Available online at http://scik.org
Adv. Fixed Point Theory, 9 (2019), No. 4, 322-332
https://doi.org/10.28919/afpt/4185
ISSN: 1927-6303

PERIODIC POINT AND FIXED POINT RESULTS FOR MONOTONE MAPPINGS IN COMPLETE ORDERED LOCALLY CONVEX SPACES WITH APPLICATION TO DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we establish some new periodic point and fixed point theorems of single-valued mapping operating between complete ordered locally convex spaces under weaker assumptions. As an application, we prove the existence of lower and upper solutions of differential equations.

Keywords: periodic point; fixed point; measure of noncompactness; ordered locally convex spaces; differential equations.

2010 AMS Subject Classification: 37C25, 47H08, 46A03, 47H10.

1. INTRODUCTION

A lot of research has been devoted to the study of the existence of a fixed and periodic points of single-valued and multivalued mappings in ordered Banach spaces and Metric spaces,[16],[17], [1], [9], and in complete locally convex spaces [7], [4], [5]. In the present work, we discuss an analogue of a periodic and a fixed point theorems proved in [1] in the setting of a complete ordered locally convex spaces.

The aim of this paper is to investigate the notion of order in a complete ordered locally convex

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Received June 22, 2019

spaces which will give us a new periodic and a new fixed point results for a monotone mappings in the case of singe-valued mapping.

The concept of measure of noncompactness in locally convex spaces [3, p 90] is used to define condensing operators in this new setting. Hence, we prove in Theorem 3.2 the equivalent of [[1, Theorem 2.1.1]] in complete ordered locally convex spaces and use it to prove the existence of a periodic and a fixed point in the theorem 3.5.

It is well known that fixed point theorems play an important role in differential equations, game theory and mathematical economics..., Toshio Yuasa [7], D. Guo, V. Lakshmikantham [1], S. Reich [15].

In Section 4, we prove the existence of lower and upper solutions of differential equations in a new framework.

2. NOTATIONS AND PRELIMINARIES

Let E be a real vector space. A cone K in E is a subset of E with $K+K\subset K$, $\alpha K\subset K$ for all $\alpha\geq 0$, and $K\cap(-K)=\{0\}$. As usual E will be ordered by the (partial) order relation

$$x\leq y\Leftrightarrow y-x\in K$$

and the cone K will be denoted by E^+. E is said to be an ordered topological vector space, if E is an ordered vector space equipped with a linear topology for which the positive cone E^+ is closed. For two vectors $x,y\in E$ the order interval $[x,y]$ is the set defined by

$$[x,y]=\{z\in E: x\leq z\leq y\}.$$

Note that if $x\nleq y$ then $[x,y]=\phi$.

A cone E^+ of an ordered topological vector space E is said to be normal whenever the topology of E has a base at zero consisting of order convex sets. If the topology of E is also locally convex, then E is said to be an ordered locally convex space, and in this case the topology of E has a base at zero consisting of open, circled, convex, and order convex neighborhoods.

The following two lemmas will be useful in the proofs of our results.

Lemma 2.1 ([2, Lemma 2.3]). *If E is an ordered topological vector space, then E is Hausdorff and the order intervals of E are closed.*

Lemma 2.2 ([2, Lemma 2.22 and Theorem 2.23])**.** *If the cone E^+ of an ordered topological vector space (E,τ) is normal, then the following assertions hold:*

(1) Every order interval is $\tau-$bounded.

(2) For every two nets (x_α), $(y_\alpha) \subset E$, (with the same index set I) satisfy $0 \leq x_\alpha \leq y_\alpha$ for each α and $y_\alpha \xrightarrow{\tau} 0$ imply $x_\alpha \xrightarrow{\tau} 0$.

Let E be an ordered locally convex space whose topology is defined by a family $\mathscr{P}$ of continuous semi-norms on E, $\mathscr{B}$ is the family of all bounded subsets of E, and Φ is the space of all functions $\varphi : \mathscr{P} \to \mathbb{R}^+$ with the usual partial ordering $\varphi_1 \leq \varphi_2$ if $\varphi_1(p) \leq \varphi_2(p)$ for all $p \in \mathscr{P}$. The measure of noncompactness on E is the function $\alpha : \mathscr{B} \to \Phi$ such that for every $B \in \mathscr{B}$, $\alpha(B)$ is the function from $\mathscr{P}$ into $\mathbb{R}^+$ defined by

$$\alpha(B)(p) = \inf\{d > 0 : \sup\{p(x-y) : x,y \in B_i\} \leq d \ \ \forall i\}$$

where the infimum is taken on all subsets B_i such that B is finite union of B_i. Properties of measure of noncompactness in locally convex spaces are presented in [4, Proposition 1.4].

An operator $T : Q \subset E \to E$ is called to be countably condensing if $T(Q)$ is bounded and if for any countably bounded set A of Q with $\alpha(A)(p) > 0$ we have

$$\alpha(T(A))(p) < \alpha(A)(p)$$

Definition 2.3. *Let E be a complete ordered locally convex space with a normal cone E^+. An element $x \in E$ is said to be a fixed point of a mapping $T : E \to E$ if $x = T(x)$.*

Definition 2.4. *Let E be a complete ordered locally convex space with a normal cone E^+. An element $x \in E$ is said to be a periodic point of a mapping $T : E \to E$ if $T^n(x) = x$ the smallest such positive integer n is called the period of x (with respect to T). We denote the set of all periodic points of T by $Per(T)$.*

For each intrger $n \geq 1, T^n$ denotes the n^{th} iterate of T, that is, the composition $T \circ T \circ ... \circ T$ of T with itself $n-1$ times $(T^1 = T, T^2 = T \circ T ...)$. Also, T^0 is the identy map of E.

Definition 2.5. *Let E be a complete ordered locally convex space with a normal cone E^+. A map $T : E \to E$ is said to be nondecreasing if for $x,y \in E$ and $x \leq y$ we have $Tx \leq Ty$.*
A map $T : E \to E$ is said to be nonincreasing if for $x,y \in E$ and $x \leq y$ we have $Tx \geq Ty$.

Definition 2.6. *Let E be an ordered locally convex spaces and let $x \in E$. A mapping $f : E \to E$ is said to be order continuous in x if $f(x_\alpha) \to f(x)$ for each increasing or decreasing net $\{x_\alpha\}$ that converges to x.*

It is evident that continuity implies order continuity

3. Main results

The following results generalize the results of [1] in complete ordered locally convex spaces, and we add another results whith low condtions.

Lemma 3.1. *Let E be an ordered topological vector space with a normal cone E^+. Then a monotone net $(u_\alpha) \subset E$ is convergent if and only if it has a weakly convergent subnet.*

Proof. The "only if" part is obvious. For the " if " part, assume that $(u_\alpha)_{\alpha \in (\alpha)}$ is nondecreasing and let $(u_{\alpha_i})_{i \in (i)} \subset (u_\alpha)$ be a subnet such that $u_{\alpha_i} \to u$ weakly for some $u \in E$, where (α) stands for the indexed set of the net (u_α). Let $\beta \in (\alpha)$ be fixed. For each $\alpha \geq \beta$, let $i_0 \in (i)$ such that $\alpha_{i_0} \geq \alpha$. Thus, for each $i \geq i_0$ we have

$$u_\beta \leq u_\alpha \leq u_{\alpha_i}. \tag{3.1}$$

Thus, since $u_{\alpha_i} \to u$ weakly and the cone E^+ is weakly closed (being a closed and convex set) we see that $u_\beta \leq u$ for each $\beta \in (\alpha)$. Thus, it follows from [2, Lemma 2.28] that $\lim u_{\alpha_i} = u$. Now, let $V \in V(0)$ be arbitrary. Since the cone E^+ is normal we may assume that V is an order convex set. Let $j \in (i)$ such that $u - u_{\alpha_i} \in V$ for each $i \geq j$. If $\beta \geq \alpha_j$ then $0 \leq u - u_\beta \leq u - u_{\alpha_j}$, and hence $u - u_\beta \in V$. That is $\lim u_\beta = u$ as required. The desired conclusion is proved similarly when (u_α) is nonincreasing. □

in the following theorem,a Hausdorff locally convex space is regular, [8, see Chapter VI, Section 1]

Theorem 3.2. *Let E be a complete ordered locally convex space with a normal cone E^+. Let Ω be an order convex subset of E,*

and let $u_0, v_0 \in \Omega$, $u_0 \leq v_0$ and $T : \Omega \to \Omega$ be a continuous and nondecreasing mappings such that :

$u_0 \leq T^k(u_0)$ and $T^k(v_0) \leq v_0$ where k is a positive integer.

Suppose that T is condensing from Ω in to itself.

Then, T has a minimal periodic point u and a maximal periodic point v in Ω.

Proof. We pose : $S = T^k$. Consider the sequences (u_n) and (v_n) defined by:

$$u_n = Su_{n-1}, \quad v_n = Sv_{n-1}, \quad n \in \mathbb{N} \tag{3.2}$$

Since T is nondecreasing and fixes the interval $[u_0, v_0]$. Then from (3.2) it follows that

$$u_0 \leq u_1 \leq .... \leq u_n \leq ..... \leq v_n \leq ... \leq v_1 \leq v_0 \tag{3.3}$$

And $[u_0, v_0] \subset \Omega$ because Ω is a order convex subset of E.

Let $A = \{u_0, u_1, ...\}$, we have $A = \{u_0\} \cup S(A)$ and the set A is bounded since S is condensing (because T is condensing and in $T(\Omega)$ is bounded).

So $\overline{A}$ is compact, by [3, p 89],

$\{u_n\}$ has a convergent subnet which converges to $u \in [u_0, v_0]$, and by (3.3), $\{u_n\}$ is nondecreasing, so by lemma 3.1, the original sequence $\{u_n\}$ converges to $u \in [u_0, v_0] \subset \Omega$. Also we have

$$u = \lim_{n\to\infty} u_n$$

Since S is continuous mapping, so, $u = Su \Leftrightarrow u = T^k u$

Similarly, we can prove that $\{v_n\}$ converges to some $v \in E$ and $v = T^k v$.

Finally, we prove that u and v are the maximal and minimal periodic points of T in $[u_0, v_0] \subset \Omega$. Indeed, let $x \in [u_0, v_0]$ and $x = T^k x$, Since T is nondecreasing, we have $u_n \leq x \leq v_n$, taking limit $n \to \infty$, we obtain $u \leq x \leq v$. □

Remark 1. *this theorem remains true if continuity is replaced by ordered continuity.*

Corollary 3.3. *Let E be a complete ordered locally convex space with a normal cone E^+. Let Ω be an order convex subset of E,*

and let $u_0, v_0 \in \Omega$, $u_0 \leq v_0$ and $T : \Omega \to \Omega$ be a order continuous and nondecreasing mappings such that : $u_0 \leq T(u_0)$ and $T(v_0) \leq v_0$.

Suppose that T is condensing from Ω in to itself.

Then, T has a minimal fixed point u and a maximal fixed point v in Ω.

Proof. It is obtained by taking $k = 1$ in Theorem 3.2. □

Corollary 3.4. *Let E be a complete ordered locally convex space with a normal cone E^+. Let $u_0, v_0 \in E$ such that $u_0 \leq v_0$ and $T : [u_0, v_0] \to [u_0, v_0]$ be a continuous and nondecreasing mapping such.*
Suppose that T is condensing from $[u_0, v_0]$ in to itself.
Then, T has a minimal fixed point u and a maximal fixed point v in $[u_0, v_0]$.

Proof. It is obtained by taking $k = 1$ and $[u_0, v_0] = \Omega$ in Theorem 3.2. □

Theorem 3.5. *Let E be a complete ordered locally convex space with a normal cone E^+. Let Ω be an order convex subset of E, and let $u_0, v_0 \in \Omega$, $u_0 \leq v_0$ and let $T : \Omega \to \Omega$ be a continuous nonincreasing mappings such that $u_0 \leq T^{2k}(u_0)$ and $T^{2k}(v_0) \leq v_0$ where k is a positive integer.*
Suppose that T is condensing mapping from Ω in to itself.
Then, the set $Per(T) = \{x \in \Omega : T^k x = x\}$ is nonempty and compact.

Proof. Since T is condensing and continuous, then so is T^2, also T^2 is nondecreasing and fixes the interval $[u_0, v_0]$.
Then, from 3.2, T^2 has a minimal periodic point u and a maximal periodic point v in $[u_0, v_0]$. It is easy to see that Tu and Tv are likwise a periodics point of T^2. Therefore, we have :

$$u \leq Tv \leq Tu \leq v$$

Now, if $x \in [u, v]$, then :

$$u \leq Tv \leq Tx \leq Tu \leq v$$

It follows that T fixes the interval $[u, v]$, we pose $S = T^{k'}$,
with $k' \in \mathbb{N}^*$, so, S also fixes the interval $[u, v]$, then $S[u, v]$ is bounded. Now, because the cone E^+ is normal, the interval $[u, v]$ is a convex, closed, and bounded subset of E.
Then applying [4, Theorem 2.7] for the set $[u, v]$ in the case where $T_i = Id_E$, it follows that S has a fixed point in $[u, v] \subset \Omega$.

Then, T has a periodic point in $[u,v] \subset \Omega$.

For the compacity of $Per(T)$, note that $Per(T) \subset [u,v]$. Therefore, $Per(T)$ is a bounded set. If $\alpha(Per(T))(p) \neq 0$ for all $p \in \mathscr{P}$.

Then we have :

$$\alpha(Per(T))(p) = \alpha(T^{k'}(Per(T)))(p) < \alpha(T^{k'-1}(Per(T)))(p) < ... < \alpha(Per(T))(p),$$

which is a contradiction. Therefore $\alpha(Per(T))(p) = 0$,

that is by [4, Proposition 1.4] and by continuity of T, $Per(T)$ is a compact set in Ω. □

Corollary 3.6. *Let E be a complete ordered locally convex space with a normal cone E^+. Let $u_0, v_0 \in E$, $u_0 \leq v_0$ and let $T : E \to E$ be a continuous nonincreasing mappings such that $u_0 \leq T^{2k}(u_0)$ and $T^{2k}(v_0) \leq v_0$ where k is a positive integer. Suppose that T is condensing mapping from E in to itself.*

Then, the set $PerT = \{x \in E : T^k x = x\}$ is nonempty and compact.

Proof. It is obtained by taking $[u_0, v_0] = \Omega$ in Theorem 3.5 since $[u_0, v_0]$ is order convex. □

Corollary 3.7. *Let E be a complete ordered locally convex space with a normal cone E^+. Let Ω be an order convex subset of E, and let $u_0, v_0 \in \Omega$, $u_0 \leq v_0$ and let $T : \Omega \to \Omega$ be a continuous nonincreasing mappings such that $u_0 \leq T(u_0)$ and $T(v_0) \leq v_0$.*

Suppose that T is condensing mapping from Ω in to itself.

Then, the set $FixT = \{x \in \Omega : Tx = x\}$ is nonempty and compact.

Proof. It is obtained by taking $k = 1$ in Theorem 3.5. □

4. APPLICATION TO DIFFERENTIAL EQUATIONS

In this section we will give an application of Corollary 3.4 to the following equation differential :

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad x_0 \in A. \tag{4.1}$$

Where X be a complete ordred Hausdorff locally convex space and $A \subset X$ be open, $J = [t_0, t_0 + a] \subset \mathbb{R}$ be an interval, $C(J,X)$ be the space of continuous functions from J to X, $f(t,x) \in C(J \times$

$A,X)$;

In this section, $\leq$ and $<$ mean the total order relation of $\mathbb{R}$.

We define an order relation $\preceq$ in C(J, X) by the order cone P in

C(J, X) defined by the cone $P = \{x \in C(J,X)/x(t) \in X^+, \forall t \in J\}$ where X^+ is a normal cone in X.

$C(J,X)$ is a complete ordred Hausdorff locally convex space with a normal cone P.

The equation (4.1) is equivalent to the integral equation :

$$x(t) = x_0 + \int_{t_0}^{t} f(s,x(s))ds \tag{4.2}$$

where the integral is in the Riemann sense. see [12, II,p.29.Theorem 21].

Proposition 4.1. $\int_{t_0}^{t_1} f(s,x(s))ds \in \overline{co}(\{f(s,x(s))/s \in [t_0,t_1]\}$

This proposition directly follows from the definition of the Riemann integral.

The following proposition characterizes the measure of nonprecompactness of a bounded, equicontinuous subset H of $C(J,X)$. Similar results are obtained for $\alpha(A)$ and $\omega(A)$ by [13, Ambrosetti] and [13, Mitchell, Smith] respectively.

Proposition 4.2. [7, proposition 2]

Let X be a complete Kausdorff locally convex space and let $J = [t_o, t_0 + a] \subset \mathbb{R}$ be a interval. Let $H \subset C(J,X)$ be a bounded equicontinuous set. Then we have :

$$\alpha(H)(p) = \alpha(H(J))(p) = \bigcup_{t \in J} \alpha(H(t))(p)$$

for all $p \in \mathscr{P}$

Definition 4.3. *A function $f(t,x)$ is said to be nondecreasing with respect to x if for any $x,y \in X$ with $x \preceq y$ we have that $f(t,x) \preceq f(t,y)$ for all $t \in J$.*

Theorem 4.4. *Assume the following hypotheses :*

(1) $f(t,x)$ increasing in x.

(2) There exists a order convex set F such as $x_0 \in F \subset A$ and $B_0 = \overline{co}((f(J \times F) \cup (\{0\})$ is bounded and $x_0 + \alpha_0 B_0 \subset F$ for some $\alpha_0 > 0$.

(3) *For any bounded set* $B_1 \subset \overline{B_1} \subset A$ *there exist an intercal* $J' = [t_0, t_0 + a'] \subset J$ *and a constant* $\lambda > 0$ *such that for any countably bounded set* $B \subset B_1$ *with* $\alpha(B)(p) > 0$ *we have:*

$$\alpha(f\left(J' \times B\right))(p) < \alpha\left(B\right)(p)$$

(4) *there exists* $\gamma, \delta \in C(J', X)$ *such that* $\gamma \preceq \delta$*:*

$$\gamma(t) \preceq x_0 + \int_{t_0}^{t} f(s, x(s))ds \preceq \delta(t)$$

Then, $\exists \beta \in]0, a]$ *such that the equationa(4.1) has a lower and upper solution in the order interval* $[\gamma, \delta] \subset C(I, X)$ $\forall t \in I = [t_0, t_0 + \beta]$.

Proof. Let $\beta = \inf\{\alpha_0, a'\}$ and let $I = [t_0, t_0 + \beta]$.
Since $I \subset J'$, it follows that :

$$\alpha(f\left(I \times B\right))(p) < \alpha\left(B\right)(p)$$

for any countably bounded set $B \subset B_1$ with $\alpha(B)(p) > 0$.
By hypotheses (4), we have :

$$[\gamma, \delta] = \{x \in C(I, X) / x(t_0) = x_0, x(t) - x(t') \in (t - t')B_0, \gamma(t) \preceq x(t) \preceq \delta(t), \forall t, t' \in I\}$$

Clearly, $[\gamma, \delta]$ is a nonempty, order convex, equicontinuous set
in $C(I, F) \subset C(J, X)$.
We define the operator $T : [\gamma, \delta] \to [\gamma, \delta]$ by :

$$Tx(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$$

T is well defined , it remains to show that the operator T satisfies the conditions of Corollary 3.4.
First, the proof of the continuity of T is similar to that of [7, p543].

Second, for any countably bounded set $B \subset H$ with $\alpha(B)(p) > 0.$, we have :

$$\begin{aligned}
\alpha(T(B))(p) &= \alpha\Big(\bigcup_{t\in I} T(B(t))\Big)(p) \\
&= \alpha\Big(\bigcup_{t\in I}\Big\{x_0 + \int_{t_0}^{t_1} f(s,x(s))ds : x \in B\Big\}\Big)(p) \\
&= \alpha\Big(\bigcup_{t\in I}\Big\{\int_{t_0}^{t_1} f(s,x(s))ds : x \in B\Big\}\Big)(p) \\
&\leq \alpha(\bigcup_{t\in I}\Big\{(t-t_0)\overline{conv} f(I \times B(I))\Big\})(p) \\
&\leq \alpha(\overline{conv} f(I \times B(I)))(p) \\
&= \alpha(f(I \times B(I)))(p) \\
&< \alpha(B(I))(p) \\
&= \alpha(B)(p)
\end{aligned}$$

Finally, by hypotheses (1) and the monotonicity of integral, we have T is nondecreasing.

Thus the conditions of Corollary 3.4 are satisfied. Consequently, T has a minimal fixed point u and a maximal fixed point v in $[\omega, \delta]$.

This further implies that differential equation (4.1) has a lower and upper solution in the order interval $[\gamma, \delta]$. This completes the proof □

Conflict of Interests

The authors declare that there is no conflict of interests.

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