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SIMULTANEOUS EXTRAGRADIENT ITERATIVE METHOD TO A SPLIT EQUALITY VARIATIONAL INEQUALITY PROBLEM AND A MULTIPLE-SETS SPLIT EQUALITY FIXED POINT PROBLEM FOR MULTI-VALUED DEMICONTRACTIVE MAPPINGS

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Abstract. This paper deals with a strong convergence theorem for a simultaneous extragradient iterative method to approximate a common solution to a split equality variational inequality problem and a multiple-sets split equality fixed point problem for two countable families of multi-valued demicontractive mappings in real Hilbert spaces. Further, we give a numerical example to justify the main result. The method and results presented in this paper extend and unify some recent known results in the literature.

Keywords: split equality variational inequality problem,; multiple-sets split equality fixed point problem; multi-valued demicontractive mapping; simultaneous extragradient iterative method; monotone mapping.

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1. INTRODUCTION

Let H_1, H_2 and H_3 be real Hilbert spaces, let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed and convex sets. We denote the inner products and induced norms of H_1, H_2 and H_3 by notations $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively.

The split feasibility problem (in short, S_pFP) is to find:

$$x^* \in C \text{ such that } Ax^* \in Q, \tag{1}$$

where $A : H_1 \to H_2$ is a bounded linear operator. The S_pFP(1) in finite dimensional Hilbert spaces was introduced by Censor and Elfving [5] for modeling inverse problem which arises from retrievals and in medical image reconstruction [4]. Since then various iterative methods have been proposed to solve S_pFP(1); see for instance [1, 3, 12, 21].

Censor *et al*. [7] proposed the following multiple-sets split feasibility problem (in short, MSS_pFP), which arises in applications such as intensity modulated radiation therapy [20]:

$$x^* \in \bigcap_{i=1}^{N} C_i \text{ such that } Ax^* \in \bigcap_{j=1}^{M} Q_j, \tag{2}$$

where N and M are positive integers, for each i, j, $C_i \subset H_1$ and $Q_j \subset H_2$ are nonempty, closed and convex sets.

A mapping $F_1 : H_1 \to H_1$ is said to be firmly quasi-nonexpansive if $\text{Fix}(F_1) \neq \emptyset$ and

$$\|F_1 x - x^*\|^2 \leq \|x - x^*\|^2 - \|x - F_1 x\|^2, \ \forall x^* \in \text{Fix}(F_1), \ x \in H_1, \tag{3}$$

where $\text{Fix}(F_1) := \{x \in H_1 : x = F_1 x\}$, the set of fixed points of F_1.

A mapping $F_1 : C \to C$ is said to be k-demicontractive if $\text{Fix}(F_1) \neq \emptyset$ and there exists a constant $k \in (0,1)$ such that

$$\|F_1 x - x^*\|^2 \leq \|x - x^*\|^2 + k\|x - F_1 x\|^2, \ \forall x^* \in \text{Fix}(F_1), \ x \in C. \tag{4}$$

Evidently, the class of demicontractive mappings properly includes the class of firmly quasi-nonexpansive mappings.

Remark 1.1. [8] For negative values of k the class of demicontractive mappings is diminished to a great extent. Such class with negative value of k was considered under the name of strongly attracting mapping. In particular, the mapping F_1 which satisfies (4) with $k=-1$ is called pseudo-contractive. Note that a mapping F_1 satisfying (4) with $k=1$ is usually called hemicontractive and used in connection with the strong convergence of the implicit Mann-type iteration method.

Example 1.1. [11] Let f be a real function defined by $f(x)=-x^2-x$; it can be seen that $f:[-2,1]\to[-2,1]$. This function is demicontractive on $[-2,1]$ and continuous. It is not quasi-nonexpansive and is not pseudo-contractive on $[-2,1]$.

Rest of the paper, unless specified, let $A:H_1\to H_2$ and $B:H_2\to H_3$ be bounded linear operators.

In 2013, Moudafi *et al.* [17] introduced and studied the following split equality fixed point problem (in short, $\mathrm{S_pEFPP}$) which is a generalization of $\mathrm{S_pFP}$ (1): Find $x^*\in C$ and $y^*\in Q$ such that

$$x^*\in \mathrm{Fix}(F_1),\ y^*\in \mathrm{Fix}(F_2) \text{ and } Ax^*=By^*, \tag{5}$$

where for each $i=1,2$, $F_i:H_i\to H_i$ is a quasi-nonexpansive mapping. Further, Chidume *et al.* [9] studied $\mathrm{S_pEFPP}$ (5) for demicontractive mappings F_1, F_2. For further related work, see [14].

We denote by $CB(H_1)$, the collection of all nonempty, closed and bounded subsets of H_1. The Hausdorff metric D on $CB(H_1)$ is defined by

$$D(P,Q)=\max\left\{\sup_{x\in P} d(x,Q),\sup_{y\in Q} d(y,P)\right\},\ \forall P,Q\in CB(H_1),$$

where $d(x,P):=\inf\limits_{y\in P} d(x,y)$ and $d(\cdot,\cdot)$ is a metric on H_1.

Definition 1.1. Let $T_1:H_1\rightrightarrows CB(H_1)$ be a multi-valued mapping. $x^*\in H_1$ is said to be fixed point of T_1 if $x^*\in T_1x^*$. We denote by $\mathrm{Fix}(T_1)$, the set of fixed points of T_1 defined by

$$\mathrm{Fix}(T_1):=\{x\in H_1 : x\in T_1x\}.$$

Definition 1.2. A multi-valued mapping $T_1:\mathscr{D}(T_1)\subset H_1\rightrightarrows CB(H_1)$ is said to be:

(i) nonexpansive if

$$D(T_1x,T_1y)\le\|x-y\|,\ \forall x,y\in\mathscr{D}(T_1);$$

(ii) quasi-nonexpansive if $\mathrm{Fix}(T_1) \neq \emptyset$ and

(6) $$D(T_1x, T_1x^*) \leq \|x - x^*\|, \ \forall x^* \in \mathrm{Fix}(T_1), \ x \in \mathscr{D}(T_1);$$

(iii) k-demicontractive if $\mathrm{Fix}(T_1) \neq \emptyset$ and there exists a constant $k \in (0,1)$ such that

$$(D(T_1x, T_1x^*))^2 \leq \|x - x^*\|^2 + k\,(D(x, T_1x))^2, \ \forall x^* \in \mathrm{Fix}(T_1), \ x \in \mathscr{D}(T_1),$$

where $\mathscr{D}(T_1)$ denotes the domain of T_1.

Evidently, the class of multi-valued demicontractive mappings properly includes the class of multi-valued quasi-nonexpansive mappings. The class of demicontractive mappings is important because several common types of operators arising in optimization problems belong to this class, see for example, Chidume and Maruster [8], Maruster and Popirlan [16] and references therein.

Example 1.2. Let $H_1 = \mathbb{R}$, the set of all real numbers, $T_1 : \mathbb{R} \to CB(\mathbb{R})$ be defined by $T_1(x) = \{-2x\}, \forall x \in \mathbb{R}$. We have that $\mathrm{Fix}(T_1) = \{0\}$ and T_1 is a multi-valued demicontractive mapping which is not quasi-nonexpansive. In fact, for each $x \in \mathbb{R}$, we have

$$(D(T_1x, T_10))^2 = 4|x - 0|^2,$$

which implies that T_1 is not quasi-nonexpansive. Also we have

$$(D(T_1x, T_10))^2 = |x - 0|^2 + \frac{1}{3}(d(x, T_1x))^2.$$

This implies that T_1 is demicontractive with $k = \frac{1}{3}$.

In 2014, Wu *et al*. [23] introduced and studied the following multiple-sets split equality problem for finite families of multi-valued quasi-nonexpansive mappings:

(7) $$\text{Find } x^* \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \text{ and } y^* \in \bigcap_{i=1}^{N} \mathrm{Fix}(S_i) \text{ such that } Ax^* = By^*,$$

where N is a positive integer, and $\{T_i\}_{i=1}^N : H_1 \rightrightarrows CB(H_1)$, $\{S_i\}_{i=1}^N : H_2 \rightrightarrows CB(H_2)$ are families of multi-valued quasi-nonexpansive mappings.

Very recently, Chidume [11] introduced and studied the following multiple-sets split equality fixed point problem (in short, $\text{MSS}_\text{p}\text{EFPP}$) for countable families of multi-valued demicontractive mappings:

$$\text{Find } x^* \in \bigcap_{i=1}^{\infty} \text{Fix}(T_i) \text{ and } y^* \in \bigcap_{i=1}^{\infty} \text{Fix}(S_i) \text{ such that } Ax^* = By^*, \tag{8}$$

where $\{T_i\}_{i=1}^{\infty} : H_1 \rightrightarrows CB(H_1)$ and $\{S_i\}_{i=1}^{\infty} : H_2 \rightrightarrows CB(H_2)$ are countable families of multi-valued demicontractive mappings.

We consider the following split equality variational inequality problem (in short, $\text{S}_\text{p}\text{EVIP}$): Find $x^* \in C$ and $y^* \in Q$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0, \forall x \in C \tag{9}$$

$$\langle g(y^*), y - y^* \rangle \geq 0, \forall y \in Q \tag{10}$$

$$\text{and } Ax^* = By^*,$$

where $f : C \to H_1$ and $g : Q \to H_2$ be single-valued mappings. When looked separately, (9) is called variational inequality problem (in short, VIP) and its solution set is denoted by Sol(VIP(9)). The solution set of $\text{S}_\text{p}\text{EVIP}$(9)-(10) is denoted by Sol($\text{S}_\text{p}\text{EVIP}$(9)-(10))$= \{(x^*, y^*) \in C \times Q : x^* \in \text{Sol}(\text{VIP}(9)), y^* \in \text{Sol}(\text{VIP}(10)) \text{ and } Ax^* = By^*\}$. $\text{S}_\text{p}\text{EVIP}$(9)-(10) generalizes split variational inequality problem (in short, $\text{S}_\text{p}\text{VIP}$) studied by Censor *et al.* [6].

In 1976, Korpelevich [13] introduced the following iterative method which is known as extragradient iterative method:

$$\begin{cases} x_0 & \in C, \\ y_n & = P_C(x_n - \lambda f x_n), \\ x_{n+1} & = P_C(x_n - \lambda f y_n),\ n \geq 0, \end{cases} \tag{11}$$

where $\lambda > 0$ is a fixed number, f is a monotone and Lipschitz continuous mapping and P_C is the metric projection of H_1 onto C; and proved that if the Sol(VIP(9)) is nonempty then, under some suitable conditions, the sequence $\{x_n\}$ generated by algorithm (11) converges to a solution of variational inequality (9). Since then a number of generalizations of extragradient iterative method has been studied for various important classes of problems, see for instance [15, 18, 22] and the revelent references therein.

Motivated by the ongoing research work in this direction, we propose and analyze a simultaneous extragradient iterative method to approximate a common solution to $\mathrm{S_pEVIP}$(9)-(10) and $\mathrm{MSS_pEFPP}$(8) for countable families of multi-valued demicontractive mappings in real Hilbert spaces. Further, we give a numerical example to justify the main result. The method and results presented in this paper extend and unify some recent known results in the literature; see for instance [6, 9, 11, 17, 24].

2. PRELIMINARIES

We recall some definitions and results which are needed in sequel. Let $\to$ and $\rightharpoonup$ denote the strong and weak convergence, respectively and $\mathbb{N}$ denote the set of natural numbers.

For every point $x \in H_1$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \ \forall y \in C.$$

The mapping P_C is called the *metric projection* of H_1 onto C. It is known that P_C is nonexpansive and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \ \forall x \in H_1. \tag{12}$$

Moreover, $P_C x$ is characterized by the fact that $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \ \forall y \in C \tag{13}$$

which implies that

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \ \forall x \in H_1, \ y \in C. \tag{14}$$

Definition 2.1. A mapping $f : C \to H_1$ is said to be:

(i) *monotone*, if

$$\langle fx - fy, x - y\rangle \geq 0, \ \forall x, y \in H_1;$$

(ii) α-*inverse strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle fx - fy, x - y\rangle \geq \alpha\|fx - fy\|^2, \ \forall x, y \in H_1;$$

(iii) β-*Lipschitz continuous*, if there exists a constant $\beta > 0$ such that

$$\|fx - fy\| \leq \beta\|x - y\|, \ \forall x, y \in H_1.$$

We note that if f is α-inverse strongly monotone mapping, then f is monotone and $\frac{1}{\alpha}$-Lipschitz continuous but converse need not be true. For $\alpha = 1$, α-inverse strongly monotone mapping f is called firmly nonexpansive mapping.

Definition 2.2. A mapping $T_1 : H_1 \to H_1$ is said to be:

(i) demiclosed at zero if for any sequence $\{x_n\} \subset H_1$, with $x_n \rightharpoonup x^*$ and $\|x_n - T_1x_n\| \to 0$, we have $x^* = T_1x^*$.

(ii) semicompact if for any bounded sequence $\{x_n\} \subset H_1$, with $\|x_n - T_1x_n\| \to 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to some $x^* \in H_1$.

Definition 2.3. [2]. A multi-valued mapping $T_1 : H_1 \rightrightarrows 2^{H_1}$ is said to be:

(i) monotone if

$$\langle u - v, x - y\rangle \geq 0, \ \textit{whenever } u \in T_1(x), \ v \in T_1(y);$$

(ii) maximal monotone if T_1 is monotone and the graph, graph$(T_1) := \{(x,y) \in H_1 \times H_1 : y \in T_1(x)\}$, is not properly contained in the graph of any other monotone mapping.

It is well known that for each $x \in H_1$ and $\lambda > 0$ there is a unique $z \in H_1$ such that $x \in (I + \lambda T_1)z$. The mapping $J_\lambda^{T_1} := (I + \lambda T_1)^{-1}$ is called the resolvent of T_1. It is a single-valued and firmly nonexpansive mapping defined on H_1.

Definition 2.4. A multi-valued mapping $T_1 : H_1 \rightrightarrows CB(H_1)$ is said to be:

(i) demiclosed at zero if for any sequence $\{x_n\} \subset H_1$, with $x_n \rightharpoonup x^*$ and $d(x_n, T_1x_n) \to 0$, we have $x^* \in T_1x^*$.

(ii) hemicompact if for any bounded sequence $\{x_n\} \subset H_1$, with $d(x_n, T_1x_n) \to 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to some $x^* \in H_1$.

Lemma 2.1.[11] Let C be a nonempty subset of a real Hilbert space H_1 and let $T_1 : C \rightrightarrows CB(C)$ be a multi-valued k-demicontractive mapping. Let for every $z \in \text{Fix}(T_1)$, $T_1 z = \{z\}$. Then there exists $L > 0$ such that

$$D(T_1 x, T_1 z) \quad \leq \quad L\|x - z\|,\ \forall x \in C,\ z \in \text{Fix}(T_1).$$

Lemma 2.2. For all $x, y \in H_1$, we have

(i) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$;

(ii) $2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x-y\|^2 = \|x+y\|^2 - \|x\|^2 - \|y\|^2$.

Lemma 2.3.[19] (Opial's lemma) Let $\{\mu_n\}$ be a sequence in Hilbert space H_1, such that there exists a nonempty set $W \subset H_1$ satisfying:

(i) For every $\mu^* \in W$, $\lim\limits_{n\to\infty} \|\mu_n - \mu^*\|$ exists.

(ii) Any weak-cluster point of the sequence $\{\mu_n\}$ belongs to W;

Then there exists $\mu^* \in W$ such that $\{\mu_n\}$ weakly converges to μ^*.

Lemma 2.4.[10] Let $\{x_i\}_{i=1}^m$ be a set in Hilbert space H_1. For $\{\alpha_i\}_{i=1}^m \subset (0,1)$ such that $\sum\limits_{i=1}^m \alpha_i = 1$. Then the following identity holds:

$$\left\|\sum_{i=1}^m \alpha_i x_i\right\|^2 = \sum_{i=1}^m \alpha_i \|x_i\|^2 - \sum_{1\leq i<j\leq m} \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Remark 2.1. It follows from Lemma 2 that the following identity holds:

$$\left\|\sum_{i=1}^\infty \alpha_i x_i\right\|^2 = \sum_{i=1}^\infty \alpha_i \|x_i\|^2 - \sum_{i,j=1, i\neq j}^\infty \alpha_i \alpha_j \|x_i - x_j\|^2.$$

for $\sum\limits_{i=1}^\infty \alpha_i = 1$, provided that $\{x_i\}$ is bounded.

3. SIMULTANEOUS EXTRAGRADIENT ITERATIVE ALGORITHMS

We propose the following simultaneous extragradient iterative algorithm to approximate a common solution of $\text{S}_\text{p}\text{EVIP}(9) - (10)$ and $\text{MSS}_\text{p}\text{EFPP}(8)$.

Algorithm 3.1. Let $(x_1, y_1) \in H_1 \times H_2$ be given. The iteration sequences $\{(x_n, y_n)\}$ be generated by the schemes:

$$\left\{\begin{array}{lcl} p_n & = & P_C(x_n - \gamma_n A^*(Ax_n - By_n)); \\ u_n & = & P_C(I - \lambda_n f)p_n; \\ c_n & = & P_C(p_n - \lambda_n f u_n); \\ x_{n+1} & = & \alpha_0 c_n + \sum_{i=1}^{\infty} \alpha_i w_{i,n},\ w_{i,n} \in T_i c_n; \\ q_n & = & P_Q(y_n + \gamma_n B^*(Ax_n - By_n)); \\ v_n & = & P_Q(I - \lambda_n g)q_n; \\ e_n & = & P_Q(q_n - \lambda_n g v_n); \\ y_{n+1} & = & \alpha_0 e_n + \sum_{i=1}^{\infty} \alpha_i z_{i,n},\ z_{i,n} \in S_i e_n, \end{array}\right. \tag{15}$$

where $\alpha_0 \in (k, 1),\ \alpha_i \in (0, 1)$, for each $i \in \mathbb{N}$ such that $\sum_{i=0}^{\infty} \alpha_i = 1$ and the step size γ_n is chosen in such a way that for some $\varepsilon > 0$,

$$\gamma_n \in (\varepsilon, \mu_n - \varepsilon),\ n \in \Lambda, \tag{16}$$

otherwise $\gamma_n = \gamma\,(\gamma \geq 0)$, where $\mu_n := \dfrac{2\|Au_n - Bv_n\|^2}{\|A^*(Au_n - Bv_n)\|^2 + \|B^*(Au_n - Bv_n)\|^2}$ and the index set $\Lambda = \{n : Ax_n - By_n \neq 0\}$.

Remark 3.1. [[24]] It follows from condition (16) that $\inf_{n\in\Lambda}\{\mu_n - \gamma_n\} > 0$. Since $\|A^*(Au_n - Bv_n)\| \leq \|A^*\|\|Au_n - Bv_n\|$ and $\|B^*(Au_n - Bv_n)\| \leq \|B^*\|\|Au_n - Bv_n\|$, we observe that $\{\mu_n\}$ is bounded below by $\frac{2}{\|A\|^2+\|B\|^2}$ and so $\inf_{n\in\Lambda} \mu_n < +\infty$. Consequently $\sup_{n\in\Lambda} \gamma_n < +\infty$ and hence $\{\gamma_n\}$ is bounded.

For each $i \in \mathbb{N}$, if T_i and S_i are single-valued demicontractive mappings then Algorithm 3.1 is reduced to the following simultaneous extragradient iterative algorithm to approximate a common solution of S_pEVIP(9)-(10) and $\mathrm{S_p}$EFPP (5):

Algorithm 3.2. Let $(x_1, y_1) \in H_1 \times H_2$ be given. The iteration sequences $\{(x_n, y_n)\}$ be generated by the schemes:

$$\left\{\begin{array}{lcl} p_n & = & P_C(x_n - \gamma_n A^*(Ax_n - By_n)); \\ u_n & = & P_C(I - \lambda_n f)p_n; \\ c_n & = & P_C(p_n - \lambda_n f u_n); \\ x_{n+1} & = & \alpha_0 c_n + \sum\limits_{i=1}^{\infty} \alpha_i T_i c_n; \\ q_n & = & P_Q(y_n + \gamma_n B^*(Ax_n - By_n)); \\ v_n & = & P_Q(I - \lambda g)q_n; \\ e_n & = & P_Q(q_n - \lambda_n g v_n); \\ y_{n+1} & = & \alpha_0 e_n + \sum\limits_{i=1}^{\infty} \alpha_i S_i e_n, \end{array}\right. \tag{17}$$

where $\alpha_0 \in (k, 1),\ \alpha_i \in (0, 1)$, for each $i \in \mathbb{N}$ such that $\sum\limits_{i=0}^{\infty} \alpha_i = 1$ and the step size γ_n is chosen in such a way that for some $\varepsilon > 0$,

$$\gamma_n \in (\varepsilon, \mu_n - \varepsilon),\ n \in \Lambda, \tag{18}$$

otherwise $\gamma_n = \gamma\,(\gamma \geq 0)$, where $\mu_n := \dfrac{2\|Au_n - Bv_n\|^2}{\|A^*(Au_n - Bv_n)\|^2 + \|B^*(Au_n - Bv_n)\|^2}$ and the index set $\Lambda = \{n : Ax_n - By_n \neq 0\}$.

4. Main results

We prove a strong convergence theorem to approximate a common solution to S_pEVIP(9)-(10) and MSS_pEFPP(8) for countable families of multi-valued demicontractive mappings by selecting the step size in such a way that the implementation of the algorithm does not require the calculation or estimation of the operator norms.

Theorem 4.1. Let H_1, H_2 and H_3 be real Hilbert spaces and $C \subset H_1$, $Q \subset H_2$ be nonempty, closed and convex sets. Let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operators with their adjoint operators A^* and B^*, respectively. Let $f : C \to H_1$ be monotone and α-Lipschitz continuous mapping and let $g : Q \to H_2$ be monotone and β-Lipschitz continuous mapping. Let $\{T_i\}_{i=1}^{\infty} : H_1 \rightrightarrows CB(H_1)$ and $\{S_i\}_{i=1}^{\infty} : H_2 \rightrightarrows CB(H_2)$ be families of multi-valued demicontractive mappings with demicontractive constants k_i and s_i, respectively and let $k_1 = \sup\limits_{i \geq 1}\{k_i\} \in (0, 1)$

and $k_2 = \sup_{i\geq 1}\{s_i\} \in (0,1)$. For each $i \in \mathbb{N}$, let T_i and S_i be demiclosed at 0. Assume that for $x \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i)$, $T_i x = \{x\}$ and for $y \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i)$, $S_i y = \{y\}$, for each $i \in \mathbb{N}$. Assume that $\Gamma := \mathrm{Sol}(\mathrm{S_pEVIP}(9) - (10)) \bigcap \left(\bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \times \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i) \right) \neq \emptyset$. If the sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset [a,b]$, for some a and b with $0 < a < b < \frac{1}{\min\{\alpha,\beta\}}$ and $k \in (0,1)$ where $k = \max\{k_1,k_2\}$, then the sequence $\{(x_n,y_n)\}$ generated by Algorithm 3.1 converges weakly to $(\bar{x},\bar{y}) \in \Gamma$. In addition, if for each $i \in \mathbb{N}$, T_i and S_i are hemicompact, then $\{(x_n,y_n)\}$ converges strongly to $(\bar{x},\bar{y}) \in \Gamma$.

Proof. Let $(x,y) \in \Gamma$, i.e., for each $i \in \mathbb{N}$, $x \in T_i(x)$, $y \in S_i(y)$, $x \in \mathrm{Sol}(\mathrm{VIP}(9))$, $y \in \mathrm{Sol}(\mathrm{VIP}(10))$ and $Ax = By$. First, we prove that $\{w_{i,n}\}_{i=0}^{\infty}$ is bounded. Indeed, it follow from Lemma 2.1 that

$$\begin{aligned} \|w_{i,n} - x\| &\leq D(T_i c_n, T_i x) \\ &\leq \frac{1+\sqrt{k_1}}{1-\sqrt{k_1}} \|c_n - x\| := M_n. \end{aligned}$$

This implies that $\{w_{i,n}\}_{i=0}^{\infty}$ is bounded. Similarly, we obtain that $\{z_{i,n}\}_{i=0}^{\infty}$ is bounded.

We estimate

$$\begin{aligned} \|p_n - x\|^2 &= \|P_C(x_n - \gamma_n A^*(Ax_n - By_n)) - x\|^2 \\ &\leq \|x_n - \gamma_n A^*(Ax_n - By_n)) - x\|^2 \\ &\leq \|x_n - x\|^2 - 2\gamma_n \langle x_n - x, A^*(Ax_n - By_n)\rangle + \gamma_n^2 \|A^*(Ax_n - By_n)\|^2 \\ &\leq \|x_n - x\|^2 - 2\gamma_n \langle Ax_n - Ax, Ax_n - By_n\rangle + \gamma_n^2 \|A^*(Ax_n - By_n)\|^2 \quad (19) \\ &\leq \|x_n - x\|^2 + 2\gamma_n \|Ax_n - Ax\| \|Ax_n - By_n\| + \gamma_n^2 \|A^*(Ax_n - By_n)\|^2. \quad (20) \end{aligned}$$

Using Lemma 2.2 (ii) in (19), we get

$$\begin{aligned} \|p_n - x\|^2 \leq \|x_n - x\|^2 - \gamma_n \|Ax_n - Ax\|^2 - \gamma_n \|Ax_n - By_n\|^2 + \gamma_n \|By_n - Ax\|^2 \\ + \gamma_n^2 \|A^*(Ax_n - By_n)\|^2. \quad (21) \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} \|q_n - y\|^2 \leq \;& \|y_n - y\|^2 - \gamma_n \|By_n - By\|^2 - \gamma_n \|Ax_n - By_n\|^2 + \gamma_n \|Ax_n - By\|^2 \\ & + \gamma_n^2 \|B^*(Ax_n - By_n)\|^2. \quad (22) \end{aligned}$$

Adding (21) and (22), and using the fact that $Ax = By$, we get

$$\begin{aligned} \|p_n - x\|^2 + \|q_n - y\|^2 &\leq \|x_n - x\|^2 + \|y_n - y\|^2 - \gamma_n[2\|Ax_n - By_n\|^2 \\ &\quad -\gamma_n(\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2)]. \end{aligned} \tag{23}$$

From condition (16) on γ_n, we obtain from (23) that

$$\|p_n - x\|^2 + \|q_n - y\|^2 \leq \|x_n - x\|^2 + \|y_n - y\|^2. \tag{24}$$

Since $c_n = P_C(p_n - \lambda_n f u_n)$, it follows from (14) that

$$\begin{aligned} \|c_n - x\|^2 &= \|P_C(p_n - \lambda_n f u_n) - x\|^2 \\ &\leq \|p_n - \lambda_n f u_n - x\|^2 - \|p_n - \lambda_n f u_n - c_n\|^2 \\ &\leq \|p_n - x\|^2 - \|p_n - c_n\|^2 + 2\lambda_n \langle f u_n, x - c_n \rangle \\ &\leq \|p_n - x\|^2 - \|p_n - c_n\|^2 + 2\lambda_n [\langle f u_n - f x, x - u_n \rangle \\ &\quad + \langle f x, x - u_n \rangle + \langle f u_n, u_n - c_n \rangle] \end{aligned} \tag{25}$$

Since f is monotone and the fact that $x \in \mathrm{Sol}(\mathrm{VIP}(9))$, we obtain from (25) that

$$\begin{aligned} \|c_n - x\|^2 &\leq \|p_n - x\|^2 - \|p_n - c_n\|^2 + 2\lambda_n \langle f u_n, u_n - c_n \rangle \\ &= \|p_n - x\|^2 - \|p_n - u_n\|^2 - \|u_n - c_n\|^2 - 2\langle p_n - u_n, u_n - c_n \rangle + 2\lambda_n \langle f u_n, u_n - c_n \rangle \\ &= \|p_n - x\|^2 - \|p_n - u_n\|^2 - \|u_n - c_n\|^2 + 2\lambda_n \langle p_n - \lambda_n f u_n - u_n, c_n - u_n \rangle \end{aligned} \tag{26}$$

From (13), we have

$$\begin{aligned} \langle p_n - \lambda_n f u_n - u_n, c_n - u_n \rangle &= \langle p_n - \lambda_n f p_n - u_n, c_n - u_n \rangle + \lambda_n \langle f p_n - f u_n, c_n - u_n \rangle \\ &\leq \lambda_n \langle f p_n - f u_n, c_n - u_n \rangle \end{aligned} \tag{27}$$

Since f is α-Lipschitz-continuous, we obtain

$$\begin{aligned} 2\langle p_n - \lambda_n f u_n - u_n, c_n - u_n \rangle &\leq 2\lambda_n \|f p_n - f u_n\| \|c_n - u_n\| \\ &\leq 2\lambda_n \alpha \|p_n - u_n\| \|c_n - u_n\| \qquad (28) \\ &\leq (\lambda_n \alpha)^2 \|p_n - u_n\|^2 + \|c_n - u_n\|^2. \qquad (29) \end{aligned}$$

Hence, from (26) and (29), we obtain

$$\|c_n - x\|^2 \leq \|p_n - x\|^2 - (1-(\lambda_n\alpha)^2)\|p_n - u_n\|^2 \tag{30}$$

Similarly, we obtain

$$\|e_n - y\|^2 \leq \|q_n - y\|^2 - (1-(\lambda_n\beta)^2)\|q_n - v_n\|^2 \tag{31}$$

On adding (30) and (31), we get

$$\|c_n - x\|^2 + \|e_n - y\|^2 \leq \|p_n - x\|^2 + \|q_n - y\|^2 - (1-(\lambda_n\eta)^2)(\|p_n - u_n\|^2 + \|q_n - v_n\|^2), \tag{32}$$

where $\eta = \min\{\alpha, \beta\}$. Next, we estimate

$$\begin{aligned}
\|x_{n+1} - x\|^2 &= \left\|\alpha_0 c_n + \sum_{i=1}^{\infty}\alpha_i w_{i,n} - x\right\|^2 \\
&= \left\|\alpha_0(c_n - x) + \sum_{i=1}^{\infty}\alpha_i(w_{i,n} - x)\right\|^2 \\
&= \left\|\alpha_0(c_n - x) + (1-\alpha_0)\sum_{i=1}^{\infty}\frac{\alpha_i}{1-\alpha_0}(w_{i,n} - x)\right\|^2 \\
&= \alpha_0\|c_n - x\|^2 + (1-\alpha_0)\left\|\sum_{i=1}^{\infty}\frac{\alpha_i}{1-\alpha_0}(w_{i,n} - x)\right\|^2 \\
&\quad -\alpha_0(1-\alpha_0)\left\|\sum_{i=1}^{\infty}\frac{\alpha_i}{1-\alpha_0}(w_{i,n} - c_n)\right\|^2 \\
&= \alpha_0\|c_n - x\|^2 + (1-\alpha_0)\left[\sum_{i=1}^{\infty}\frac{\alpha_i}{1-\alpha_0}\|w_{i,n} - x\|^2\right. \\
&\quad \left. - \sum_{i,j=1,i\neq j}^{\infty}\frac{\alpha_i\alpha_j}{1-\alpha_0}\|w_{i,n} - w_{j,n}\|^2\right] - \alpha_0(1-\alpha_0)\sum_{i=1}^{\infty}\frac{\alpha_i}{1-\alpha_0}\|w_{i,n} - c_n\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_0\|c_n - x\|^2 + \sum_{i=1}^{\infty}\alpha_i\|w_{i,n} - x\|^2 \\
&\quad - \sum_{i=1}^{\infty}\alpha_0\alpha_i\|c_n - w_{i,n}\|^2 - \sum_{i,j=1,i\neq j}^{\infty}\alpha_i\alpha_j\|w_{i,n} - w_{j,n}\|^2 \\
&\leq \alpha_0\|c_n - x\|^2 + \sum_{i=1}^{\infty}\alpha_i\|w_{i,n} - x\|^2 - \sum_{i=1}^{\infty}\alpha_0\alpha_i d^2(c_n, T_i c_n) \\
&\leq \alpha_0\|c_n - x\|^2 + \sum_{i=1}^{\infty}\alpha_i D^2(T_i c_n, T_i x) - \sum_{i=1}^{\infty}\alpha_0\alpha_i d^2(c_n, T_i c_n) \\
&\leq \alpha_0\|c_n - x\|^2 + \sum_{i=1}^{\infty}\alpha_i\|c_n - x\|^2 \\
&\quad + \sum_{i=1}^{\infty}\alpha_i k_i d^2(c_n, T_i c_n) - \sum_{i=1}^{\infty}\alpha_0\alpha_i d^2(c_n, T_i c_n) \\
&= \|c_n - x\|^2 - \sum_{i=1}^{\infty}\alpha_i(\alpha_0 - k_i) d^2(c_n, T_i c_n) \qquad (33) \\
&\leq \|c_n - x\|^2 - (\alpha_0 - k_1)\sum_{i=1}^{\infty}\alpha_i d^2(c_n, T_i c_n). \qquad (34)
\end{aligned}
$$

Similarly, we obtain

$$
\|y_{n+1} - y\|^2 \leq \|e_n - y\|^2 - (\alpha_0 - k_2)\sum_{i=1}^{\infty}\alpha_i d^2(e_n, S_i e_n). \qquad (35)
$$

On adding the inequalities (34), (35) and using $k = \max\{k_1, k_2\}$, we get

$$
\begin{aligned}
\|x_{n+1} - x\|^2 + \|y_{n+1} - y\|^2 &\leq \|c_n - x\|^2 + \|e_n - y\|^2 - (\alpha_0 - k)\Bigg(\sum_{i=1}^{\infty}\alpha_i d^2(c_n, T_i c_n) \\
&\quad + \sum_{i=1}^{\infty}\alpha_i d^2(e_n, S_i e_n)\Bigg). \qquad (36)
\end{aligned}
$$

On using (23) and (32) in (36), we obtain

$$
\begin{aligned}
\|x_{n+1} - x\|^2 &+ \|y_{n+1} - y\|^2 \leq \|x_n - x\|^2 + \|y_n - y\|^2 - (1 - (\lambda_n\eta)^2)(\|p_n - u_n\|^2 + \|q_n - v_n\|^2) \\
&- \gamma_n[2\|Ax_n - By_n\|^2 - \gamma_n(\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2)] \\
&- (\alpha_0 - k)\Bigg(\sum_{i=1}^{\infty}\alpha_i d^2(c_n, T_i c_n) + \sum_{i=1}^{\infty}\alpha_i d^2(e_n, S_i e_n)\Bigg). \qquad (37)
\end{aligned}
$$

Now, setting $\rho_n(x,y) := \|x_n - x\|^2 + \|y_n - y\|^2$ in (37), we obtain

$$\begin{aligned}\rho_{n+1}(x,y) &\leq \rho_n(x,y) - (1-(\lambda_n\eta)^2)(\|p_n - u_n\|^2 + \|q_n - v_n\|^2) \\ &\quad -\gamma_n[2\|Ax_n - By_n\|^2 - \gamma_n(\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2)] \\ &\quad -(\alpha_0 - k)\left(\sum_{i=1}^{\infty}\alpha_i d^2(c_n, T_i c_n) + \sum_{i=1}^{\infty}\alpha_i d^2(e_n, S_i e_n)\right). \qquad (38)\end{aligned}$$

Since $\lambda_n < \frac{1}{\eta}$ and $\alpha_0 \in (k,1)$ then $(\alpha_0 - k) > 0$ and hence it follows from condition (16) on γ_n that

$$\rho_{n+1}(x,y) \leq \rho_n(x,y).$$

This implies that the sequence $\{\rho_n(x,y)\}$ is non-increasing and bounded below and hence it converges to $\rho(x,y)$ (say). Thus, condition (i) of Lemma 2.3 is satisfied with $\mu_n = (x_n, y_n)$, $\mu^* = (x,y)$ and $W := \Gamma \subset H = H_1 \times H_2$ with norm $\|(x,y)\| = \left(\|x\|^2 + \|y\|^2\right)^{\frac{1}{2}}$.

Since $\|x_n - x\|^2 \leq \rho_n(x,y)$, $\|y_n - y\|^2 \leq \rho_n(x,y)$ and $\lim_{n\to\infty}\rho_n(x,y)$ exists, we observe that $\{x_n\}$ and $\{y_n\}$ are bounded and $\limsup_{n\to\infty}\|x_n - x\|$ and $\limsup_{n\to\infty}\|y_n - y\|$ exist. Also, $\limsup_{n\to\infty}\|Ax_n - Ax\|$ and $\limsup_{n\to\infty}\|By_n - By\|$ exist. Further, from (24) and (32), we easily observe that the sequences $\{c_n\}$, $\{e_n\}$, $\{p_n\}$ and $\{q_n\}$ are bounded.

Now, since $\{\gamma_n\}$ is bounded, $(1-(\lambda_n\eta)^2) > 0$ and $(\alpha_0 - k) > 0$ then it follows from the convergence of the sequence $\{\rho_n(x,y)\}$ and (38) that

$$\lim_{n\to\infty}(\|p_n - u_n\|^2 + \|q_n - v_n\|^2) = 0, \qquad (39)$$

$$\lim_{n\to\infty}\left(d^2(c_n, T_i c_n) + d^2(e_n, S_i e_n)\right) = 0, \text{ for each } i \in \mathbb{N}, \qquad (40)$$

and

$$\lim_{n\to\infty}\left(\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2\right) = 0. \qquad (41)$$

Note that $Ax_n - By_n = 0$, if $n \notin \Lambda$. Hence, we obtain

$$\lim_{n\to\infty}\|p_n - u_n\| = \lim_{n\to\infty}\|q_n - v_n\| = 0, \qquad (42)$$

$$\lim_{n\to\infty} d(c_n, T_i c_n) = 0, \text{ for each } i \in \mathbb{N}, \qquad (43)$$

$$\lim_{n\to\infty} d(e_n, S_i e_n) = 0, \text{ for each } i \in \mathbb{N}, \tag{44}$$

and

$$\lim_{n\to\infty} \|Ax_n - By_n\| = 0. \tag{45}$$

Let $\bar{x},\ \bar{y}$ be weak cluster points of the bounded sequences $\{x_n\},\ \{y_n\}$, respectively. It follows from Lemma 2.2 (i) that

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x - x_n + x\|^2 \\ &= \|x_{n+1} - x\|^2 - \|x_n - x\|^2 - 2\langle x_{n+1} - x_n, x_n - x\rangle \\ &= \|x_{n+1} - x\|^2 - \|x_n - x\|^2 - 2\langle x_{n+1} - \bar{x}, x_n - x\rangle + 2\langle x_n - \bar{x}, x_n - x\rangle. \end{aligned} \tag{46}$$

Since $\limsup_{n\to\infty} \|x_n - x\|$ exists, it follows from (46) that

$$\limsup_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Consequently,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{47}$$

Similarly, we obtain

$$\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0. \tag{48}$$

Since P_C is firmly nonexpansive, we have

$$\begin{aligned} \|p_n - x\|^2 &= \|P_C(x_n - \gamma_n A^*(Ax_n - By_n)) - x\|^2 \\ &\leq \langle p_n - x, x_n - \gamma_n A^*(Ax_n - By_n) - x\rangle \\ &= \frac{1}{2}\Big\{ \|p_n - x\|^2 + \|x_n - \gamma_n A^*(Ax_n - By_n) - x\|^2 \\ &\quad - \|p_n - x_n + \gamma_n A^*(Ax_n - By_n)\|^2 \Big\}, \end{aligned}$$

which implies that

$$\begin{aligned}\|p_n - x\|^2 &\leq \|x_n - x\|^2 - 2\gamma_n\langle x_n - x, A^*(Ax_n - By_n)\rangle + \gamma_n^2\|A^*(Ax_n - By_n)\|^2 \\ &\quad -\|p_n - x_n\|^2 - \gamma_n^2\|A^*(Ax_n - By_n)\|^2 - 2\gamma_n\langle p_n - x_n, A^*(Ax_n - By_n)\rangle \\ &\leq \|x_n - x\|^2 + 2\gamma_n\|Ax_n - Ax\|\|Ax_n - By_n\| - \|p_n - x_n\|^2 \\ &\quad +2\gamma_n\|Ap_n - Ax_n\|\|Ax_n - By_n\|. \end{aligned} \tag{49}$$

Since (33) can also be written as

$$\|x_{n+1} - x\|^2 \leq \|c_n - x\|^2 + \sum_{i=1}^{\infty}\alpha_i k_i d^2(c_n, T_i c_n). \tag{50}$$

Using (30) and (49) in (50), we get

$$\begin{aligned}\|x_{n+1} - x\|^2 &\leq \|x_n - x\|^2 + 2\gamma_n\|Ax_n - Ax\|\|Ax_n - By_n\| - \|p_n - x_n\|^2 \\ &\quad +2\gamma_n\|Ap_n - Ax_n\|\|Ax_n - By_n\| + \sum_{i=1}^{\infty}\alpha_i k_i d^2(c_n, T_i c_n), \\ &\quad -(1-(\lambda_n\alpha)^2)\|p_n - u_n\|^2,\end{aligned}$$

which, in turn, implies that

$$\begin{aligned}\|p_n - x_n\|^2 &\leq (\|x_n - x\| + \|x_{n+1} - x\|)\|x_{n+1} - x_n\| + 2\gamma_n\|Ax_n - Ax\|\|Ax_n - By_n\| \\ &\quad +2\gamma_n\|Ap_n - Ax_n\|\|Ax_n - By_n\| + \sum_{i=1}^{\infty}\alpha_i k_i d^2(c_n, T_i c_n). \end{aligned} \tag{51}$$

Since $\{p_n\}$, $\{x_n\}$ are bounded and A is a bounded linear operator, then $\{Ap_n - Ax_n\}$ is bounded. Now, using (45), (43) and (47) in (51), we have

$$\lim_{n\to\infty}\|p_n - x_n\| = 0. \tag{52}$$

Since

$$\|u_n - x_n\| \leq \|u_n - p_n\| + \|p_n - x_n\|, \tag{53}$$

using (42), (52) in (53), we have

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{54}$$

Similarly, we obtain

$$\lim_{n\to\infty}\|q_n-y_n\|=0, \tag{55}$$

$$\lim_{n\to\infty}\|v_n-y_n\|=0. \tag{56}$$

It follows from (21), (26), (28) and (50) that

$$\begin{aligned}\|x_{n+1}-x\|^2 \;\le\; & \|x_n-x\|^2+2\gamma_n\|Ax_n-Ax\|\|Ax_n-By_n\|-\|p_n-x_n\|^2\\ &+2\gamma_n\|Ap_n-Ax_n\|\|Ax_n-By_n\|-\|p_n-u_n\|^2-\|u_n-c_n\|^2\\ &+2\lambda_n\alpha\|p_n-u_n\|\|c_n-u_n\|+\sum_{i=1}^{\infty}\alpha_ik_id^2(c_n,T_ic_n),\end{aligned}$$

which, in turn, implies that

$$\begin{aligned}\|u_n-c_n\|^2 \;\le\; & (\|x_n-x\|+\|x_{n+1}-x\|)\|x_{n+1}-x_n\|+2\gamma_n\|Ax_n-Ax\|\|Ax_n-By_n\|\\ &+2\gamma_n\|Ap_n-Ax_n\|\|Ax_n-By_n\|+\sum_{i=1}^{\infty}\alpha_ik_id^2(c_n,T_ic_n)\\ &+2\lambda_n\alpha\|p_n-u_n\|\|c_n-u_n\|.\end{aligned} \tag{57}$$

Since $\{c_n\}$, $\{u_n\}$ are bounded, using (45), (43) and (47) in (57), we have that

$$\lim_{n\to\infty}\|u_n-c_n\|=0. \tag{58}$$

Since

$$\|c_n-x_n\|\le\|c_n-u_n\|+\|u_n-x_n\|, \tag{59}$$

using (58), (54) in (59), we have

$$\lim_{n\to\infty}\|c_n-x_n\|=0. \tag{60}$$

Similarly, we obtain

$$\lim_{n\to\infty}\|v_n-e_n\|=0, \tag{61}$$

$$\lim_{n\to\infty}\|e_n-y_n\|=0. \tag{62}$$

Since every Hilbert space satisfies Opial's condition, Opial's condition guarantees that the weakly subsequential limit of $\{x_n\}$ and $\{y_n\}$ is unique. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \bar{x}$ and hence it follows from (60) that there is a subsequence $\{c_{n_i}\}$ of $\{c_n\}$ such that $c_{n_i} \rightharpoonup \bar{x}$. Further, demiclosedness of T_i at 0 for each $i \in \mathbb{N}$ and (43) imply that $\bar{x} \in T_i\bar{x}$ for each $i \in \mathbb{N}$. Hence $\bar{x} \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i)$. Also, it follows from boundedness of $\{y_n\}$ and (62) that there exist subsequences $\{y_{n_i}\}$ of $\{y_n\}$ and $\{e_{n_i}\}$ of $\{e_n\}$ such that $y_{n_i} \rightharpoonup \bar{y}$ and $e_{n_i} \rightharpoonup \bar{y}$ and hence demiclosedness of S_i at 0 along with (44) yield that $\bar{y} \in S_i\bar{y}$ for each $i \in \mathbb{N}$. Thus $\bar{y} \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i)$.

Now, we show that $\bar{x} \in \mathrm{Sol}(\mathrm{VIP}(9))$. Since $\lim_{n\to\infty} \|p_n - u_n\| = 0$ and $\lim_{n\to\infty} \|p_n - x_n\| = 0$, there exist subsequences $\{p_{n_i}\}$ and $\{u_{n_i}\}$ of $\{p_n\}$ and $\{u_n\}$, respectively such that $p_{n_i} \rightharpoonup \bar{x}$ and $u_{n_i} \rightharpoonup \bar{x}$. Let

$$Tv = \begin{cases} fv + N_C(v), \text{ if } v \in C\ ; \\ \emptyset, \text{ if } v \notin C\ , \end{cases}$$

where $N_C(v)$ is the normal cone to C at $v \in H_1$. In this case, the mapping T is maximal monotone and hence $0 \in Tv$ if and only if $v \in \mathrm{Sol}(\mathrm{VIP}(9))$. Let $(v,w) \in \mathrm{graph}(T)$. Then, we have $w \in Tv = fv + N_C(v)$ and hence $w - fv \in N_C(v)$. So, we have $\langle v - u, w - fv\rangle \geq 0$, for all $u \in C$. On the other hand, from $u_n = P_C(I - \lambda_n f)p_n$ and $v \in C$, we have

$$\langle (I - \lambda_n f)p_n - u_n, u_n - v\rangle \ \geq \ 0.$$

This implies that

$$\langle v - u_n, \frac{u_n - p_n}{\lambda_n} + fp_n\rangle \ \geq \ 0.$$

Since $\langle v - u, w - fv\rangle \geq 0$, for all $u \in C$ and $u_{n_i} \in C$, using monotonicity of f, we have

$$\begin{aligned} \langle v - u_{n_i}, w\rangle &\geq \langle v - u_{n_i}, fv\rangle \\ &\geq \langle v - u_{n_i}, fv\rangle - \left\langle v - u_{n_i}, \frac{u_{n_i} - p_{n_i}}{\lambda_n} + fp_{n_i}\right\rangle \\ &= \langle v - u_{n_i}, fv - fu_{n_i}\rangle + \langle v - u_{n_i}, fu_{n_i} - fp_{n_i}\rangle - \left\langle v - u_{n_i}, \frac{u_{n_i} - p_{n_i}}{\lambda_n}\right\rangle \\ &\geq \langle v - u_{n_i}, fu_{n_i} - fp_{n_i}\rangle - \left\langle v - u_{n_i}, \frac{u_{n_i} - p_{n_i}}{\lambda_n}\right\rangle. \end{aligned}$$

Since f is continuous, then on taking limit $i \to \infty$, we have $\langle v-\bar{x}, w\rangle \geq 0$. Since T is maximal monotone, we have $\bar{x} \in T^{-1}0$ and hence $\bar{x} \in \mathrm{Sol}(\mathrm{VIP}(9))$. Similarly, one can show that $\bar{y} \in \mathrm{Sol}(\mathrm{VIP}(10))$.

Again, since A and B are bounded linear operators, we have $Ax_n \rightharpoonup A\bar{x}$ and $By_n \rightharpoonup B\bar{y}$. Further, since $\|\cdot\|^2$ is weakly lower semicontinuous, we have

$$\|A\bar{x}-B\bar{y}\|^2 \leq \liminf_{n_i\to\infty} \|Ax_{n_i}-By_{n_i}\|^2 = 0, \tag{63}$$

i.e., $A\bar{x}=B\bar{y}$. Thus, $(\bar{x},\bar{y}) \in \Gamma$ and hence $w_w(x_{n_i}, y_{n_i}) \subset \Gamma$. Now, it follows from Lemma 2.3 that the sequence $\{(x_n, y_n)\}$ generated by iterative Algorithm 3.1 converges weakly to $(\bar{x},\bar{y}) \in \Gamma$.

Now, since T_i and S_i for each $i \in \mathbb{N}$, are hemi-compact, $\{x_n\}$ and $\{y_n\}$ are bounded and $\lim_{n\to\infty} d(c_n, T_i c_n) = 0$ and $\lim_{n\to\infty} d(e_n, S_i e_n) = 0$ for each $i \in \mathbb{N}$, there exist (without loss of generality) subsequences $\{x_{n_i}\}$ of $\{x_n\}$ and $\{y_{n_i}\}$ of $\{y_n\}$ such that $\{x_{n_i}\}$ and $\{y_{n_i}\}$ converge strongly to some points $\bar{u}$ and $\bar{v}$, respectively. It follows from the demiclosedness of T_i and S_i, for each $i \in \mathbb{N}$ that $\bar{u} \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i)$ and $\bar{v} \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i)$. Since $\{x_{n_i}\}$ and $\{y_{n_i}\}$ converge weakly to $\bar{x}$ and $\bar{y}$, respectively, we then have $\bar{u}=\bar{x}$ and $\bar{v}=\bar{y}$. On the other hand, since $\rho_n(x,y) = \|x_n - x\|^2 + \|y_n - y\|^2$, for any $(x,y) \in \Gamma$ then $\lim_{i\to\infty} \rho_{n_i}(\bar{x},\bar{y}) = 0$. Further, since $\lim_{n\to\infty} \rho_n(\bar{x},\bar{y})$ exists then $\lim_{n\to\infty} \rho_n(\bar{x},\bar{y}) = 0$ and hence $\lim_{n\to\infty} \|x_n - \bar{x}\| = 0$ and $\lim_{n\to\infty} \|y_n - \bar{y}\| = 0$. Thus, $\{(x_n, y_n)\}$ converges strongly to $(\bar{x},\bar{y}) \in \Gamma$. This completes the proof.

Now, we present a consequence of Theorem 4.1.

For each $i \in \mathbb{N}$, if T_i and S_i are single-valued demicontractive mappings then we have the following result to approximate a common solution of S_pEVIP(9)-(10) and $\mathrm{S_p}$EFPP (5) for two countable families of single-valued demicontractive mappings:

Corollary 4.1. Let H_1, H_2 and H_3 be real Hilbert spaces and $C \subset H_1$, $Q \subset H_2$ be nonempty, closed and convex sets. Let $A: H_1 \to H_3$, $B: H_2 \to H_3$ be bounded linear operators with their adjoint operators A^* and B^*, respectively. Let $f: C \to H_1$ be monotone and α-Lipschitz continuous mapping and let $g: Q \to H_2$ be monotone and β-Lipschitz continuous mapping. Let $\{T_i\}_{i=1}^{\infty}: H_1 \to H_1$ and $\{S_i\}_{i=1}^{\infty}: H_2 \to H_2$ be families of single-valued demicontractive mappings with demicontractive constants k_i and s_i, respectively and let $k_1 = \sup_{i\geq 1}\{k_i\} \in (0,1)$

and $k_2 = \sup_{i\geq 1}\{s_i\} \in (0,1)$. For each $i \in \mathbb{N}$, let T_i and S_i be demiclosed at 0. Assume that $\Gamma := \mathrm{Sol}(\mathrm{S_pEVIP}(9)-(10)) \cap \left(\bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \times \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i)\right) \neq \emptyset$. If the sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset [a,b]$, for some a and b with $0 < a < b < \frac{1}{\min\{\alpha,\beta\}}$ and $k \in (0,1)$ where $k = \max\{k_1,k_2\}$, then the sequence $\{(x_n,y_n)\}$ generated by Algorithm 3.2 converges weakly to $(\bar{x},\bar{y}) \in \Gamma$. In addition, if for each $i \in \mathbb{N}$, T_i and S_i are semicompact, then $\{(x_n,y_n)\}$ converges strongly to $(\bar{x},\bar{y}) \in \Gamma$.

We remark that it is of further research effort to extend the iterative method presented in this paper, to split equality mixed equilibrium problem and split equality monotone variational inclusion problem [14].

5. Numerical Example

Finally, we give a numerical example which justifies Theorem 4.1.

Example 5.1. Let $H_1 = H_2 = H_3 = \mathbb{R}$ with the inner product defined by $\langle x,y\rangle = xy$, $\forall\, x,y \in \mathbb{R}$, and induced usual norm $|\cdot|$. Let $C = [-10,10]$ and $Q = [-10,10]$; let $\{T_i\}_{i=1}^{\infty}, \{S_i\}_{i=1}^{\infty} : \mathbb{R} \to CB(\mathbb{R})$ by $T_i(x) = \left\{\left(-\dfrac{1+i}{i}\right)x\right\}$, $S_i(y) = \left\{\left(-\dfrac{1+2i}{2i}\right)y\right\}$, for each $i \in \mathbb{N}$; let $f : C \to \mathbb{R}$ and $g : Q \to \mathbb{R}$ be defined by $f(x) = 2x$, $\forall x \in C$ and $g(y) = 3y$, $\forall y \in Q$; let $A, B : \mathbb{R} \to \mathbb{R}$ be defined by $A(x) = 2x, \forall x \in \mathbb{R}$, $B(y) = 4y, \forall y \in \mathbb{R}$. If we set $\alpha_i = \dfrac{1}{2^{i+1}}$, $\forall i \in \mathbb{N} \cup \{0\}$, then there is a unique sequence $\{(x_n,y_n)\}$ generated by the iterative schemes:

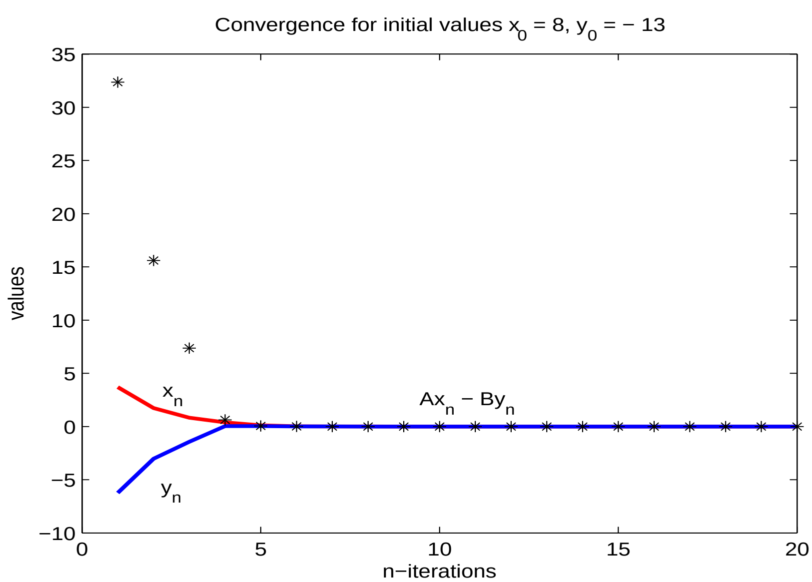
$$(64)\qquad \begin{cases} p_n &= P_C(x_n - 4\gamma_n(x_n - 2y_n)); \\ u_n &= P_C(p_n - 2\lambda_n p_n); \\ c_n &= P_C(p_n - 2\lambda_n u_n); \\ x_{n+1} &= \frac{1}{2}c_n + \sum_{i=1}^{\infty} \frac{1}{2^{i+1}}\left(-\frac{1+i}{i}\right)c_n; \\ q_n &= P_Q(y_n + 8\gamma_n(x_n - 2y_n)); \\ v_n &= P_Q(q_n - 3\lambda_n q_n); \\ e_n &= P_Q(q_n - 3\lambda_n v_n); \\ y_{n+1} &= \frac{1}{2}e_n + \sum_{i=1}^{\infty} \frac{1}{2^{i+1}}\left(-\frac{1+2i}{2i}\right)e_n. \end{cases}$$

Then the sequence $\{(x_n,y_n)\}$ converges to a point $(\bar{x},\bar{y}) \in \Gamma$.

Proof. Evidently, A and B are bounded linear operators on $\mathbb{R}$ with adjoint operators A^*, B^*, respectively with $\|A\| = \|A^*\| = 2$, $\|B\| = \|B^*\| = 4$, and hence $\gamma_n \in \left(\varepsilon, \frac{1}{10} - \varepsilon\right)$. Therefore, for $\varepsilon = \frac{1}{100}$, we choose $\gamma_n = \frac{1}{20}$. We also assume $\lambda_n = \frac{2}{3}$. Furthermore, we observe that for each $i \in \mathbb{N}$, T_i is demicontractive with $k_i = \frac{1}{1+2i}$, $\text{Fix}(T_i) = \{0\}$ and $(T_i - I)$ is demiclosed at 0, and S_i is demicontractive with $s_i = \frac{1}{1+4i}$, $\text{Fix}(S_i) = \{0\}$ and $(S_i - I)$ is demiclosed at 0. Since $k_1 = \sup_{i\geq 1}\{k_i\} = \frac{1}{3}$ and $k_2 = \sup_{i\geq 1}\{s_i\} = \frac{1}{5}$ then $k = \max\{k_1, k_2\} = \frac{1}{3}$. Next,, we observe that $\Gamma := \text{Sol}(\text{S}_\text{p}\text{EVIP}(9) - (10)) \bigcap \left(\bigcap_{i=1}^{\infty} \text{Fix}(T_i) \times \bigcap_{i=1}^{\infty} \text{Fix}(S_i)\right) = \{(0,0)\} \neq \emptyset$.
After simplification, iterative schemes (64) are reduced to the following:

$$(65) \qquad \begin{cases} p_n = \frac{4}{5}x_n + \frac{2}{5}y_n;\ u_n = -\frac{p_n}{3};\ c_n = p_n - \frac{4}{3}u_n; \\ q_n = -\frac{2}{5}x_n + \frac{9}{5}y_n;\ v_n = -q_n;\ e_n = q_n - 2v_n; \\ x_{n+1} = \frac{1}{2}c_n - \sum_{i=1}^{\infty} \frac{1+i}{i2^{i+1}}c_n;\ y_{n+1} = \frac{1}{2}e_n - \sum_{i=1}^{\infty} \frac{1+2i}{i2^{i+2}}e_n; \end{cases}$$

Next, using the software Matlab 7.8.0, we have following figure and table which shows that $\{(x_n, y_n)\}$ converges to the point $(\bar{x}, \bar{y}) = (0,0)$.



Table

No. of iterations	x_n $x_0 = 8$	y_n $y_0 = -13$	$Ax_n - By_n$		**No. of iterations**	x_n	y_n	$Ax_n - By_n$
1	3.699636	-6.240070	32.359552		**11**	0.000031	0.000015	0.000004
2	1.733757	-3.033392	15.601081		**12**	0.000008	0.000004	0.000001
3	0.823414	-1.430052	7.367037		**13**	0.000002	0.000001	0.000000
4	0.390003	0.039671	0.621323		**14**	0.000000	0.000000	0.000000
5	0.112934	0.041824	0.058573		**15**	0.000000	0.000000	0.000000
6	0.029665	0.013086	0.006988		**16**	0.000000	0.000000	0.000000
7	0.007582	0.003505	0.001146		**17**	0.000000	0.000000	0.000000
8	0.001922	0.000901	0.000240		**18**	0.000000	0.000000	0.000000
9	0.000486	0.000229	0.000057		**19**	0.000000	0.000000	0.000000
10	0.000123	0.000058	0.000014		**20**	0.000000	0.000000	0.000000

This completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

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