_C(v), & \text{if } v \in C; \\
\emptyset, & \text{if } v \notin C,
\end{cases}
\]
where \( N_C(v) \) is the normal cone to \( C \) at \( v \in H_1 \). In this case, the mapping \( T \) is maximal monotone
and hence \( 0 \in Tv \) if and only if \( v \in \text{Sol}(\text{VIP}(9)) \). Let \( (v, w) \in \text{graph}(T) \). Then, we have \( w \in 
Tv = fv + N_C(v) \) and hence \( w - fv \in N_C(v) \). So, we have \( \langle v - u, w - fv \rangle \geq 0 \), for all \( u \in C \).
On the other hand, from \( u_n = P_C(I - \lambda_n f)p_n \) and \( v \in C \), we have
\[
\langle (I - \lambda_n f)p_n - u_n, u_n - v \rangle \geq 0.
\]
This implies that
\[
\langle v - u_n, \frac{u_n - p_n}{\lambda_n} + fp_n \rangle \geq 0.
\]
Since \( \langle v - u, w - fv \rangle \geq 0 \), for all \( u \in C \) and \( u_{n_i} \in C \), using monotonicity of \( f \), we have
\[
\langle v - u_{n_i}, w \rangle \geq \langle v - u_{n_i}, fv \rangle \\
\geq \langle v - u_{n_i}, fv \rangle - \left\langle v - u_{n_i}, \frac{u_{n_i} - p_{n_i}}{\lambda_n} + fp_{n_i} \right\rangle \\
= \langle v - u_{n_i}, fv - fu_{n_i} \rangle + \langle v - u_{n_i}, fu_{n_i} - fp_{n_i} \rangle - \left\langle v - u_{n_i}, \frac{u_{n_i} - p_{n_i}}{\lambda_n} \right\rangle \\
\geq \langle v - u_{n_i}, fu_{n_i} - fp_{n_i} \rangle - \left\langle v - u_{n_i}, \frac{u_{n_i} - p_{n_i}}{\lambda_n} \right\rangle.
\]
Since $f$ is continuous, then on taking limit $i \to \infty$, we have $\langle \nu - \bar{x}, w \rangle \geq 0$. Since $T$ is maximal monotone, we have $\bar{x} \in T^{-1}0$ and hence $\bar{x} \in \text{Sol}(\text{VIP}(9))$. Similarly, one can show that $\bar{y} \in \text{Sol}(\text{VIP}(10))$.

Again, since $A$ and $B$ are bounded linear operators, we have $Ax_n \rightharpoonup A\bar{x}$ and $By_n \rightharpoonup B\bar{y}$. Further, since $\| \cdot \|^2$ is weakly lower semicontinuous, we have

$$
\|A\bar{x} - B\bar{y}\|^2 = \liminf_{n_i \to \infty} \|Ax_{n_i} - By_{n_i}\|^2 = 0,
$$
i.e., $A\bar{x} = B\bar{y}$. Thus, $(\bar{x}, \bar{y}) \in \Gamma$ and hence $w_w(x_{n_i}, y_{n_i}) \subset \Gamma$. Now, it follows from Lemma 2.3 that the sequence $\{(x_n, y_n)\}$ generated by iterative Algorithm 3.1 converges weakly to $(\bar{x}, \bar{y}) \in \Gamma$.

Now, since $T_i$ and $S_i$ for each $i \in \mathbb{N}$, are hemi-compact, $\{x_n\}$ and $\{y_n\}$ are bounded and

$$
\lim_{n \to \infty} d(c_n, T_i c_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(e_n, S_i e_n) = 0
$$
for each $i \in \mathbb{N}$, there exist (without loss of generality) subsequences $\{x_{n_i}\}$ of $\{x_n\}$ and $\{y_{n_i}\}$ of $\{y_n\}$ such that $\{x_{n_i}\}$ and $\{y_{n_i}\}$ converge strongly to some points $\bar{u}$ and $\bar{v}$, respectively. It follows from the demiclosedness of $T_i$ and $S_i$, for each $i \in \mathbb{N}$ that $\bar{u} \in \bigcap_{i=1}^{\infty} \text{Fix}(T_i)$ and $\bar{v} \in \bigcap_{i=1}^{\infty} \text{Fix}(S_i)$. Since $\{x_{n_i}\}$ and $\{y_{n_i}\}$ converge weakly to $\bar{x}$ and $\bar{y}$, respectively, we then have $\bar{u} = \bar{x}$ and $\bar{v} = \bar{y}$. On the other hand, since $\rho_n(x, y) = \|x_n - x\|^2 + \|y_n - y\|^2$, for any $(x, y) \in \Gamma$ then $\lim_{n \to \infty} \rho_n(x, y) = 0$. Further, since $\lim_{n \to \infty} \rho_n(\bar{x}, \bar{y})$ exists then $\lim_{n \to \infty} \rho_n(\bar{x}, \bar{y}) = 0$ and hence $\lim_{n \to \infty} \|x_n - \bar{x}\| = 0$ and $\lim_{n \to \infty} \|y_n - \bar{y}\| = 0$. Thus, $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y}) \in \Gamma$. This completes the proof.

Now, we present a consequence of Theorem 4.1.

For each $i \in \mathbb{N}$, if $T_i$ and $S_i$ are single-valued demicontractive mappings then we have the following result to approximate a common solution of $S_p$-EVIP(9)-(10) and $S_p$-EFPP (5) for two countable families of single-valued demicontractive mappings:

**Corollary 4.1.** Let $H_1$, $H_2$ and $H_3$ be real Hilbert spaces and $C \subset H_1$, $Q \subset H_2$ be nonempty, closed and convex sets. Let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operators with their adjoint operators $A^*$ and $B^*$, respectively. Let $f : C \to H_1$ be monotone and $\alpha$-Lipschitz continuous mapping and let $g : Q \to H_2$ be monotone and $\beta$-Lipschitz continuous mapping. Let $\{T_i\}_{i=1}^{\infty} : H_1 \to H_1$ and $\{S_i\}_{i=1}^{\infty} : H_2 \to H_2$ be families of single-valued demicontractive mappings with demicontractive constants $k_i$ and $s_i$, respectively and let $k_1 = \sup_{i \geq 1} \{k_i\} \in (0, 1)$
and \(k_2 = \sup_{i \geq 1} \{s_i\} \in (0, 1)\). For each \(i \in \mathbb{N}\), let \(T_i\) and \(S_i\) be demiclosed at 0. Assume that 

\[
\Gamma := \text{Sol}(S_p\text{EVIP}(9) - (10)) \cap \left( \bigcap_{i=1}^{\infty} \text{Fix}(T_i) \times \bigcap_{i=1}^{\infty} \text{Fix}(S_i) \right) \neq \emptyset.
\]

If the sequence \(\{\lambda_n\}_{n \in \mathbb{N}} \subset [a, b]\), for some \(a\) and \(b\) with \(0 < a < b < \frac{1}{\min\{a, b\}}\) and \(k \in (0, 1)\) where \(k = \max\{k_1, k_2\}\), then the sequence \(\{(x_n, y_n)\}\) generated by Algorithm 3.2 converges weakly to \((\bar{x}, \bar{y}) \in \Gamma\). In addition, if for each \(i \in \mathbb{N}\), \(T_i\) and \(S_i\) are semicompact, then \(\{(x_n, y_n)\}\) converges strongly to \((\bar{x}, \bar{y}) \in \Gamma\).

We remark that it is of further research effort to extend the iterative method presented in this paper, to split equality mixed equilibrium problem and split equality monotone variational inclusion problem [14].

## 5. Numerical Example

Finally, we give a numerical example which justifies Theorem 4.1.

**Example 5.1.** Let \(H_1 = H_2 = H_3 = \mathbb{R}\) with the inner product defined by \(\langle x, y \rangle = xy, \forall x, y \in \mathbb{R}\), and induced usual norm \(|\cdot|\). Let \(C = [-10, 10]\) and \(Q = [-10, 10]\); let \(\{T_i\}_{i=1}^{\infty}, \{S_i\}_{i=1}^{\infty} : \mathbb{R} \rightarrow CB(\mathbb{R})\) by \(T_i(x) = \left\{ -\frac{1+i}{i}x \right\}, S_i(y) = \left\{ -\frac{1+2i}{2i}y \right\}\), for each \(i \in \mathbb{N}\); let \(f : C \rightarrow \mathbb{R}\) and \(g : Q \rightarrow \mathbb{R}\) be defined by \(f(x) = 2x, \forall x \in C\) and \(g(y) = 3y, \forall y \in Q\); let \(A, B : \mathbb{R} \rightarrow \mathbb{R}\) be defined by \(A(x) = 2x, \forall x \in \mathbb{R}\), \(B(y) = 4y, \forall y \in \mathbb{R}\). If we set \(\alpha_i = \frac{1}{2i+1}, \forall i \in \mathbb{N} \cup \{0\}\), then there is a unique sequence \(\{(x_n, y_n)\}\) generated by the iterative schemes:

\[
\begin{align*}
    p_n &= P_C(x_n - 4\gamma_n(x_n - 2y_n)), \\
    u_n &= P_C(p_n - 2\lambda_n p_n), \\
    c_n &= P_C(u_n - 2\lambda_n u_n), \\
    x_{n+1} &= \frac{1}{2}c_n + \sum_{i=1}^{\infty} \frac{1+i}{2i+1} c_n, \\
    q_n &= P_Q(y_n + 8\gamma_n(x_n - 2y_n)), \\
    v_n &= P_Q(q_n - 3\lambda_n q_n), \\
    e_n &= P_Q(q_n - 3\lambda_n v_n), \\
    y_{n+1} &= \frac{1}{2}e_n + \sum_{i=1}^{\infty} \frac{1+2i}{2i+1} e_n.
\end{align*}
\]

Then the sequence \(\{(x_n, y_n)\}\) converges to a point \((\bar{x}, \bar{y}) \in \Gamma\).
Proof. Evidently, $A$ and $B$ are bounded linear operators on $\mathbb{R}$ with adjoint operators $A^*$, $B^*$, respectively with $\|A\| = \|A^*\| = 2$, $\|B\| = \|B^*\| = 4$, and hence $\gamma_n \in \left(\varepsilon, \frac{1}{10} - \varepsilon\right)$. Therefore, for $\varepsilon = \frac{1}{100}$, we choose $\gamma_n = \frac{1}{20}$. We also assume $\lambda_n = \frac{2}{3}$. Furthermore, we observe that for each $i \in \mathbb{N}$, $T_i$ is demicontractive with $k_i = \frac{1}{1 + 2i}$, Fix($T_i$) = $\{0\}$ and $(T_i - I)$ is demiclosed at 0, and $S_i$ is demicontractive with $s_i = \frac{1}{1 + 4i}$, Fix($S_i$) = $\{0\}$ and $(S_i - I)$ is demiclosed at 0. Since $k_1 = \sup_{i \geq 1} \{k_i\} = \frac{1}{3}$ and $k_2 = \sup_{i \geq 1} \{s_i\} = \frac{1}{5}$ then $k = \max\{k_1, k_2\} = \frac{1}{3}$. Next, we observe that $\Gamma := \text{Sol}(S_p\text{EVIP}(9) - (10)) \cap \left(\bigcap_{i=1}^{\infty} \text{Fix}(T_i) \times \bigcap_{i=1}^{\infty} \text{Fix}(S_i)\right) = \{(0, 0)\} \neq \emptyset$.

After simplification, iterative schemes (64) are reduced to the following:

$$
\begin{align*}
p_n &= \frac{4}{5}x_n + \frac{2}{5}y_n; \quad u_n = -\frac{p_n}{3}; \quad c_n = p_n - \frac{4}{3}u_n; \\
q_n &= -\frac{2}{5}x_n + \frac{9}{5}y_n; \quad v_n = -q_n; \quad e_n = q_n - 2v_n; \\
x_{n+1} &= \frac{1}{2}c_n - \sum_{i=1}^{\infty} \frac{1 + i}{1 + 2i}c_n; \quad y_{n+1} = \frac{1}{2}e_n - \sum_{i=1}^{\infty} \frac{1 + 2i}{1 + 2i}e_n.
\end{align*}
$$

Next, using the software Matlab 7.8.0, we have following figure and table which shows that $\{(x_n, y_n)\}$ converges to the point $(\bar{x}, \bar{y}) = (0, 0)$.

![Convergence for initial values $x_0 = 8, y_0 = -13$](image-url)
This completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES


