FIXED POINT THEOREMS FOR CYCLIC CONTRACTION ON $b$-METRIC SPACES WITH $wt$-DISTANCE

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Abstract. In this paper, some fixed point theorems for cyclic generalized $\varphi$-contraction and $(\psi, \varphi)$–weakly contraction on $b$-metric spaces with $wt$-distance are proved, which extend some results in the literature.

Keywords: cyclic; fixed point; $wt$-distance; $b$-metric space.

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1. INTRODUCTION AND PRELIMINARIES

Since the concept of $b$-metric space as a generalization of metric space was given by Czerwik [1], many fixed point results in metric spaces were generalized in $b$-metric spaces (see [2, 6], etc.). In 2014, the concept of $wt$-distance on $b$-metric spaces was given by N. Hussain et al. [3], we shall use $wt$-distance on $b$-metric spaces to extend some results by others.

In the section one, we give some elementary definitions and lemmas. In the section two, inspired by H.K. Nashine and Z. Kadelburg [8] and H.P. Huang [5], we define cyclic generalized $\varphi$-contraction and $(\psi, \varphi)$–weakly contraction on $b$-metric spaces with $wt$-distance and related fixed point results are proved, which extend some results in the literature.

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Throughout, we denote all natural number by $\mathbb{N}$.

**Definition 1.1.** [1] Let $X$ be a nonempty set and constant $s \geq 1$ be a fixed real number. Suppose that the mapping $d : X \times X \rightarrow [0, \infty)$ satisfies the following conditions:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then $(X, d)$ is called a $b$-metric space with coefficient $s$.

**Definition 1.2.** [3, 4] Let $(X, d)$ be a $b$-metric space with constant $s \geq 1$, then a function $p : X \times X \rightarrow [0, \infty)$ is called a $wt$-distance on $X$ if the following conditions are satisfied:

1. $p(x, z) \leq s[p(x, y) + p(y, z)]$ for any $x, y, z \in X$;
2. $p(x, \cdot) : X \rightarrow [0, \infty)$ is $s$-lower semi-continuous for any $x \in X$, if
   \[
   \liminf_{n \to \infty} p(x, x_n) = \infty, \quad \text{or} \quad p(x, x_0) \leq \liminf_{n \to \infty} sp(x, x_n),
   \]
   where $\lim_{n \to \infty} d(x_0, x_n) = 0$;
3. for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

The $wt$-distance $p$ is called symmetric if $p(x, y) = p(y, x)$ for any $x, y \in X$. We say that

(a) The sequence $\{x_n\}$ converges to $x \in X$ if and only if $\lim_{n \to \infty} d(x_n, x) = 0$, i.e., $x_n \to x$;
(b) The sequence $\{x_n\}$ is Cauchy if and only if $\lim_{n, m \to \infty} d(x_n, x_m) = 0$;
(c) $(X, d)$ is complete if and only if any Cauchy sequence in $X$ is convergent.

**Lemma 1.3.** [3, 4] Let $(X, d)$ be a $b$-metric space with constant $s \geq 1$ and $p$ be a $wt$-distance on $X$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $X$, $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, \infty)$ converging to zero. Then for any $x, y, z \in X$, the following properties hold:

1. If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$;
2. If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $\lim_{n \to \infty} d(y_n, z) = 0$;
3. If $p(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;
4. If $p(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

2. MAIN RESULTS

In this part, we will show our lemmas, theorems and corollaries.
Lemma 2.1. Let \((X, d)\) be a \(b\)-metric space with constant \(s \geq 1\) and \(p\) be a \(wt\)-distance on \(X\), \(\{x_n\}\) be sequence in \(X\), then the inequality

\[
p(x_0, x_k) \leq s^n \sum_{i=0}^{k-1} p(x_i, x_{i+1})
\]  

(2.1)

is valid for every \(n \in \mathbb{N}\) and every \(k \in \{1, 2, \ldots, 2^n-1, 2^n\}\).

Proof. Let us use mathematical induction, denote (2.1) by \(P(n)\), then we have

\[
P(0) : p(x_0, x_1) \leq p(x_0, x_1) = s^0 \sum_{i=0}^{0} p(x_i, x_{i+1}),
\]

\[
P(1) : p(x_0, x_2) \leq s[p(x_0, x_1) + p(x_1, x_2)] = s^1 \sum_{i=0}^{1} p(x_i, x_{i+1})
\]

Now, we assume that

\[
P(n) : p(x_0, x_k) \leq s^n \sum_{i=0}^{k-1} p(x_i, x_{i+1})
\]  

(2.2)

is valid for every \(x_0, x_1, \ldots, x_{2^n} \in X\) for every \(k \in \{1, 2, \ldots, 2^n-1, 2^n\}\), then we will prove that \(P(n+1)\) is also valid.

Indeed, for \(k \in \{2^n + 1, 2^n + 2, \ldots, 2^{n+1} - 1, 2^{n+1}\}, \) by (2.2), we have

\[
p(x_0, x_k) \leq s[p(x_0, x_{2^n}) + p(x_{2^n}, x_k)]
\]

\[
\leq s \left[ s^n \sum_{i=0}^{2^n-1} p(x_i, x_{i+1}) + s^2 \sum_{i=2^n}^{k-1} p(x_i, x_{i+1}) \right]
\]

\[
= s^{n+1} \sum_{i=0}^{k-1} p(x_i, x_{i+1}). \quad \Box
\]

Lemma 2.2. Let \((X, d)\) be a \(b\)-metric space with constant \(s \geq 1\) and \(p\) be a \(wt\)-distance on \(X\), \(\{x_n\}\) be sequence in \(X\), we say the \(\{x_n\}\) is a Cauchy sequence if there exists \(c \in [0, 1)\), such that \(p(x_n, x_{n+1}) \leq cp(x_{n-1}, x_n)\) for every \(n \in \mathbb{N}\).
Proof. We note that \( p(x_n, x_{n+1}) \leq c^n p(x_0, x_1) \) for every \( n \in \mathbb{N} \). For all \( m, k \in \mathbb{N} \) with \( r = \lfloor \log_2 k \rfloor \), we have

\[
p(x_{m+1}, x_{m+k}) \leq s \left[ p(x_{m+1}, x_{m+2}) + p(x_{m+2}, x_{m+k}) \right]
\]

\[
\leq sp(x_{m+1}, x_{m+2}) + s^2 p(x_{m+2}, x_{m+2^2}) + s^2 p(x_{m+2}, x_{m+k})
\]

\[
\leq sp(x_{m+1}, x_{m+2}) + s^2 p(x_{m+2}, x_{m+2^2}) + s^3 p(x_{m+2^2}, x_{m+2^3}) + s^3 p(x_{m+2^3}, x_{m+k})
\]

\[
\ldots
\]

(2.3)

\[
\leq \sum_{n=1}^{r} s^n p(x_{m+2^{n-1}}, x_{m+2^n}) + s^{r+1} p(x_{m+2^r}, x_{m+k})
\]

Then by (2.3) and Lemma 2.1, we have

\[
p(x_{m+1}, x_{m+k}) \leq \sum_{n=1}^{r} s^{2n} \left( \sum_{i=m}^{m+2^{n-1}-1} p(x_{2^{n-1}+i}, x_{2^{n-1}+i+1}) \right)
\]

\[
+ s^{2(r+1)} \left( \sum_{i=m}^{m+k-2^r-1} p(x_{2^r+i}, x_{2^r+i+1}) \right)
\]

\[
\leq \sum_{n=1}^{r+1} s^{2n} \left( \sum_{i=m}^{m+2^{n-1}-1} p(x_{2^{n-1}+i}, x_{2^{n-1}+i+1}) \right)
\]

\[
\leq p(x_0, x_1) \sum_{n=1}^{r+1} s^{2n} \left( \sum_{i=0}^{2^{n-1}-1} c^{m+2^{n-1}+i} \right)
\]

\[
\leq \frac{p(x_0, x_1)}{1 - c} \sum_{n=1}^{r+1} s^{2n} c^{m+2^{n-1}}
\]

\[
= \frac{p(x_0, x_1)}{1 - c} c^m \sum_{n=1}^{r+1} s^{2n} c^{2^{n-1}}
\]

(2.4)

\[
= \frac{p(x_0, x_1)}{1 - c} c^m \sum_{n=1}^{r+1} c^{2n \log_c s + 2^{n-1}} \to 0 \ (m \to \infty),
\]

where \( 0 < c < 1 \) and \( \sum_{n=1}^{\infty} c^{2n \log_c s + 2^{n-1}} \) is convergent.

Then by lemma 1.3, the proof is immediate. \( \square \)

Now, we denote by \( \Phi \) the set of functions \( \varphi : [0, \infty) \to [0, \infty) \) with \( \varphi(t) < \frac{t}{\omega t} \) for each \( t > 0, \alpha > 1 \) and \( \varphi(0) = 0 \).
Definition 2.3. [7, 9] Let \((X, d)\) be a \(b\)-metric space, \(k\) be a positive integer, \(A_1, A_2, \cdots, A_k\) be nonempty subsets of \(X\), \(V = \bigcup_{i=1}^{k} A_i\), \(f : V \to V\), then \(f\) is called a cyclic operator if
\[
\begin{align*}
(1) & \ A_i, i = 1, 2, \cdots, k \text{ are nonempty subsets;} \\
(2) & \ f(A_1) \subseteq A_2, \cdots, f(A_{p-1}) \subseteq A_p, f(A_p) \subseteq A_1.
\end{align*}
\]

Definition 2.4. Let \((X, d)\) be a \(b\)-metric space with constant \(s \geq 1\) and \(p\) be a \(wt\)-distance on \(X\), \(k\) be a positive integer, \(A_1, A_2, \cdots, A_k\) be nonempty subsets of \(X\), \(V = \bigcup_{i=1}^{k} A_i\), \(f : V \to V\) satisfies a cyclic generalized \(\varphi\)-contraction for some \(\varphi \in \Phi\), if
\[
\begin{align*}
(1) & \ V = \bigcup_{i=1}^{k} A_i \text{ is a cyclic representation of } V \text{ with respect to } f; \\
(2) & \text{for any } (x, y) \in A_i \times A_{i+1}, i = 1, 2, \cdots, k, (A_{k+1} = A_1), \text{ there exist } L \geq 0 \text{ and constant } \lambda_1 > 0, \\
& \quad 0 < \lambda_2 < \frac{as}{2}, a > 1 \text{ and } a > \lambda_2 \text{ such that}
\end{align*}
\]
\[
(2.5) \quad p(fx, fy) \leq M_s(x, y) + L \min \{\varphi(p(x, fx)), \varphi(p(y, fy)), \varphi(p(x, fy)), \varphi(p(y, fx))\}
\]
where
\[
M_s(x, y) = \max \{\varphi(p(x, y)), \varphi(p(x, fx)), \varphi(\lambda_1 p(x, fx) + (1 - \lambda_1) p(y, fy)), \\
\varphi(\frac{\lambda_2 p(x, fy) + (1 - \lambda_2) p(fx, y)}{s})\}.
\]

Theorem 2.5. Let \((X, d)\) be a complete \(b\)-metric space with \(s \geq 1\) and \(p\) be a \(wt\)-distance on \(X\), \(p(x, x) = 0\) for any \(x \in X\), \(V = \bigcup_{i=1}^{k} A_i\) and \(A_1, A_2, \cdots, A_k\) be nonempty closed subsets of \(X\), \(k\) be a positive integer, \(f : V \to V\) is a cyclic generalized \(\varphi\)-contraction mapping for some \(\varphi \in \Phi\).

Suppose that either
\[
(1) \inf \{p(x, w) + p(x, fx) : x \in X\} > 0 \text{ for every } w \in X \text{ with } w \neq fw;
\]
or
\[
(2) \text{the mapping } f \text{ is continuous.}
\]

Then \(f\) has a unique fixed point. Moreover, the fixed point of \(f\) belongs to \(\bigcap_{i=1}^{k} A_i\).

Proof. For any \(x_0 \in A_1\) (such a point exists since \(A_1 \neq \emptyset\)), we can construct the sequence \(\{x_n\}\) in \(X\) by \(x_{n+1} = fx_n\) \((n \in \mathbb{N} \cup \{0\})\). If \(x_n = x_{n+1}\) for some \(n \in \mathbb{N} \cup \{0\}\), then \(f\) has fixed point. Now, suppose that \(x_n \neq x_{n+1}\) for any \(n \in \mathbb{N} \cup \{0\}\).

Next, we shall prove that
\[
\lim_{n \to \infty} p(x_n, x_{n+1}) = 0.
\]
Indeed, if not, suppose that \( p(x_n, x_{n+1}) > 0 \) for all \( n \in \mathbb{N} \cup \{0\} \), there exists \( i = i(n) \in \{1, 2, \ldots, k\} \) such that \( (x_n, x_{n+1}) \in A_i \times A_{i+1} \), then we claim that \( \xi_n \leq c \xi_{n-1} \) for all \( n \in \mathbb{N} \) \( (0 < c < 1) \), where \( \xi_n = p(x_n, x_{n+1}) \).

By (2.5), we have

\[
M_s(x_n, x_{n+1}) = p(fx_n, fx_{n+1}) = M_s(x_n, x_{n+1}) + L \min \{ \varphi(p(x_n, fx_{n-1})), \varphi(p(x_n, f^2x_{n-1})), \varphi(p(x_n, f^3x_{n-1})), \ldots, \varphi(p(x_n, f^nx_{n-1})) \}
\]

\[
= M_s(x_n, x_{n+1}) + L \min \{ \varphi(p(x_n, x_{n+1})), \varphi(p(x_n, x_{n+1})) \}
\]

(2.6)

\[
= M_s(x_n, x_{n+1})
\]

from Definition 2.4, we have

\[
M_s(x_n, x_{n+1}) = \max \{ \varphi(p(x_n, x_{n+1})), \varphi(\lambda_1 p(x_n, x_{n+1})), \varphi(1 - \lambda_1) p(x_n, x_{n+1}), \varphi(\frac{\lambda_2 p(x_n, x_{n+1})}{s}) \}.
\]

Consider the following possibilities.

If \( M_s(x_n, x_{n+1}) = \varphi(p(x_n, x_{n+1})) \), then by (2.6) and \( \varphi(t) < \frac{t}{as} \), we have

\[
\xi_n = p(x_n, x_{n+1}) \leq M_s(x_n, x_{n+1}) = \varphi(p(x_n, x_{n+1})) < \frac{p(x_n, x_{n+1})}{as} = r_1 \xi_{n-1},
\]

where \( r_1 \equiv \frac{1}{as} \in (0, 1) \).

If \( M_s(x_n, x_{n+1}) = \varphi(\lambda_1 p(x_n, x_{n+1}) + (1 - \lambda_1) p(x_n, x_{n+1})) \), then by (2.6) and \( \varphi(t) < \frac{t}{as} \), we have

\[
\xi_n = p(x_n, x_{n+1}) \leq M_s(x_n, x_{n+1}) = \varphi(\lambda_1 p(x_n, x_{n+1}) + (1 - \lambda_1) p(x_n, x_{n+1})) \leq \frac{\lambda_1 p(x_n, x_{n+1}) + (1 - \lambda_1) p(x_n, x_{n+1})}{as} \]

i.e.,

\[
\xi_n = p(x_n, x_{n+1}) < r_2 p(x_n, x_{n+1}) = r_2 \xi_{n-1},
\]
where $r_2 \doteq \frac{\lambda_1}{as + A_1} \in (0, 1)$.

If $M_s(x_{n-1}, x_n) = \varphi \left( \frac{\lambda_2 p(x_{n-1}, x_{n+1})}{s} \right)$, then by (2.6) and $\varphi(t) < \frac{t}{as}$, we have

$$
\xi_n = p(x_n, x_{n+1}) \leq M_s(x_{n-1}, x_n) = \varphi \left( \frac{\lambda_2 p(x_{n-1}, x_{n+1})}{s} \right)
$$

$$
< \frac{\lambda_2 p(x_{n-1}, x_{n+1})}{as^2} \leq \frac{\lambda_2}{as} \left[ p(x_{n-1}, x_n) + p(x_n, x_{n+1}) \right]
$$

i.e.,

$$
\xi_n = p(x_n, x_{n+1}) < r_3 p(x_{n-1}, x_n) = r_3 \xi_{n-1}.
$$

where $r_3 \doteq \frac{\lambda_3}{as - A_2} \in (0, 1)$.

Let $\hat{c} = \max \{ r_1, r_2, r_3 \}$, then we have

(2.7) \quad $0 < \xi_n < \hat{c} \xi_{n-1} < (\hat{c})^2 \xi_{n-2} < \cdots < (\hat{c})^n \xi_0$.

Since $\hat{c} \in (0, 1)$, then we have

(2.8) \quad $\lim_{n \to \infty} \xi_n = \lim_{n \to \infty} p(x_n, x_{n+1}) = 0$.

By (2.7) and Lemma 2.2, then $\{x_n\}$ is a Cauchy sequence.

Since $X$ is a complete space, there exists $u \in X$ such that

(2.9) \quad $\lim_{n \to \infty} x_n = u$.

We shall prove that $u \in \bigcap_{i=1}^k A_i$. By Definition, we have $x_0 \in A_1$ and $\{x_{nk}\} \subseteq A_1$. Since $A_1$ is closed, we get that $u \in A_1$. Similarly, we have $\{x_{nk+1}\} \subseteq A_2$ and $u \in A_2$. By mathematical induction, we get that $u \in \bigcap_{i=1}^k A_i$.

By (2.4), we obtain that $\lim_{n,m \to \infty} p(x_n, x_m) = 0$. Then for any $\varepsilon > 0$, there exists a $n > N_\varepsilon \in \mathbb{N}$ such that $p(x_{N_\varepsilon}, x_n) < \frac{\varepsilon}{s}$.

By (2.9) and $p(x, \cdot)$ is $s$-lower semi-continuous, thus we have

$$
p(x_{N_\varepsilon}, u) \leq \liminf_{n \to \infty} sp(x_{N_\varepsilon}, x_n) \leq \varepsilon
$$

Let $\varepsilon = \frac{1}{t}$ and $N_\varepsilon = n_t \ (t \in \mathbb{N})$, then we have

(2.10) \quad $\lim_{t \to \infty} p(x_n, u) = 0$. 

Next, we shall prove that the \( u \) is a fixed point of \( f \).

Case (1), suppose that \( fu \neq u \), then by (2.8) and (2.10), we have

\[
0 < \inf \{ p(x, u) + p(x, fx) : x \in X \} \leq \inf \{ p(x_n, u) + p(x_n, x_{n+1}) : n \in N \} \to 0 \ (n \to \infty)
\]

which is a contradiction, thus \( fu = u \).

Case (2), suppose that there exists a \( w \in X \) with \( fw \neq w \) such that \( \inf \{ p(x, w) + p(x, fx) : x \in X \} = 0 \), then there exists a sequence \( \{ y_n \} \subset X \) such that \( p(y_n, w) + p(y_n, fy_n) \to 0 \) as \( n \to \infty \), thus we have

\[
(2.11) \quad \lim_{n \to \infty} p(y_n, w) = \lim_{n \to \infty} p(y_n, fy_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(fy_n, w) = 0 \ (\text{by Lemma 1.3}).
\]

Since

\[
M_s(y_n, fy_n) = \max \{ \varphi(p(y_n, fy_n)), \varphi(p(y_n, fy_n)), \varphi(\lambda_1 p(y_n, fy_n)) + (1 - \lambda_1) p(fy_n, f^2y_n)), \varphi(\frac{\lambda_2 p(y_n, f^2y_n)}{s}) \}
\]

\[
\leq \frac{1}{as} \max \{ p(y_n, fy_n), \lambda_1 p(y_n, fy_n) + (1 - \lambda_1) p(fy_n, f^2y_n), \frac{\lambda_2 p(y_n, f^2y_n)}{s} \}
\]

\[
\leq \frac{1}{as} \max \{ p(y_n, fy_n), \lambda_1 p(y_n, fy_n) + (1 - \lambda_1) p(fy_n, f^2y_n), \lambda_2 (p(y_n, fy_n) + p(fy_n, f^2y_n))) \}
\]

\[
\to \max \{ \frac{1 - \lambda_1}{as}, \frac{\lambda_2}{as} \} \lim_{n \to \infty} p(fy_n, f^2y_n),
\]

and

\[
p(fy_n, f^2y_n) \leq M_s(y_n, fy_n) + L \min \{ \varphi(p(y_n, fy_n)), \varphi(p(fy_n, f^2y_n)), \varphi(p(y_n, f^2y_n)), \varphi(p(fy_n, fy_n)) \}
\]

\[
= M_s(y_n, fy_n) \to \max \{ \frac{1 - \lambda_1}{as}, \frac{\lambda_2}{as} \} \lim_{n \to \infty} p(fy_n, f^2y_n),
\]

which is contradistive with \( \max \{ \frac{1 - \lambda_1}{as}, \frac{\lambda_2}{as} \} \in (0, 1) \). Thus we have

\[
(2.12) \quad \lim_{n \to \infty} p(fy_n, f^2y_n) = 0.
\]

By (2.11) and (2.13), we have

\[
(2.13) \quad \lim_{n \to \infty} p(fy_n, f^2y_n) = 0.
\]

By (2.11) and (2.13), we have

\[
(2.14) \quad p(y_n, f^2y_n) \leq s(p(y_n, fy_n) + p(fy_n, f^2y_n)) \to 0 \ (n \to \infty).
\]
Thus by (2.11), (2.14) and Lemma 1.3, we obtain that \( \lim_{n \to \infty} d(f^n x_n, w) = 0 \).

By the continuity of \( f \), we have \( f w = f(\lim_{n \to \infty} f x_n) = \lim_{n \to \infty} f^2 x_n = w \), which is a contradiction with the hypothesis. So case (1) always holds, and by case (1), \( u = fu \).

Finally, we shall prove the uniqueness of fixed point \( u \) of \( f \).

Assume that there exists \( v \in X \) such that \( f v = v \) with \( v \neq u \), then we have

\[
p(u, v) = p(fu, fv)
\]

\[
\leq M_s(u, v) + L \min \{ \phi(p(u, fu)), \phi(p(v, fv)), \phi(p(u, fv)), \phi(p(v, fu)) \}
\]

\[
= M_s(u, v) + L \min \{ \phi(p(u, u)), \phi(p(v, v)), \phi(p(u, v)), \phi(p(v, u)) \}
\]

\[
= M_s(u, v),
\]

where

\[
M_s(u, v) = \max \{ \phi(p(u, v)), \phi(p(u, fu)), \phi(\lambda_1 p(u, fu))
\]

\[
+ (1 - \lambda_1) p(v, fu), \phi(\frac{\lambda_2 p(u, fv) + (1 - \lambda_2) p(fu, v)}{s}) \}
\]

\[
= \max \{ \phi(p(u, v)), \phi(p(u, u)), \phi(\lambda_1 p(u, u))
\]

\[
+ (1 - \lambda_1) p(v, v), \phi(\frac{\lambda_2 p(u, v) + (1 - \lambda_2) p(u, v)}{s}) \}
\]

\[
= \max \{ \phi(p(u, v)), \phi(\frac{p(u, v)}{s}) \}
\]

\[
\leq \frac{1}{as} p(u, v)
\]

Then, we get that \( p(u, v) \leq \frac{1}{as} p(u, v) \) (as \( > 1 \)), a contradiction. Thus we have \( p(u, v) = 0 \).

Similarly, we get that \( p(u, u) = 0 \), and by Lemma 1.3, we have \( u = v \). \( \square \)

We can get a more comfortable theorem if \( wt \)-distance \( p \) is symmetric.

**Theorem 2.6.** Let \( (X, d) \) be a complete \( b \)-metric space with constant \( s \geq 1 \) and \( p \) be a symmetric \( wt \)-distance on \( X \), \( p(x, x) = 0 \) for \( x \in X \), \( V = \bigcup_{i=1}^k A_i \) and \( A_1, A_2, \cdots, A_k \) be nonempty closed subsets of \( X \), \( k \) be a positive integer, \( f : V \to V \) is a cyclic generalized \( \phi \)-contraction mapping for some \( \phi \in \Phi \), then \( f \) has a unique fixed point. Moreover, the fixed point of \( f \) belongs to \( \bigcap_{i=1}^k A_i \).

**Proof.** By comparing Theorem 2.6 with Theorem 2.5, we find that we can omit the condition "case (1) and case (2)" by the condition that \( wt \)-distance \( p \) is symmetric. By observing the proof
of Theorem 2.5, we find that the condition "case (1) and case (2)" is only used to prove the existence of fixed point $u$. So we continue using the similar notations in Theorem 2.5 to prove the existence of fixed point $u$ by the condition that wt-distance $p$ is symmetric.

Next, we shall prove that the $u$ is a fixed point of $f$.

Since Cauchy sequence $\{x_n\} \subset X$ with $x_{n+1} = fx_n$ converges to $u \in X$. And by the symmetry of wt-distance $p$ and (2.10), we have

\[
(*) \quad \lim_{n \to \infty} p(u, x_n) = 0.
\]

Then by (*), (2.8) and (2.10), we have

\[
p(u, fu) \leq s(p(u, fx_n) + p(fx_n, fu)) \leq sp(u, fx_n) + sM_s(x_n, u) + \]

\[
sL\min\{\phi(p(x_n, fx_n)), \phi(p(u, fu)), \phi(p(x_n, fu)), \phi(p(u, fx_n))\}\]

\[
= sp(u, x_{n+1}) + s\max\{\phi(p(x_n, u)), \phi(p(x_n, x_{n+1})), \phi(\lambda_1 p(x_n, x_{n+1}) + (1 - \lambda_1)p(u, fu)), \phi(\lambda_2 p(x_n, fu) + (1 - \lambda_2)p(x_{n+1}, u))\} + \]

\[
sL\min\{\phi(p(x_n, x_{n+1})), \phi(p(u, fu)), \phi(p(x_n, fu)), \phi(p(u, x_{n+1}))\}\]

\[
\leq sp(u, x_{n+1}) + \frac{1}{a}\max\{p(x_n, u), p(x_n, x_{n+1}), \lambda_1 p(x_n, x_{n+1}) + (1 - \lambda_1)p(u, fu), \lambda_2 p(x_n, fu) + (1 - \lambda_2)p(x_{n+1}, u)\} + \]

\[
\frac{1}{a}\min\{p(x_n, x_{n+1}), p(u, fu), p(x_n, fu), p(u, x_{n+1})\}\]

\[
= \frac{1}{a}\max\{(1 - \lambda_1)p(u, fu), \lim_{n \to \infty} \frac{\lambda_2 p(x_n, fu)}{s}\} (n \to \infty)\]

\[
\leq \frac{1}{a}\max\{(1 - \lambda_1)p(u, fu), \lim_{n \to \infty} \lambda_2 [p(x_n, u) + p(u, fu)]\} \]

\[
= \max\{\frac{1 - \lambda_1}{a}, \frac{\lambda_2}{a}\} p(u, fu)
\]

which is contradicitive with $\max\{\frac{1 - \lambda_1}{a}, \frac{\lambda_2}{a}\} \in (0, 1)$. So $p(u, fu) = 0$. Similarly, we obtain that $p(u, u) = 0$. By Lemma 1.3 again, we have that $u = fu$. \qed

Since $b$-metric $d$ is also a wt-distance on $(X, d)$, then we obtain the following corollary.

Corollary 2.7. Let $(X, d)$ be a complete $b$-metric space with constant $s \geq 1$ and $V = \bigcup_{i=1}^k A_i$ and $A_1, A_2, \cdots, A_k$ be nonempty closed subsets of $X$, $k$ be a positive integer, $f : V \to V$ is a
cyclic generalized \( \varphi \)-contraction mapping for some \( \varphi \in \Phi \) (where let \( p = d \) in (2.5)), then \( f \) has a unique fixed point. Moreover, the fixed point of \( f \) belongs to \( \bigcap_{i=1}^{k} A_i \).

If let \( \lambda_1 = \lambda_2 = \frac{1}{2} \) in Corollary 2.7, we obtain the Theorem 2.2 by given by H.K. Nashine and Z. Kadelburg [8].

**Corollary 2.8.** [8] Let \( (X, d) \) be a complete \( b \)-metric space with constant \( s \geq 1 \) and \( V = \bigcup_{i=1}^{k} A_i \) and \( A_1, A_2, \ldots, A_k \) be nonempty closed subsets of \( X \), \( k \) be a positive integer, \( f : V \to V \) is a cyclic generalized \( \varphi \)-contraction mapping for some \( \varphi \in \Phi \) (where let \( p = d \) and \( \lambda_1 = \lambda_2 = \frac{1}{2} \) in (2.5)), then \( f \) has a unique fixed point. Moreover, the fixed point of \( f \) belongs to \( \bigcap_{i=1}^{k} A_i \).

**Definition 2.9.** Let \( (X, d) \) be a complete \( b \)-metric space with constant \( s \geq 1 \), \( p \) be a \( wt \)-distance on \( X \) and \( p(x, x) = 0 \) for any \( x \in X \). \( V = \bigcup_{i=1}^{k} A_i \) and \( A_1, A_2, \ldots, A_k \) be nonempty closed subsets of \( X \), \( k \) be a positive integer. If there exists \( f : V \to V \) with \( fA_i = A_{i+1} \) and \( A_{k+1} = A_1 \) such that

\[
(2.15) \quad \psi(s^{\alpha}p(fx, fy)) \leq \psi \left( \frac{p(x, fy) + p(fx, y)}{s^\varepsilon} \right) - \varphi(p(fx, y)), \quad \forall x, y \in V,
\]

where \( s^{\alpha+\varepsilon-1} > 2 \), \( \psi : [0, \infty) \to [0, \infty) \) is nondecreasing and \( \varphi : [0, \infty) \to [0, \infty) \) is a continuous function such that \( \varphi(x) = 0 \) implies \( x = 0 \), then \( f \) is called the \( (\psi, \varphi) \)-weakly contractive.

**Theorem 2.10.** Let \( (X, d) \) be a complete \( b \)-metric space with constant \( s \geq 1 \) and \( p \) be a \( wt \)-distance on \( X \), \( p(x, x) = 0 \) for any \( x \in X \). \( V = \bigcup_{i=1}^{k} A_i \) and \( A_1, A_2, \ldots, A_k \) be nonempty closed subsets of \( X \), \( k \) be a positive integer. If \( f : V \to V \) is \( (\psi, \varphi) \)-weakly contractive, and suppose that either

(1) \( \inf \{ p(x, w) + p(x, fx) : x \in X \} > 0 \) for every \( w \in X \) with \( w \neq fw \);

or

(2) the mapping \( f \) is continuous.

Then \( f \) has a unique fixed point. Moreover, the fixed point of \( f \) belongs to \( \bigcap_{i=1}^{k} A_i \).

**Proof.** For any \( x_0 \in A_1 \) (such a point exists since \( A_1 \neq \emptyset \)), we can construct the sequence \( \{x_n\} \) in \( X \) by \( x_{n+1} = fx_n, n \in \mathbb{N} \cup \{0\} \). If \( x_n = x_{n+1} \) for some \( n \in \mathbb{N} \cup \{0\} \), then \( f \) has fixed point.

Now, suppose that \( x_n \neq x_{n+1} \) for any \( n \in \mathbb{N} \cup \{0\} \), we shall prove that \( \lim_{n \to \infty} p(x_n, x_{n+1}) = 0 \).

Indeed, if not, we have that \( p(x_n, x_{n+1}) > 0 \) for all \( n \in \mathbb{N} \), there exists \( i = i(n) \in \{1, 2, \ldots, k\} \), such that \( (x_n, x_{n+1}) \in A_i \times A_{i+1} \), then we have
\[
\psi(s^\alpha p(f x_{n-1}, f x_n)) \leq \psi\left(p(x_{n-1}, f x_n) + p(f x_{n-1}, x_n)\right) - \varphi(p(f x_{n-1}, x_n)) \\
= \psi\left(p(x_{n-1}, x_{n+1})\right) - \varphi(0) \\
\leq \psi\left(p(x_{n-1}, x_n) + p(x_n, x_{n+1})\right) \\
\]

since \(\psi\) is nondecreasing, we have

\[
s^\alpha p(f x_{n-1}, f x_n) = s^\alpha p(x_n, x_{n+1}) \leq \frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1})}{s^{e-1}}
\]
i.e.,

\[
p(x_n, x_{n+1}) \leq \dot{c} p(x_{n-1}, x_n)
\]

where \(\dot{c} = \frac{1}{s^{\alpha+\epsilon-1}} < 1\), then we have

\[(2.16) \quad \lim_{n \to \infty} p(x_n, x_{n+1}) = 0\]

by Lemma 2.2 we have that \(\{x_n\}\) is a Cauchy sequence.

Since \(X\) is a complete space, there exists \(u \in X\) such that

\[(2.17) \quad \lim_{n \to \infty} x_n = u.\]

We shall prove that \(u \in \bigcap_{i=1}^{k} A_i\). By Definition, we have \(x_0 \in A_1\) and \(\{x_{nk}\} \subseteq A_1\). Since \(A_1\) is closed, we get that \(u \in A_1\). Similarly, we have \(\{x_{nk+1}\} \subseteq A_2\) and \(u \in A_2\). By mathematical induction, we get that \(u \in \bigcap_{i=1}^{k} A_i\).

By (2.4), we obtain that \(\lim_{n,m \to \infty} p(x_n, x_m) = 0\). Then for any \(\epsilon > 0\), there exists a \(n > N_\epsilon \in \mathbb{N}\) such that \(p(x_{N_\epsilon}, x_n) < \frac{\epsilon}{s}\).

By (2.17) and \(p(x, \cdot)\) is \(s\)-lower semi-continuous, thus we have

\[
p(x_{N_\epsilon}, u) \leq \liminf_{n \to \infty} sp(x_{N_\epsilon}, x_n) \leq \epsilon
\]

Let \(\epsilon = \frac{1}{t}\) and \(N_\epsilon = n_t\), then we have

\[(2.18) \quad \lim_{t \to \infty} p(x_n, u) = 0.\]

Next, we shall prove that the \(u\) is a fixed point of \(f\).
Case (1), suppose that $fu \neq u$, then by (2.16) and (2.18), we have

$$0 < \inf \{p(x,u) + p(x,fx) : x \in X\} \leq \inf \{p(x_n,u) + p(x_n,x_{n+1}) : n \in N\} \to 0 \ (n \to \infty)$$

which is a contradiction, thus $fu = u$.

Case (2), suppose that there exists a $w \in X$ with $fw \neq w$ such that $\inf \{p(x,w) + p(x,fx) : x \in X\} = 0$, then there exists a sequence $\{y_n\} \subset X$ such that $p(y_n,w) + p(y_n,fy_n) \to 0$ as $n \to \infty$, thus we have

$$\lim_{n \to \infty} p(y_n,w) = 0 \text{ and } \lim_{n \to \infty} p(y_n,fy_n) = 0. \ (2.19)$$

Then by Lemma 1.3 (2), $fy_n \to w$ as $n \to \infty$.

Since

$$\psi(s^\alpha p(fy_n,f^2y_n)) \leq \psi\left(\frac{p(y_n,f^2y_n) + p(fy_n,fy_n)}{s^\epsilon}\right) - \varphi(p(fy_n,fy_n))$$

$$= \psi\left(\frac{p(y_n,f^2y_n)}{s^\epsilon}\right)$$

and by the condition that $\psi$ is nondecreasing, then we have

$$s^\alpha p(fy_n,f^2y_n) \leq \frac{p(y_n,f^2y_n)}{s^\epsilon} \leq \frac{p(y_n,fy_n) + p(fy_n,f^2y_n)}{s^\epsilon-1},$$

i.e.,

$$p(fy_n,f^2y_n) \leq \frac{1}{(s^\alpha + \epsilon - 1)} p(y_n,fy_n) \to 0 \ (n \to \infty) \ (by \ (2.19)) \ (2.20)$$

and by (2.19) and (2.20), we have

$$p(y_n,f^2y_n) \leq s(p(y_n,fy_n) + p(fy_n,f^2y_n)) \to 0 \ (n \to \infty). \ (2.21)$$

Thus by (2.19), (2.21) and Lemma 1.3 (2), we obtain that $\lim_{n \to \infty} f^2y_n = w$.

By the continuity of $f$, we have $fw = f(\lim_{n \to \infty} fy_n) = \lim_{n \to \infty} f^2y_n = w$, which is a contradiction with the hypothesis. So case (1) always holds, and $u = fu$.

Finally, we shall prove the uniqueness of fixed point $u$ of $f$. 

\[\text{\textit{Fixed Point Theorems in } b\text{-Metric Spaces}}\]
Assume that there exists \( v \in X \) such that \( f v = v \) with \( v \neq u \), then we have

\[
\psi(s^\alpha p(u,v)) = \psi(s^\alpha p(fu,fv)) \leq \psi\left(\frac{p(u,fv) + p(fu,v)}{s^\epsilon}\right) - \varphi(p(fu,v)) = \psi\left(\frac{p(u,v) + p(u,v)}{s^\epsilon}\right) - \varphi(p(u,v)) \leq \psi\left(\frac{2p(u,v)}{s^\epsilon}\right)
\]

then we have

\[
\frac{s^{\alpha+\epsilon}}{2} p(u,v) \leq p(u,v)
\]

thus we get that

\[
p(u,v) = 0,
\]

where \( s^{\alpha+\epsilon} > 2s \geq 2 \).

Similarly, we get that \( p(u,u) = 0 \), and by Lemma 1.3, we have \( u = v \). \( \Box \)

We can get a more comfortable theorem if \( wt \)-distance \( p \) is symmetric.

**Theorem 2.11.** Let \((X,d)\) be a complete \( b \)-metric space with constant \( s \geq 1 \) and \( p \) be a symmetric \( wt \)-distance on \( X \), \( p(x,x) = 0 \) for any \( x \in X \). \( V = \bigcup_{i=1}^k A_i \) and \( A_1, A_2, \ldots, A_k \) be nonempty closed subsets of \( X \), \( k \) be a positive integer. If \( f : V \to V \) is the \((\psi, \varphi)\)-weakly contractive, then \( f \) has a unique fixed point. Moreover, the fixed point of \( f \) belongs to \( \bigcap_{i=1}^k A_i \).

**Proof.** By comparing Theorem 2.11 with Theorem 2.10, we find that we can omit the condition ”case (1) and case (2)” by the condition that \( wt \)-distance \( p \) is symmetric. By observing the proof of Theorem 2.10, we find that the condition ”case (1) and case (2)” is only used to prove the existence of fixed point \( u \). So we continue using the similar notations in Theorem 2.10 to prove the existence of fixed point \( u \) by the condition that \( wt \)-distance \( p \) is symmetric.

Next, we shall prove that the \( u \) is a fixed point of \( f \).

Since Cauchy sequence \( \{x_n\} \subset X \) with \( x_{n+1} = fx_n \) converges to \( u \in X \). And by the symmetry of \( wt \)-distance \( p \) and (2.18), we have

\[
\lim_{n \to \infty} p(u,x_n) = 0.
\]
By (2.15), we have

$$
\psi(s^\alpha p(x_n, fu)) = \psi(s^\alpha p(fx_{n-1}, fu)) \\
\leq \psi\left(\frac{p(x_{n-1}, fu) + p(fx_{n-1}, u)}{se}\right) - \varphi(p(fx_{n-1}, u)) \\
= \psi\left(\frac{p(x_{n-1}, fu) + p(x_n, u)}{se}\right) - \varphi(p(x_n, u)) \\
\leq \psi\left(\frac{p(x_{n-1}, fu) + p(x_n, u)}{se}\right),
$$

and by the condition that $\psi$ is nondecreasing, then we have

$$
s^\alpha p(x_n, fu) \leq \frac{p(x_{n-1}, fu) + p(x_n, u)}{se} \\
\leq \frac{p(x_{n-1}, x_n) + p(x_n, fu)}{se} + \frac{p(x_n, u)}{se},
$$

i.e.,

$$
p(x_n, fu) \leq \frac{s[p(x_{n-1}, x_n) + p(x_n, u)]}{s^\alpha + s - s} \\
\rightarrow 0 \ (n \rightarrow \infty) \ (\text{by } (2.16), (2.18))
$$

(2.23)

where $s^\alpha + s - s > s \geq 1$.

By (2.22) and (2.23), we have

$$
p(u, fu) \leq s[p(u, x_n) + p(x_n, fu)] \rightarrow 0 \ (n \rightarrow \infty).
$$

Since $p(u, u) = 0$ and by Lemma 1.3 again, we have that $u = fu$. □

Similarly, let $p = d$ in Theorem 2.11, we have the following corollary.

**Corollary 2.12.** Let $(X, d)$ be a complete $b$-metric space with constant $s \geq 1$, $V = \bigcup_{i=1}^{k} A_i$ and $A_1, A_2, \ldots, A_k$ be nonempty closed subsets of $X$, $k$ be a positive integer. If $f : V \rightarrow V$ is the $(\psi, \varphi)$—weakly contractive, then $f$ has a unique fixed point. Moreover, the fixed point of $f$ belongs to $\bigcap_{i=1}^{k} A_i$.

**Conflict of Interests**

The author(s) declare that there is no conflict of interests.
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