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NONLINEAR GENERALIZED φ-CONTRACTIONS IN COMPLETE G-METRIC SPACES AND APPLICATIONS

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Abstract. We will first introduce the fixed point property and a new class of operators and contraction mapping for classes of φ-contractions in $E-G$-metric spaces. Also, we will realize the study of the fixed point theory for (local and global) nonlinear contractions with an o-comparison function in $E-G$-metric spaces. We also introduce G-open ball, G-closed ball and $X(x_0,r)$ and obtain some fixed point theorems. Furthermore we give an application to a Fredholm-Volterra type differential equation where one of the integral operators satisfies φ-contraction condition.

Keywords: φ-contraction; G-open ball; G-closed ball; fixed point.

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1. INTRODUCTION

Fixed point theory has a variety of interesting applications in disciplines such as physics, chemistry, and engineering. In physics and engineering fixed point technique has been used in areas like image retrieval, signal processing and the stu dy of existence and uniqueness of solutions for a class of nonlinear integral equations. Some recent work on fixed point theorems of integral type in G-metric spaces, the stability of the functional difference equation can be

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found in [12, 13, 14] and the references therein. One of the most spaces used in many branches of Mathematical Analysis and its applications are Riesz spaces, see, for instance, [5], [1] [4].

On the other hand, in 2006, Mustafa and Sims [9, 10] introduced the notion of generalized metric spaces or simply G-metric spaces. Recently, Mohsenialhosseini in [7, 8] introduced the approximate fixed points of operators on G-metric spaces. The aim of this paper is to introduce vector G-metric spaces. Also, we introduce the new classes of operators and contraction maps regarding fixed point on a Riesz space. Furthermore, we give some illustrative example of our main results.

2. Preliminaries

This section recalls the following notations and the ones that will be used in what follows.

Definition 2.1. [9] Let X be a nonempty set and let $G : X \times X \times X \longrightarrow R^{+}$ be a function satisfying the following properties:
(G1) $G(x,y,z) = 0 \; if \; and \; only \; if \; x = y = z$;
(G2) $0 < G(x,x,y) for \; all \; x,y \in X \; with \; x \neq y$;
(G3) $G(x,x,y) \leq G(x,y,z) for \; all \; x,y,z \in X with \; z \neq y$;
(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots (symmetry \; in \; all \; three \; variables)$;
(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z) for \; all \; x,y,z,a \in X(\; rectangle \; inequality)$.
Then, the function G is called generalized metric or, more specifically $G-metric$ on X, and the pair (X,G) is called a $G-metric$ space.

Proposition 2.2. [9] *Every G-metric (X,G) defines a metric space (X,d_G) by*
1) $d_G(x,y) = G(x,y,y) + G(y,x,x)$.
if (X,G) is a symmetric G- metric space. Then
2) $d_G(x,y) = 2G(x,y,y)$.

Now, we recall the following notations and notions that will be used in what follows (see[2] [3]). A set E equipped with a partial order $\leq$ is called a partially ordered set. In a partially ordered set $(E,\leq)$ the notation $x < y$ means $x \leq y$ and $x \neq y$. A partially ordered set $(E,\leq)$ is a lattice if each pair of elements $x,y \in E$ has a supremum and an infimum. A real linear space

E with an order relation $\leq$ on E which is compatible with the algebraic structure of E, in the sense that satisfies properties:

(1) $x \leq y$ implies $x+z \leq y+z$, for each $z \in E, x, y \in E$;

(2) $x \leq y$ implies $tx \leq ty$, for each $t > 0, x, y \in E$,

is called an ordered linear space. In an ordered vector space E, the set $\{x \in E : x \geq 0\}$ is a pointed convex cone, called the positive cone of E, denoted E_+. An ordered linear space E for which $(E, \leq)$ is a lattice is called a Riesz space or linear lattice. A Riesz space E is Archimedean if $\frac{1}{n}x \downarrow 0$ holds for every $x \in E_+$. The notation $x_n \downarrow x$ means that $x_n \downarrow$ and $\inf\{x_n\} = x$. A Riesz space E is order complete or Dedekind complete if every nonempty subset of E which is bounded from above has a supremum. (Equivalently, every nonempty the subset of E which is bounded from below has an infimum).

Definition 2.3. [6] Let E, F be two Riesz spaces:

1. A sequence $\{b_n\}$ in E is called order convergent to b (we write $b_n \xrightarrow{\circ} b$), if there exists a sequence $\{a_n\}$ in E satisfying $a_n \downarrow 0$ and $|b_n - b| \leq a_n$, for each n.

2. Let $f : E \to F$. The function f is order continuous if $b_n \xrightarrow{\circ} b$ in E implies $f(b_n) \xrightarrow{\circ} f(b)$ in F.

3. A sequence $\{b_n\}$ in E is called order Cauchy, if there exists a sequence $\{a_n\}$ in E such that $an \downarrow 0$ and $|b_n - b_{n+p}| \leq a_n$, for all n and p.

4. A Riesz space E is called order complete if every order Cauchy sequence is order convergent.

Definition 2.4. [6] Let X, Y be two E-metric spaces:

(1) A sequence $\{x_n\}$ in X E-converges to some $x \in E$, written $x_n \xrightarrow{d,E} x$, if there is a sequence $\{a_n\}$ in E such that $a_n \downarrow 0$ and $d(x_n, x) \leq a_n$, for each n.

(2) Let $f : X \to Y$. The function f is E-continuous if $x_n \xrightarrow{d,E} x$ in X implies $f(x_n) \xrightarrow{d,E} f(x)$ in Y.

(3) A sequence $\{x_n\}$ in X is called to be E-Cauchy, if there is a sequence $\{a_n\}$ in E such that $a_n \downarrow 0$ and $d(x_n, x_{n+p}) \leq a_n$, for all n, p.

(4) An E-metric space X is called E-complete if each E-Cauchy sequence in X E-converges to a limit in X.

(5) Let (X, d, E) be an E-metric space. We say that a subset $Y \subset X$ is E-closed if $\{x_n\} \subset Y$ and $x_n \xrightarrow{d,E} x$ implies $x \in Y$.

Definition 2.5. [6] Let (X,d,E) be an E- metric space and $\varphi : E_{+} \to E_{+}$ be an increasing operator such that $\varphi(t) < t$ and $\varphi^{n}(t) \xrightarrow{o} 0$, for any $t > 0$. We say that the operator $T : X \to X$ is a nonlinear φ-contraction, if and only if

$$d(Tx,Ty) \leq \varphi[d(x,y)],\ for\ any \quad x,y \in X.$$

Definition 2.6. [6] Let X be a nonempty set and E be a Riesz space. The function $d : X \times X \longrightarrow E$ is said to be a vector metric or E-metric if it satisfies the following properties:

$$\begin{aligned} (a) \quad d(x,y) &= 0\ if\ and\ only\ if\ x = y; \\ (b) \quad d(x,y) &\leq d(x,z) + d(y,z)\ for\ all\ x,y,z \in X. \end{aligned}$$

Also, the triple (X,d,E) is said to be a vector metric space or an E-metric space.

3. Main result

In this section, we give some existence and uniqueness results for nonlinear $\varphi-$ contractions in $G-$-metric spaces, using the Riesz space E.

Definition 3.1. Let X be a nonempty set and E be a Riesz space. The function $G : X \times X \times X \longrightarrow E$ is said to be a vector G-metric or E-G-metric if it satisfies the following properties:
(a) $G(x,x,y) = 0\ if\ and\ only\ if\ x = y = z$;
(b) $G(x,x,y) + G(y,y,x) \leq G(x,x,z) + G(z,z,x) + G(y,y,z) + G(z,z,y)$,
for all $x,y,z \in X$. Also, the triple (X,G,E) is said to be a vector G-metric space or an E-G-metric space.

Definition 3.2. Let (X,G,E) be an E-G-metric space.

(1) A sequence $\{x_n\}$ in X E-G-converges to some $x \in E$, written $x_n \xrightarrow{G,E} x$, if there is a sequence $\{a_n\}$ in E such that $a_n \downarrow 0$ and $G(x_n,x_n,x) + G(x,x,x_n) \leq a_n$, for each n.
(2) A sequence $\{x_n\}$ in X is called to be E-G-Cauchy, if there is a sequence $\{a_n\}$ in E such that $a_n \downarrow 0$ and $G(x_n,x_n,x_{n+p}) + G(x_{n+p},x_{n+p},x_n) \leq a_n$, for all n,p.
(3) An E-G-metric space X is called E-G-complete if each E-G-Cauchy sequence in X E-G-converges to a limit in X.

Definition 3.3. Let (X,G,E) be an E-G-metric space and $\varphi : E_+ \to E_+$ be an increasing operator such that $\varphi(t) < t$ and $\varphi^n(t) \underset{\rightarrow}{\circ} 0$, for any $t > 0$. We say that the operator $T : X \to X$ is a nonlinear φ-contraction, if and only if

$$G(Tx,Tx,Ty)+G(Ty,Ty,Tx) \leq \varphi[G(x,x,y)+G(y,y,x)],\ for\ any \quad x,y \in X.$$

Definition 3.4. By definition, an operator $\varphi : E_+ \to E_+$ which satisfies the above properties is called an o-comparison function.

Example 3.5. Let $X = [-1,2]$ and $G : X \times X \times X \to \mathbb{R}^+$ be defind as follows:

$$G(x,y,z) = \mid x-y \mid + \mid y-z \mid + \mid z-x \mid .$$

Define the mapping Let $T : X \to X$ by $Tx = 1$ with ,

$$\varphi(t) = \begin{cases} \frac{t}{2k} & if\ t > \frac{1}{2} \\ 0 & if\ t \in (0, \frac{1}{2}] \end{cases}$$

for $k \geq t$ where $k \in \mathbf{N}$. it is easy to show φ is a $\varphi - contraction$ mapping.

Theorem 3.6. *Let X be a E-G-complete of a E-G-metric space. Suppose that the mapping $T : X \to X$ is a nonlinear φ-contraction. Then:*
i) there exists a unique fixed point $z \in X$ for T and for any $x \in X$, $T^n x \underset{\longrightarrow}{G,E} z$;
ii) $[G(z,z,T^n(x))+G(T^n(x),T^n(x),z)] \leq \varphi^n[G(z,x,x)+G(z,z,x)], for\ any\ n \in N.$

Proof. Let $x \in X$ be arbitrarily. Wa have

$$[G(T^n x, T^n x, T^{n+p}(x)) + G(T^n x, T^{n+p} x, T^{n+p}(x))] \leq$$

$$\varphi^n[G(x,x,T^p(x)) + G(x,T^p x,T^p(x))],$$

for any n, p. Given that $\varphi^n[G(x,x,T^p(x)) + G(x,T^p x,T^p(x))] \longrightarrow 0$ as $n \longrightarrow \infty$, we have $\exists \eta_n \in G$ such that $\eta_n \downarrow 0$ and $\varphi^n[G(x,x,T^p(x)) + G(x,T^p x,T^p(x))] \leq \eta_n$, for any n. Thus,

$$[G(T^n x, T^n x, T^{n+p}(x)) + G(T^n x, T^{n+p} x, T^{n+p}(x))] \leq \eta_n$$

for any n, p. Letting $n \longrightarrow 0$, we obtain that sequence $\{T^n(x)\}$ is $G - Cauchy$ in X. By the G-completeness of X, it follows that there exists $z \in X$ such that for any $x \in X$ $T^n x \underset{\longrightarrow}{G,E} z$, Thus, there exists a sequence ε_n in G such that $\varepsilon_n \downarrow 0$ and $G(T^n x, T^n x, z) + G(T^n x, x, x) \leq \varepsilon_n$, for any n. We have

$$\begin{aligned} G(z,z,Tz) + G(Tz,Tz,z) &\leq G(z,z,T^{n+1}(x) + G(T^{n+1}(x), T^{n+1}(x), z) \\ &+ G(T^{n+1}x, T^{n+1}x, Tz) + G(T^{n+1}x, Tz, Tz) \\ &+ \varepsilon_{n+1} + \varphi G(T^n x, T^n x, z) + G(T^n x, x, x) \\ &+ \varepsilon_{n+1} + \varphi(\varepsilon_n) \leq 2\varepsilon_n \downarrow 0, \end{aligned}$$

when $n \to \infty$. Thus, z is a fixed point of T in X. For the uniqueness, we suppose that $y \in X$ is another fixed point of T with $y \neq z$. Then

$$G(y,y,z) + G(y,z,z) = G(Ty,Ty,Tz) + G(Tz,Tz,Ty) \leq \varphi[G(z,z,y) + G(y,y,z)].$$

Thus, by definition 3.3 we have $G(y,y,z) + G(z,z,y) = 0$ and so, $z = y$.

Therefore for any n, we have

$$\begin{aligned} G(z,z,T^n(x)) + G(T^n(x), T^n(x), z) &= G(T^n(z), T^n(z), T^n(x)) \\ &+ G(T^n(z), T^n(x), T^n(x)) \\ &\leq \varphi^n[G(z,z,x) + G(x,x,z)]. \end{aligned}$$

□

Example 3.7. Let $X = [-1,2]$ and $G : X \times X \times X \to \mathrm{R}^+$ be defind as follows:

$$G(x,y,z) = \mid x - y \mid + \mid y - z \mid + \mid z - x \mid .$$

Let $T : X \to X$ and $Tx = 1$ with

$$\varphi(t) = \begin{cases} \frac{t}{2k} & if\ t > \frac{1}{2} \\ 0 & if\ t \in (0, \frac{1}{2}] \end{cases}$$

for $k \geq t$ where $k \in \mathbf{N}$.

By Example 3.5 φ is a $\varphi - contraction$ mapping. So T satisfies all the conditions of Theorem 3.6. Thus there exists a unique fixed point $z \in X$ for T and for any $x \in X$, $T^n x \underrightarrow{G,E} z$. ALso for every $n \in N$ inequality following is hold: $[G(z,z,T^n(x)) + G(T^n(x),T^n(x),z)] \leq \varphi^n[G(z,x,x) + G(z,z,x)]$.

Example 3.8. Let $I = [0,a], a > 0$ be an interval of the real axis. Suppose that $T \in C(I^2 \times B, B), g \in C(I,B)$ and we consider the Fredholm type integral equation

$$x(t) = \int_I T(t,s,x(s))ds + g(t), \tag{3.1}$$

in $C(I,B)$, i.e., in the space of all continuous functions defined on I, with values in a Banach space B, with the uniform convergence $\to$ and with the G-metric defined by

$$G(x,y,y) + G(y,x,x) = \|x(t) - y(t)\| \; for\ any\ x,y \in C(I,B),$$

where $\|.\| : B \to \mathbb{R}_+$ is the norm of B. In $C(I,\mathbb{R}_+)$ we choose the usual partial order and the usual operations (addition and multiplication) and for the convergence relation $\downarrow$ we consider the pointwise convergence of decreasing sequences in $C(I,\mathbb{R}_+)$. Then, in this case, we can easily observe that the Riesz space E is $C(I,\mathbb{R}_+)$ and the abstract space X is $C(I,B)$. Moreover, if we assume that there exists a continuous function $\nu \in (I^2,\mathbb{R}_+)$ with $sup_{t\in I} \int_I \nu(t,s)ds \leq 1$, such that

$$\|T(t,s,x) - T(t,s,y)\| \leq \nu(t,s)\varphi(\|x-y\|), \; for\ each\ t,s \in I, x,y \in B.$$

where $\varphi : E_+ \to E_+$ is an $\circ$-comparison operator. Then, the integral equation 3.1 has a unique solution in $C(I,B)$.

Using proposotion 2.2, Theorem 3.6, and example 2.5 of [6] we find that the integral equation 3.1 has a unique solution in $C(I,B)$.

Definition 3.9. Let (X,G,E) be an E-G-metric space. For given $x_0 \in X$ *and* $r \in E_+$ *with* $r > 0$, we define G-open ball $B_G(x_0,r)$ to be a subset of X given by $B_G(x_0,r) = \{y \in X \mid G(x_0,x_0,y) + G(y,y,x_0) < r\}$ and the G-closed ball in $\overline{B}_G(x_0,r)$ as

$\overline{B}_G(x_0,r) = \{y \in X \mid G(x_0,x_0,y) + G(y,y,x_0) \leq r\}$.

Theorem 3.10. *Let X be a E-G-complete of a G-metric space, $x_0 \in X, r \in E_+$, let $T : \overline{B}_G(x_0,r) \to X$ be an oprator and there exists an increasing operator $\varphi : [0,r] \to [0,r] \subset E_+$ such that $\varphi^n \overset{\circ}{\to} 0$ for any $t \in (0,r]$, with the property $G(Tx,Tx,Ty) + G(Ty,Ty,Tx) \leqslant$*

$\varphi[G(x,x,y)]+G(y,y,x)]$ and $G(x,x,y)+G(y,y,x)\leq r$, for any $x,y\in\overline{B}_G(x_0,r)$. We assume that $G(x_0,Tx_0,Tx_0)+G(x_0,x_0,Tx_0)\leq r-\varphi(r)$. Then:

i) T has a unique fixed point $z\in\overline{B}_G(x_0,r)$ and for any $x\in\overline{B}_G(x_0,r)$ we have $T^n x \underrightarrow{G,E} z$,
ii) $[G(z,z,T^n(x))+G(T^n(x),T^n(x),z)]\leq\varphi^n(r)$, for any n.

Proof. We will show that $T(\overline{B}_G(x_0,r))\subset\overline{B}_G(x_0,r)$. Let $x\in\overline{B}_G(x_0,r)$ and by the estimation

$$\begin{aligned}
G(x_0,x_0,Tx)+G(x_0,Tx,Tx) &\leq G(x_0,x_0,Tx_0)+G(x_0,Tx_0,Tx_0)\\
&+ G(Tx_0,x_0,Tx)+G(Tx_0,Tx,Tx)\\
&\leq r-\varphi(r)+\varphi[G(x_0,x_0,x)+G(x,x,x_0)]\\
&\leq r-\varphi(r)+\varphi(r)=r
\end{aligned}$$

we get that $Tx\in\overline{B}_G(x_0,r)$. Thus, $T:\overline{B}_G(x_0,r)\to\overline{B}_G(x_0,r)$ and since X is G-complete, by Theorem 3.5, we get that there exists a unique fixed point z for T in $\overline{B}_G(x_0,r)$ and for any $x\in\overline{B}_G(x_0,r)$, $T^n x \underrightarrow{G,E} z$. Therefore for any n, we have

$$\begin{aligned}
G(z,z,T^n(y_0))+G(T^n(y_0),T^n(y_0),z) &= G(T^n(z),T^n(z),T^n(y_0))\\
&+ G(T^n(z),T^n(y_0),T^n(y_0))\\
&\leq \varphi^n[G(z,z,x)+G(x,x,z)].
\end{aligned}$$

□

Now, other results with equivalent conclusions with Theorems 3.5 and 3.10 can be obtained in the following space:

Definition 3.11. Let $\overline{B}_G(x_0,r)=\{y\in X\mid G(x_0,x_0,y)+G(y,y,x_0)\leq r\}$ be a G-closed ball with center x_0 and radius r. For any G-closed ball, we can define, $X(x_0,r):=\cup_{\lambda\in E^+}\overline{B}_G(x_0,\lambda r)=\cup_{\lambda\in E^+}\{x\in X:G(x,x,x_0)+G(x_0,x_0,x)\leq\lambda r\}$.

Theorem 3.12. *Let (X,G,E) be a E-G-metric compplete space, $x_0\in X$. Suppose that the mapping $T:X(x_0,r)\to X$ be an operator and there exists an increasing operator $\varphi:E^+\to E^+$ such that $\varphi^n \underset{\to}{\circ} 0$ for $t>0$, with properties:*

i) $\varphi(\lambda r) \leq \varphi(\lambda)r$, *for* $\lambda \in E^+$,

ii) $G(Tx,Tx,Ty)+G(Ty,Ty,Tx) \leq \varphi[G(x,x,y)+G(y,y,x)]$ *and* $G(x,x,y)+G(y,y,x) \leq \lambda r$, *for any* $x,y \in X(x_0,r)$ *and for* $\lambda \in E^+$

iii) $G(x_0,x_0,Tx_0)+G(Tx_0,Tx_0,x_0) \leq \lambda_0 r$, *for* $\lambda_0 \in E^+$.

Then:

i) there exists a unique fixed point z for T in $X(x_0,r)$ *and for any* $x \in X(x_0,r)$,

$$T^n x \underset{\longrightarrow}{G,E} z,$$

ii) $G(z,z,T^n(x))+G(T^n(x),T^n(x),z) \leq \varphi^n[G(x,x,z)+G(z,z,x)]$, *for any n.*

Proof. We have to prove that $X(x_0,r)$ is invariant with respect to T, i.e., $T(X(x_0,r)) \subset X(x_0,r)$. Let $x \in X(x_0,r)$, then there exists $\lambda \in E^+$ such that $G(x,x,x_0)+G(x_0,x_0,x) \leq \lambda r$.

$$\begin{aligned}
[G(Tx,Tx,x_0)+G(x_0,x_0,Tx)] &\leq [G(Tx,Tx,Tx_0)+G(Tx_0,Tx_0,Tx)] \\
&+ [G(Tx_0,Tx_0,x_0)+G(x_0,x_0,Tx_0)] \\
&\leq \varphi[G(x,x,x_0)+G(x_0,x_0,x)]+\lambda_0 r \\
&\leq \varphi(\lambda r)+\lambda_0 r \\
&\leq [\varphi(\lambda)+\lambda_0]r
\end{aligned}$$

thus, there exists $\lambda' := [\varphi(\lambda)+\lambda_0] \in E^+$ such that $G(Tx,Tx,x_0)+G(x_0,x_0,Tx) \leq \lambda' r$, i.e, $Tx \in X(x_0,r)$.Then, the conclusion follows by Theorem 3.5. □

Lemma 3.13. *If* $\{y_n\} \subset X(x_0,r)$ *and* $y_n \underset{\longrightarrow}{G,E} y$, *then* $y \in X(x_0,r)$, *i.e.,* $X(x_0,r)$ *is E- closed in X with respect to the convergence* $\underset{\longrightarrow}{G,E}$.

Proof. Let $y_n \in X(x_0,r)$, then there exists $\lambda \in E^+$ such that

$$G(y_n,y_n,x_0)+G(x_0,x_0,y_n) \leq \lambda r$$

On the other hand since $y_n \underrightarrow{G,E} y$, there exists a sequence $\{\varepsilon_n\}$ in E such that $\varepsilon_n \downarrow 0$ and $G(y_n,y_n,y)+G(y,y,y_n) \leq \varepsilon_n r$, for any n. We have

$$\begin{aligned}[G(y,y,x_0)+G(x_0,x_0,y)] &\leq [G(y_n,y_n,y)+G(y,y,y_n)]\\ &+ [G(y_n,y_n,x_0)+G(x_0,x_0,y_n)]\\ &\leq \lambda r+\varepsilon_n r=(\lambda+\varepsilon_n)r\end{aligned}$$

we get $\lambda' := (\lambda+\varepsilon_n) \in E^+$ such that

$$G(y,y,x_0)+G(x_0,x_0,y) \leq \lambda' r,$$

Hence $y \in X(x_0,r)$. $\square$

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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