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AN IMPROVEMENT OF KNOWN UNIQUE COMMON FIXED POINT THEOREMS FOR FOUR MAPPINGS ON MULTIPLICATIVE METRIC SPACES

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Abstract. Removing the continuity of four mappings and replacing the compatibility and φ-weak commutativity by the weak compatibility, we obtain a generalized result of some well-known unique common fixed point theorems for four mappings defined on multiplicative metric spaces. We also give several deformed common fixed point theorems under non-continuous conditions.

Keywords: multiplicative metric space; common fixed point; ϕ-weakly commuting map; weakly compatible.

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1. Introduction and preliminaries

The Banach contraction principle[1] is one basic and simple fixed point theorem, it is widely applied in mathematics and other fields. Hence the principle is very goodly generalized and improved on various metric spaces. Especially, in 2008, Bashirov et al[2] introduced the concept of multiplicative metric spaces and studied some basic properties. Afterwards, Florack and Assen[3] and Bashirove et al[4] also gave some other properties in this space. In 2012, Özavsar

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and Çevikel[5] introduced the concept of multiplicative contraction mappings on multiplicative metric spaces and obtained several existence theorems of fixed points; in 2013, He et al[6] proved some existence theorems of common fixed points for four mappings using the weakly commuting condition; in 2015, Abbas et al[7], Kang et al[8] and Gu et al[9] obtained common fixed point theorems using the locally contractive condition, compatible condition and weakly compatible condition respectively. Gu et al[9] gave a example ([9], Example 2.3) to support the main theorem(Theorem 2.1 in [9]). The condition of λ in Theorem 2.1 is $\lambda \in (0, \frac{1}{2})$ in [9], but the given λ in Example 2.3 is $\lambda = \frac{2}{3}$ in [9], hence the example is not persuasive. Recently, Jiang and Gu[10] introduce the concept of ϕ-weakly commutative mappings and obtained common fixed point theorems for four mappings. These obtained results in [10] generalized and improved the corresponding conclusions in [6-9].

Definition 1.1.[2] Let X be a nonempty set, A multiplicative metric is a mapping $d : X \times X \to [0,\infty)$ satisfying:

(i) $d(x,y) \geq 1$ for all $x,y \in X$ and $d(x,y) = 1 \Longleftrightarrow x = y$;

(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(iii) $d(x,z) \leq d(x,y)\,d(y,z)$ for all $x,y,z \in X$.(multiplicative triangle inequality).

The pair (X,d) is called a multiplicative metric space.

Example 1.1.[5] Let $\mathbb{R}_+{}^n = \{(a_1,a_2,\cdots,a_n)|a_1,a_2,\cdots,a_n > 0\}$. Define $d : \mathbb{R}_+{}^n \times \mathbb{R}_+{}^n \to [0,\infty)$ as follows

$$d(x,y) = \left| \frac{x_1}{y_1} \right| \cdot \left| \frac{x_2}{y_2} \right| \cdots \left| \frac{x_2}{y_2} \right|,$$

where $x = (x_1,x_2,\cdots,x_n)$, $y = (y_1,y_2,\cdots,y_n) \in \mathbb{R}_+{}^n$, $|\cdot| : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by :

$$| a | = \begin{cases} a, & \text{if} a \geq 1 \\ \dfrac{1}{a}, & \text{if} a < 1. \end{cases}$$

Then $(\mathbb{R}_+{}^n, d)$ is a multiplicative metric space.

Example 1.2.[9] Let $X = \mathbb{R}$ and define $d(x,y) = e^{|x-y|}$, $\forall x,y \in X$. Then $(\mathbb{R},d)$ is also a multiplicative metric space.

Definition 1.2.[2] Let (X,d) be a multiplicative metric space, $\{x_n\}$ a sequence in X and $x \in X$. If for every multiplicative open ball $B_\varepsilon(x) = \{y \in X | d(x,y) < \varepsilon\}$, $\varepsilon > 1$, there exists a natural number N such that $x_n \in B_\varepsilon(x)$ for all $n > N$, then the sequence $\{x_n\}$ is said to be multiplicative converging to x, denoted by $x_n \to x(n \to \infty)$.

Lemma 1.1.[5] Let (X,d) be a multiplicative metric space, $\{x_n\}$ a sequence in X and $x \in X$. Then

$$x_n \to x\,(n \to \infty) \Longleftrightarrow d(x_n,x) \to 1\,(n \to \infty).$$

Definition 1.3.[5] Let (X,d) be a multiplicative metric space, $\{x_n\}$ a sequence in X. The sequence $\{x_n\}$ is called multiplicative Cauchy sequence if, for each $\varepsilon > 1$, there exists a natural number N such that $d(x_n,x_m) < \varepsilon$ for all $n,m > N$.

Lemma 1.2.[5] Let (X,d) be a multiplicative metric space, $\{x_n\}$ a sequence in X. Then $\{x_n\}$ is a multiplicative Cauchy sequence if and only if $d(x_m,x_n) \to 1(m,n \to \infty)$.

Definition 1.4.[5] A multiplicative metric space (X,d) is said to be multiplicative complete if, every multiplicative Cauchy sequence in (X,d) is multiplicative convergent in X.

Definition 1.5.[5] Let (X,d_X) and (Y,d_Y) be two multiplicative metric spaces, $f : X \to Y$ a mapping and $x \in X$. If for every $\varepsilon > 1$, there exists $\delta > 1$ such that $f(B_\delta(x)) \subset B_\varepsilon(f(x))$, then we call f multiplicative continuous at x. If f is multiplicative continuous at all $x \in X$, then we say that f is multiplicative continuous on X.

Lemma 1.3.[5] Let (X,d_X) and (Y,d_Y) be two multiplicative metric spaces, $f : X \to Y$ a mapping and $x \in X$. Then f is multiplicative continuous at x if and only if $fx_n \to fx$ for every sequence $\{x_n\} \subset X$ with $x_n \to x$.

Lemma 1.4.[5] Let (X,d) be a multiplicative metric space, $\{x_n\}$ and $\{y_n\}$ two sequences and $x,y \in X$. Then

$$x_n \to x,\, y_n \to y\,(n \to \infty) \Longrightarrow d(x_n,y_n) \to d(x,y)\,(n \to \infty).$$

Definition 1.6.[5] Let (X,d) be a multiplicative metric space. A mapping $f : X \to X$ is called a multiplicative contraction if there exists a real constant $\lambda \in (0,1)$ such that $d(fx,fy) \leq [d(x,y)]^\lambda$ for all $x,y \in X$.

Theorem 1.1.[5] Let (X,d) be a complete multiplicative metric space. If $f: X \to X$ is a multiplicative contraction, then f has a unique fixed point.

He et al obtained the following existence theorem of common fixed point for four mappings with the weakly commutativity on multiplicative metric spaces:

Theorem 1.2.[6] If four self-mappings S,T,A,B on a multiplicative metric space X satisfy the following conditions:

(i) $SX \subset BX$, $TX \subset AX$;

(ii) A and S are weakly commutative mappings, B and T are also weakly commutative mappings;

(iii) one of $\{A,B,S,T\}$ is continuous;

(iii) there exists $\lambda \in (0,\frac{1}{2})$ such that for every $x,y \in X$,

$$d(Sx,Ty) \le \{\max\{d(Ax,By),d(Ax,Sx),d(Ry,Ty),d(Ax,Ty),d(By,Sx)\}\}^{\lambda},$$

Then S,T,A,B have a unique common fixed point.

Gu an Cho[9] introduced the concepts of the compatibility and weak compatibility of two self-mappings on multiplicative metric spaces and also gave the following fact:

commutativity $\Longrightarrow$ weak commutativity $\Longrightarrow$ compatibility $\Longrightarrow$ weak compatibility.

But the converse of the above fact is not true (see, Remark 2.1 in [9]).

$\phi \in \Phi$[9] if and only if $\phi : [1,\infty)^5 \to [0,\infty)$ satisfy

(ϕ_1) ϕ is non-decreasing and continuous in each coordinate variable;

(ϕ_2) for each $t \ge 1$,

$$\max\{\phi(t,t,t,1,t),\phi(t,t,t,t,1),\phi(t,1,1,t,t),\phi(1,t,1,t,1),\phi(1,1,t,1,t)\} \le t.$$

Using $\phi \in \Phi$ and the concepts of compatibility and weak compatibility, Gu and Cho obtained the next generalization of Theorem 1.2:

Theorem 1.3.[9] If four self-mappings S,T,A,B on a complete multiplicative metric space X satisfy the following conditions:

(i) $SX \subset BX$, $TX \subset AX$;

(ii) there exists $\lambda \in (0,\frac{1}{2})$ such that for each $x,y \in X$,

$$d(Sx,Ty) \le \phi(d^{\lambda}(Ax,By),d^{\lambda}(Ax,Sx),d^{\lambda}(By,Ty),d^{\lambda}(Ax,Ty),d^{\lambda}(By,Sx));$$

(iii) one of the following conditions is satisfied:

(a) either A or S is continuous, the pair (A,S) is compatible and the pair (T,B) is weakly compatible;

(b) either B or T is continuous, the pair (T,B) is compatible and the pair (A,S) is weakly compatible.

Then S,T,A,B have a unique common fixed point.

In 2017, Jiang and Gu[10] introduced the concept of φ-weak commutativity of two mappings and obtained the following result:

Theorem 1.4.[10] If four self-mappings S,T,A,B on a complete multiplicative metric space X satisfy the following conditions:

(i) $SX \subset BX$, $TX \subset AX$;

(ii) there exists $\lambda \in (0, \frac{1}{2})$ such that for each $x, y \in X$,

$$d(Sx,Ty) \leq \phi(d^{\lambda}(Ax,By), d^{\lambda}(Ax,Sx), d^{\lambda}(By,Ty), d^{\lambda}(Ax,Ty), d^{\lambda}(By,Sx));$$

(iii) one of the following conditions is satisfied:

(a) either A or S is continuous, the pair (A,S) is compatible and the pair (T,B) is φ-weakly commutative;

(b) either B or T is continuous, the pair (T,B) is compatible and the pair (A,S) is φ-weakly commutative.

Then S,T,A,B have a unique common fixed point.

In [11], Piao introduced a 5-dimensional real function ψ and obtained several unique common fixed point theorems for four noncontinuous mappings on multiplicative metric spaces, where ψ is very different from ϕ in Theorem 1.3 and Theorem 1.4.

Note that at least one mapping must be continuous and at least one pair of mappings must be weakly commutative or compatible or φ-weakly commutative in Theorem 1.2–Theorem 1.4. Hence the main aim in this paper is to obtain the same conclusions as the results in Theorem 1.2–Theorem 1.4 removing the continuity of four mappings and replacing the compatibility and (φ)-weak commutativity by the weak compatibility. The obtained new result is a generalization and improvement of Theorem 1.2–Theorem 1.4 and is very different from the conclusion in

[11]. Finally, we give some deformed common fixed point theorems for four mappings with commutativity but without continuity.

Definition 1.7.[9] Let (X,d) be a multiplicative metric space and $f,g : X \to X$ be two mappings. If $fgx = gfx$ whenever $fx = gx\,(x \in X)$, i.e., $d(fx,gx) = 1\,(x \in X) \Longrightarrow d(fgx,gfx) = 1$, then the pair (f,g) is called weakly compatible.

Definition 1.8.[12] Let X be a nonempty set and $f,g : X \to X$ two mappings. If there exist $w,x \in X$ such that $w = fx = gx$, then x is said to be a coincidence point of the pair (f,g) and w is said to be a point of coincidence of the pair (f,g).

Lemma 1.5.[11] Let (X,d) be a multiplicative metric space and $f,g : X \to X$ be the pair of weakly compatible mappings. If $w = fx = gx$ is the unique point of coincidence of the pair (f,g), then w is the unique common fixed point of the pair (f,g).

2. An improvement of the well known theorems

Theorem 2.1. If four self-mappings S,T,A,B on a multiplicative metric space X satisfy the following conditions:

(i) $SX \subset BX$, $TX \subset AX$;

(ii) there exists $\lambda \in (0,\frac{1}{2})$ such that for each $x,y \in X$,

$$d(Sx,Ty) \le \phi(d^{\lambda}(Ax,By), d^{\lambda}(Ax,Sx), d^{\lambda}(By,Ty), d^{\lambda}(Ax,Ty), d^{\lambda}(By,Sx)); \tag{2.1}$$

(iii) the pair (A,S) is weakly compatible and the pair (T,B) is also weakly compatible;

(iv) one of $\{SX,AX,BX,TX\}$ is complete.

Then S,T,A,B have a unique common fixed point.

Proof. Take any element $x_0 \in X$. In view of (i), we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X satisfying

$$y_{2n} = Sx_{2n} = Bx_{2n+1}, y_{2n+1} = Tx_{2n+1} = Ax_{2n+2}, n = 0,1,2,\cdots. \tag{2.2}$$

Using the condition (ii) and (2.2) and modifying the proof of the corresponding theorem in [9-10], we can prove that the sequence $\{y_n\}$ is a multiplicative Cauchy sequence, i.e., $d(y_m,y_n) \to 1$ as $m,n \to \infty$.

Suppose that either SX or BX is complete. Since $y_{2n} \in SX \subset BX$ for all $n = 0, 1, 2, \cdots$ and $\{y_{2n}\}$ is also a multiplicative Cauchy sequence as a subsequence of $\{y_n\}$, there exist $u, v \in X$ such that $y_{2n} \to u = Bv$ as $n \to \infty$, i.e., $d(y_{2n}, u) \to 1$ as $n \to \infty$. Hence by the following fact

$$d(y_{2n+1}, u) \le d(y_{2n+1}, y_{2n}) \cdot d(y_{2n}, u),$$

we obtain $d(y_{2n+1}, u) \to 1$ as $n \to \infty$.

For each $n \in \mathbb{N}$, by (2.1),

$$\begin{aligned} & d(y_{2n}, Tv) = d(Sx_{2n}, Tv) \\ & \le \phi(d^\lambda(Ax_{2n}, Bv), d^\lambda(Ax_{2n}, Sx_{2n}), d^\lambda(Bv, Tv), d^\lambda(Ax_{2n}, Tv), d^\lambda(Bv, Sx_{2n})) \\ & = \phi(d^\lambda(y_{2n-1}, u), d^\lambda(y_{2n-1}, y_{2n}), d^\lambda(u, Tv), d^\lambda(y_{2n-1}, Tv), d^\lambda(u, y_{2n})). \end{aligned} \tag{2.3}$$

Letting $n \to \infty$ in (2.3) and using Lemma 1.4 and the conditions (ϕ_1)-(ϕ_2), we obtain

$$\begin{aligned} & d(u, Tv) \\ & \le \phi(1, 1, d^\lambda(u, Tv), d^\lambda(u, Tv), 1) \\ & \le \phi(d^\lambda(u, Tv), d^\lambda(u, Tv), d^\lambda(u, Tv), d^\lambda(u, Tv), 1) \\ & \le d^\lambda(u, Tv), \end{aligned}$$

hence $d(u, Tv) = 1$ since $0 < \lambda < 1$ and $d(u, Tv) \ge 1$, therefore $Tv = u = Bv$. This show that u is a point of coincidence of T and B.

On the other hand, $u = Tv \in TX \subset AX$ implies that there exists $w \in X$ such that $u = Aw$. For each $n \in \mathbb{N}$, by (2.1),

$$\begin{aligned} & d(Sw, y_{2n+1}) = d(Sw, Tx_{2n+1}) \\ & \le \phi(d^\lambda(Aw, Bx_{2n+1}), d^\lambda(Aw, Sw), d^\lambda(Bx_{2n+1}, Tx_{2n+1}), d^\lambda(Aw, Tx_{2n+1}), d^\lambda(Bx_{2n+1}, Sw)) \\ & = \phi(d^\lambda(u, y_{2n}), d^\lambda(u, Sw), d^\lambda(y_{2n}, y_{2n+1}), d^\lambda(u, y_{2n+1}), d^\lambda(y_{2n}, Sw)). \end{aligned} \tag{2.4}$$

Letting $n \to \infty$ in (2.4) and using Lemma 1.4 and conditions (ϕ_1)-(ϕ_2), we have

$$\begin{aligned}
&d(Sw,u)\\
&\leq \phi(1, d^{\lambda}(u,Sw), 1, 1, d^{\lambda}(u,Sw))\\
&\leq \phi(d^{\lambda}(u,Sw), d^{\lambda}(u,Sw), d^{\lambda}(u,Sw), 1, d^{\lambda}(u,Sw))\\
&\leq d^{\lambda}(u,Sw).
\end{aligned}$$

Hence $d(Sw,u) = 1$, therefore $Sw = u = Aw$. This show that u is also a point of coincidence of S and A.

If $z = Sx = Ax$ is also a point of coincidence of S and A, then by (2.1) and the properties of ϕ,

$$\begin{aligned}
&d(z,u) = d(Sx,Tv)\\
&\leq \phi(d^{\lambda}(Ax,Bv), d^{\lambda}(Ax,Sx), d^{\lambda}(Bv,Tv), d^{\lambda}(Ax,Tv), d^{\lambda}(Bv,Sx))\\
&= \phi(d^{\lambda}(z,u), 1, 1, d^{\lambda}(z,u), d^{\lambda}(z,u))\\
&\leq d^{\lambda}(z,u).
\end{aligned}$$

Hence $d(z,u) = 1$, i.e., $z = u$. This completes that u is the unique point of coincidence of A and S. Similarly, u is also the unique point of coincidence of B and T. By (iii) and Lemma 1.5, u is the unique common fixed point of the pair (A,S) and the pair (B,T) respectively, hence u is a common fixed point of $\{A,B,S,T\}$.

If u' is also a common fixed point of $\{A,B,S,T\}$, then by (2.1),

$$\begin{aligned}
&d(u',u) = d(Su',Tu)\\
&\leq \phi(d^{\lambda}(Au',Bu), d^{\lambda}(Au',Su'), d^{\lambda}(Bu,Tu), d^{\lambda}(Au',Tu), d^{\lambda}(Bu,Su'))\\
&= \phi(d^{\lambda}(u',u), 1, 1, d^{\lambda}(u',u), d^{\lambda}(u',u))\\
&\leq d^{\lambda}(u',u).
\end{aligned}$$

Hence $u' = u$, therefore u is the unique common fixed point of $\{A,B,S,T\}$. Similarly, we can give the same conclusion for either TX or AX being complete.

Remark 2.1. Theorem 2.1 generalize and improve Theorem 1.2-Theorem 1.4 in the following aspects:

(1) The space X is complete in Theorem 1.2-Theorem 1.4, but we need that one of $\{SX, AX, BX, TX\}$ is complete in Theorem 2.1;

(2) Four mappings A, B, S, T in Theorem 2.1 need not be continuous, but at least one of $\{A, B, S, T\}$ must be continuous in Theorem 1.2-Theorem 1.4;

(3) $\{A, S\}$ and $\{B, T\}$ are all weakly commutative in Theorem 1.2, and either $\{A, S\}$ or $\{B, T\}$ is compatible in Theorem 1.3 and Theorem 1.4, but $\{A, S\}$ are $\{B, T\}$ are all weakly compatible in Theorem 2.1;

(4) If (ϕ_1) holds, then $\phi(1,t,1,t,1) \leq \phi(t,t,t,t,1)$ and $\phi(1,1,t,1,t) \leq \phi(t,t,t,1,t)$ for all $t \geq 1$, hence the condition (ϕ_2) in Theorem 2.1 and Theorem 1.3-Theorem 1.4 can be replaced by the next equivalent condition (ϕ_3):

(ϕ_3) $\max\{\phi(t,t,t,1,t), \phi(t,t,t,t,1), \phi(t,1,1,t,t)\} \leq t$ for all $t \geq 1$.

Let $S = B$ and $T = A$ in Theorem 2.1, then we obtain the following common fixed point theorem for two mappings.

Theorem 2.2. If two self-mappings S, T on a multiplicative metric space X satisfy the following conditions:

(i) there exists $\lambda \in (0, \frac{1}{2})$ such that for each $x, y \in X$,

$$d(Sx, Ty) \leq \phi(d^\lambda(Tx, Sy), d^\lambda(Sx, Tx), d^\lambda(Sy, Ty), d^\lambda(Tx, Ty), d^\lambda(Sx, Sy)); \tag{2.5}$$

(ii) the pair (T, S) is weakly compatible;

(iii) one of $\{SX, TX\}$ is complete.

Then S, T have a unique common fixed point.

Theorem 2.3. Suppose that (X, d) is a multiplicative metric space, $S, T, A, B : X \to X$ four mappings, $\{p, q, m, n\}$ are four natural numbers. If

(i) $S^p X \subset B^n X$, $T^q X \subset A^m X$;

(ii) there exists $\lambda \in (0, \frac{1}{2})$ such that for each $x, y \in X$,

$$d(S^p x, T^q y) \leq \phi(d^\lambda(A^m x, B^n y), d^\lambda(A^m x, S^p x), d^\lambda(B^n y, T^q y), d^\lambda(A^m x, T^q y), d^\lambda(B^n y, S^p x)); \tag{2.6}$$

(iii) the pair (A, S) and the pair (T, B) are all commutative respectively;

(iv) one of $\{S^p X, A^m X, B^n X, T^q X\}$ is complete.

Then S,T,A,B have a unique common fixed point.

Proof. (iii) implies that the pair (A^m,S^p) and the pair (T^q,B^n) are all commutative respectively, hence (A^m,S^p) and (T^q,B^n) are all weakly compatible respectively. Obviously, $\{S^p,A^m,B^n,T^q\}$ satisfy all conditions of Theorem 2.1, hence $\{S^p,A^m,B^n,T^q\}$ have a unique common fixed point u, i.e., $S^pu=A^mu=B^nu=T^qu=u$. Now, we will prove that u is also the unique common fixed point of $\{S,A,B,T\}$. In fact, by (2.6) and (ϕ_2),

$$\begin{aligned}
d(Su,u) &= d(S^pSu,T^qu)\\
&\le \phi(d^\lambda(A^mSu,B^nu),d^\lambda(A^mSu,S^pSu),d^\lambda(B^nu,T^qu),d^\lambda(A^mSu,T^qu),d^\lambda(B^nu,S^pSu))\\
&= \phi(d^\lambda(Su,u),1,1,d^\lambda(Su,u),d^\lambda(u,Su))\\
&\le d^\lambda(Su,u).
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
d(u,Tu) &= d(S^pu,T^qTu)\\
&\le \phi(d^\lambda(A^mu,B^nTu),d^\lambda(A^mu,S^pu),d^\lambda(B^nTu,T^qTu),d^\lambda(A^mu,T^qTu),d^\lambda(B^nTu,S^pu))\\
&= \phi(d^\lambda(u,Tu),1,1,d^\lambda(u,Tu),d^\lambda(Tu,u))\\
&\le d^\lambda(Tu,u).
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
d(Au,u) &= d(S^pAu,T^qu)\\
&\le \phi(d^\lambda(A^mAu,B^nu),d^\lambda(A^mAu,S^pAu),d^\lambda(B^nu,T^qu),d^\lambda(A^mAu,T^qu),d^\lambda(B^nu,S^pAu))\\
&= \phi(d^\lambda(Au,u),1,1,d^\lambda(Au,u),d^\lambda(u,Au))\\
&\le d^\lambda(Au,u).
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
d(u,Bu) &= d(S^pu,T^qBu)\\
&\le \phi(d^\lambda(A^mu,B^nBu),d^\lambda(A^mu,S^pu),d^\lambda(B^nBu,T^qBu),d^\lambda(A^mu,T^qBu),d^\lambda(B^nBu,S^pu))\\
&= \phi(d^\lambda(u,Bu),1,1,d^\lambda(u,Bu),d^\lambda(Bu,u))\\
&\le d^\lambda(Bu,u).
\end{aligned} \tag{2.10}$$

From (2.7)-(2.10), we have $d(Su,u)=d(Au,u)=d(Bu,u)=d(Tu,u)=1$, i.e., $Au=Bu=Su=Tu=u$. Hence u is a common fixed point of $\{A,B,S,T\}$. If w is also a common fixed point of

$\{A,B,S,T\}$, then by (2.6) and (ϕ_2),

$$
\begin{aligned}
d(u,w) &= d(S^p u, T^q w) \\
&\leq \phi(d^\lambda(A^m u, B^n w), d^\lambda(A^m u, S^p u), d^\lambda(B^n w, T^q w), d^\lambda(A^m u, T^q w), d^\lambda(B^n w, S^p u)) \\
&= \phi(d^\lambda(u,w), 1, 1, d^\lambda(u,w), d^\lambda(u,w)) \\
&\leq d^\lambda(u,w).
\end{aligned}
$$

Hence $d(u,w) = 1$, i.e., $u = w$. Therefore u is the unique common fixed point of $\{A,B,S,T\}$.

Remark 2.2. The condition (i) in Theorem 2.3 can be replaced by $SX \subset B^n X$ and $TX \subset A^m X$ Since $S^p X \subset S$X and $T^q X \subset TX$. Also, (i) can be replaced by $S^p X \cup T^q X \subseteq A^m X \cap B^n X$.

Theorem 2.4. Suppose that (X,d) is a multiplicative metric space, $S,T : X \to X$ two mappings, $\{p,q,m,n\}$ are four natural numbers with $n \leq p$ and $m \leq q$. If

(i) there exists $\lambda \in (0, \frac{1}{2})$ such that for each $x,y \in X$,

$$d(S^p x, T^q y) \leq \phi(d^\lambda(T^m x, S^n y), d^\lambda(T^m x, S^p x), d^\lambda(S^n y, T^q y), d^\lambda(T^m x, T^q y), d^\lambda(S^n y, S^p x));$$

(ii) the pair (S,T) is commutative;

(iii) one of $\{S^p X, T^m X, S^n X, T^q X\}$ is complete.

Then S,T have a unique common fixed point.

Proof. Let $A = T$ and $B = S$, then (i) can be rewritten as

$$d(S^p x, T^q y) \leq \phi(d^\lambda(A^m x, B^n y), d^\lambda(A^m x, S^p x), d^\lambda(B^n y, T^q y), d^\lambda(A^m x, T^q y), d^\lambda(B^n y, S^p x)).$$

Also, $S^p X \subset B^n X$ and $T^q X \subset A^m X$ since $n \leq p$ and $m \leq q$. Hence the conclusion follows from Theorem 2.3.

Conflict of Interests

The authors declare that there is no conflict of interests.

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