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WEAK AND STRONG CONVERGENCE THEOREMS FOR TWO FINITE FAMILIES OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, an implicit iterative process with mixed errors for two finite family of asymptotically nonexpansive mappings is considered. Weak and strong convergence theorems for common fixed points of two finite family of asymptotically nonexpansive mappings are established in a uniformly convex Banach space.

Keywords: asymptotically nonexpansive mappings; fixed point; implicit iterative process with errors.

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1. Introduction

Throughout this paper, we always assume that E is a real Banach space and K is a nonempty subset of E. we denote the fixed point of the mapping T by $F(T)$, $\rightharpoonup$ and $\rightarrow$ denote weak and strong convergence, respectively.

Recall that $T : K \to K$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in K. \tag{1.1}$$

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$T: K \to K$ is said to be uniformly L-lipschitz if there exists a positive constant L such that

$$\|T^n x - T^n y\| \leq L\|x - y\|, \quad \forall x, y \in K,\, n \geq 1. \tag{1.2}$$

$T: K \to K$ is said to be asymptotically nonexpansive if there exists a sequence $\{h_n\}$ with $h_n \in [1, +\infty)$ and $\lim_{n\to\infty} h_n = 1$ such that

$$\|T^n x - T^n y\| \leq h_n\|x - y\|, \quad \forall x, y \in D(T),\, n \geq 1. \tag{1.3}$$

It is obvious, every asymptotically nonexpansive mapping is L-Lipschitzian with the uniform Lipschitz constant $L = \sup h_n : n \geq 1$. Moreover, for a finit family $\{T_1, T_2, \cdots, T_N\}$ of asymptotically nonexpansive mapping, we have, for all $x, y \in E$

$$\|T_i^k x - T_i^k y\| \leq L\|x - y\|, \quad \forall i = 1, 2, \cdots, N, k \geq 1. \tag{1.4}$$

where $L = \max\{L = L_1, L_2, \cdots, L_N\}$, and L_i is the Lipschitz constant of the mapping T_i

In 2001, Xu and Ori [11], in the framework of Hilbert spaces, introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_1, T_2, \cdots, T_N\}$ with $\{\alpha_n\}$ a real sequence in $(0, 1)$ and an initial point x_0:

$$x_1 = \alpha_1 x_0 + (1 - \alpha_1)T_1 x_1,$$

$$x_2 = \alpha_2 x_1 + (1 - \alpha_2)T_2 x_2,$$

$$\cdots$$

$$x_N = \alpha_N x_{N-1} + (1 - \alpha_N)T_N x_N,$$

$$x_{N+1} = \alpha_{N+1} x_N + (1 - \alpha_{N+1})T_1 x_{N+1},$$

$$\cdots$$

which can written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_n x_n, \quad \forall n \geq 1, \tag{1.5}$$

where $T_n = T_{n(modN)}$ (here the mod N takes values in $\{1, 2, \cdots, N\}$).

They proved the weak convergence of the process (1.5) to a common fixed point in the framework of Hilbert spaces.

Subsequently, fixed point problems based on implicit iterative processes have been considered by many authors, see, for example, [1,5-7,12]. In 2003, Sun [9] has extended the process (1.5) for a finite family of asymptotically quasi-nonexpansive mappings, with $\{\alpha_n\}$ a real sequence in $(0,1)$, which is defined as follows:

$$
\begin{aligned}
&x_1 = \alpha_1 x_0 + (1-\alpha_1)T_1 x_1,\\
&x_2 = \alpha_2 x_1 + (1-\alpha_2)T_2 x_2,\\
&\cdots\\
&x_N = \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N,\\
&x_{N+1} = \alpha_{N+1} x_N + (1-\alpha_{N+1})T_1^2 x_{N+1},\\
&\cdots\\
&x_{2N+1} = \alpha_{2N+1} x_2 N + (1-\alpha_{2N+1})T_1^3 x_{2N+1},\\
&\cdots
\end{aligned}
$$

which can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_{i(n)}^{k(n)} x_n, \quad \forall n \geq 1, \tag{1.6}$$

where, for any $n \in N$ fixed, $k(n)-1$ denotes the quotient of the division of n by N and $i(n)$ the rest, i.e.

$$n = (k(n)-1)N + i(n), \quad i(n) \in \{1, 2, \cdots, N\}.$$

Strong convergence theorems for fixed points of the mappings are obtained; see [9] for more details.

In 2006, Chang et al. [1] established an implicit iterative process with errors for a finite family of asymptotically nonexpansive mappings as follows:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_{i(n)}^{k(n)} x_n + u_n, \quad \forall n \geq 1, \tag{1.7}$$

where $n = (k(n)-1)N + i(n)$, $i(n) \in 1, 2, \cdots, N$, $\{u_n\}$ is a bounded sequence. Weak and strong convergence theorems for fixed point of the mapping were established; see [1] for more details.

In 2010, Filomena Cianciaruso et al. [2] considered the following implicit iterative process for a finite family of asymptotically nonexpansive mappings:

$$\begin{aligned} x_n &= (1-\alpha_n-\gamma_n)x_{n-1}+\alpha_n T_{i(n)}^{k(n)} y_n+\gamma_n u_n, \\ y_n &= (1-\beta_n-\delta_n)x_n+\beta_n T_{i(n)}^{k(n)} x_n+\delta_n v_n, n \geq 1. \end{aligned} \tag{1.8}$$

where $n = (k(n)-1)N+i(n)$, $i(n) \in 1,2,\cdots,N$, $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\delta_n\}$ are four real sequences in $(0,1)$ satisfying $\alpha_n+\gamma_n \leq 1$ and $\beta_n+\delta_n \leq 1$ for all $n \geq 1$. $\{u_n\},\{v_n\}$ are two bounded sequence and x_0 is a given point. They proved the implicit iterative sequence $\{x_n\}$ defined by (1.8) converges to a common fixed point for a finite family of asymptotically nonexpansive mappings in uniformly convex Banach spaces; see [2] for more details.

In this paper, motivated by the above results, an implicit iterative process with mixed errors for two finite family of asymptotically nonexpansive mappings is introduced. Weak and strong convergence theorems are obtained. The results presented in this paper improve and extend some results in Chang et al.[1], Filomena Cianciaruso et al.[2], Sun [9] and some others.

Let $\{S_i\}_{i=1}^N$,$\{T_i\}_{i=1}^N : E \to E$ be $2N$ asymptotically nonexpansive mappings. Define the sequence $\{x_n\}$ as follows: $x_0 \in K$, and

$$\begin{aligned} x_n &= (1-\alpha_n-\gamma_n)x_{n-1}+\alpha_n T_{i(n)}^{k(n)} y_n+\gamma_n u_n, \\ y_n &= (1-\beta_n-\delta_n)x_n+\beta_n S_{i(n)}^{k(n)} x_n+\delta_n v_n, n \geq 1. \end{aligned} \tag{1.9}$$

where $n = (k(n)-1)N+i(n)$, $i(n) \in 1,2,\cdots,N$, $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\delta_n\}$ are four real sequences in $(0,1)$ satisfying $\alpha_n+\gamma_n \leq 1$ and $\beta_n+\delta_n \leq 1$ for all $n \geq 1$, $\{u_n\},\{v_n\}$ are two bounded sequences. The purpose of this paper is to approximate common fixed points of the two families of mappings based on the iterative process. Weak and strong convergence theorems are established.

2. Preliminaries

Recall that a space E is said to satisfy Opial's condition [5] if, for each sequence $\{x_n\}$ in E, the convergence $x_n \to x$ weakly implies that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in E\,(y \neq x).$$

Recall that a mapping $T : E \to E$ is semicompact if any sequence $\{x_n\}$ in E satisfying $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ has a convergent subsequence.

Recall that a mapping $T : E \to E$ is demiclosed at the origin if for each sequence $\{x_n\}$ in E, the convergence $x_n \to x_0$ weakly and $Tx_n \to 0$ strongly imply that $Tx_0 = 0$.

Lemma 2.1 ([3]). *Let K be a nonempty subset of a Banach space E, $\{S_i\}_{i=1}^N$,$\{T_i\}_{i=1}^N$: $E \to E$ be $2N$ asymptotically nonexpansive mappings, then there exists a sequence $\{h_n\} \subset [1, \infty)$ with $h_n \to 1$ such that, for all $x, y \in E$,*

$$\|T_i^n x - T_i^n y\| \le h_n \|x - y\|, \quad \|S_i^n x - S_i^n y\| \le h_n \|x - y\|, \forall n \ge 1.$$

Lemma 2.2 ([10]).*Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative sequences satisfying the following condition:*

$$a_{n+1} \le (1 + b_n)a_n + c_n, \quad \forall n \ge n_0,$$

where n_0 is some nonnegative integer, $\sum_{n=1}^\infty b_n < \infty$ and $\sum_{n=1}^\infty c_n < \infty$. Then the limit $\lim_{n\to\infty} a_n$ exists.

Lemma 2.3 ([8]). *Let E be a uniformly convex Banach space and $0 < p \le t_n \le q < 1$, for all $n \in N$.Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that*

$$\limsup_{n\to\infty} \|x_n\| \le r, \limsup_{n\to\infty} \|y_n\| \le r, \lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = r,$$

hold for some $r \ge 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$

Lemma 2.4 ([12]). *Let E be a uniformly convex Banach space, K a nonempty closed convex subset of E and $T : K \to K$ be an asymptotically nonexpansive mapping. Then the mapping $I - T$ is demiclosed at zero, i.e. for each sequence $\{x_n\}$ in K,if $\{x_n\}$convergence weakly to $q \in K$ and $(I - T)x_n$ convergence strongly to 0, then $(I - T)q = 0$.*

Lemma 2.5 ([4]). *Let X be a Banach space satisfying Opial's condition and let $\{x_n\}$ be a sequence in X. let $u, v \in X$ be such that $\lim_{n\to\infty} \|x_n - u\| = 0$ and $\lim_{n\to\infty} \|x_n - v\| = 0$ exist. if $\{x_{n_k}\}$ and $\{x_{n_l}\}$ are subsequences of $\{x_n\}$ which convergence weakly to u and v, respectively, then $u = v$.*

3. Main results

Lemma 3.1. *Let E be a Banach space and K be a nonempty closed and convex subset of E. Let $\{S_i\}_{i=1}^N, \{T_i\}_{i=1}^N : K \to K$ be $2N$ asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \cap F(S_i) \neq \emptyset$. Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequence in K. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be four real sequences in $[0,1]$ satisfying the following conditions:*

(a) $\alpha_n + \gamma_n \le 1$ *and* $\beta_n + \delta_n \le 1$ $\forall n \ge 1$;

(b) $\limsup_{n\to\infty} \alpha_n < 1$;

(c) $\sum_{n=1}^\infty \gamma_n < \infty, \sum_{n=1}^\infty \delta_n < \infty, \sum_{n=1}^\infty (h_n - 1) < \infty$;

Let $\{x_n\}$ be defined by (1.9). Then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$.

Proof. In view of Lemma 2.1, we have for any $x, y \in K$,

$$\|T_{i(n)}^{k(n)} x - T_{i(n)}^{k(n)} y\| \le h_n \|x - y\|, \|S_{i(n)}^{k(n)} x - S_{i(n)}^{k(n)} y\| \le h_n \|x - y\|.$$

Picking $p \in F$, we have

$$\begin{aligned} \|x_n - p\| &= \|(1 - \alpha_n - \gamma_n)(x_{n-1} - p) + \alpha_n (T_{i(n)}^{k(n)} y_n - p) + \gamma_n (u_n - p)\| \\ &\le (1 - \alpha_n - \gamma_n)\|x_{n-1} - p\| + \alpha_n \|T_{i(n)}^{k(n)} y_n - p\| + \gamma_n \|u_n - p\|. \end{aligned}$$

Setting $M_1 = \sup_{n\ge1} \|u_n - p\|$, we have

$$\|x_n - p\| \le (1 - \alpha_n - \gamma_n)\|x_{n-1} - p\| + \alpha_n h_n \|y_n - p\| + M_1 \gamma_n, \tag{3.1}$$

Notice that

$$\|y_n - p\| = \|(1 - \beta_n - \delta_n)(x_n - p) + \beta_n (S_{i(n)}^{k(n)} x_n - p) + \delta_n (v_n - p)\|.$$

Set $M_2 = \sup_{n\geq 1} \|v_n - p\|$, we have

$$\begin{aligned} \|y_n - p\| &\leq (1-\beta_n)\|x_n - p\| + \beta_n h_n \|x_n - p\| + M_2\delta_n \\ &= (1-\beta_n+\beta_n h_n)\|x_n - p\| + M_2\delta_n \\ &\leq h_n\|x_n - p\| + M_2\delta_n. \end{aligned} \tag{3.2}$$

It follows that

$$\begin{aligned} \|x_n - p\| &\leq (1-\alpha_n-\gamma_n)\|x_{n-1} - p\| + \alpha_n h_n^2 \|x_n - p\| + M_1\gamma_n + M_2\alpha_n h_n \delta_n \\ &\leq (1-\alpha_n)\|x_{n-1} - p\| + \alpha_n h_n^2 \|x_n - p\| + M_1\gamma_n + M_2\alpha_n h_n \delta_n. \end{aligned} \tag{3.3}$$

This implies that

$$\begin{aligned} \|x_n - p\| &\leq \frac{1-\alpha_n}{1-\alpha_n h_n^2}\|x_{n-1} - p\| + \frac{M_1\gamma_n + M_2\alpha_n h_n\delta_n}{1-\alpha_n h_n^2} \\ &= (1 + \frac{\alpha_n(h_n^2-1)}{1-\alpha_n h_n^2})\|x_{n-1} - p\| + \frac{M_1\gamma_n + M_2\alpha_n h_n\delta_n}{1-\alpha_n h_n^2}. \end{aligned} \tag{3.4}$$

By hypothesis(b), it follows that there exists $\lambda < 1$, such that $\alpha_n \leq \lambda$ for big n. It follows that

$$\frac{\alpha_n(h_n^2-1)}{1-\alpha_n h_n^2} \leq \frac{\lambda(h_n^2-1)}{1-\lambda h_n^2} = \frac{\lambda(h_n+1)}{1-\lambda h_n^2}h_n - 1.$$

From $\lim_{n\to\infty} h_n = 1$, it derives that $\lim_{n\to\infty} \frac{\lambda(h_n+1)}{1-\lambda h_n^2} = \frac{2\lambda}{1-\lambda}$. Then there exists a real constant $L_1 > 0$ such that

$$\frac{\lambda(h_n+1)}{1-\lambda h_n^2} \leq L_1, \forall n \geq 1.$$

It follows from the hypothesis that $\sum_{n\geq 1} \frac{\alpha_n(h_n^2-1)}{1-\alpha_n h_n^2}$ convergence. Similarly, we can prove that $\sum_{n\geq 1} \frac{M_1\gamma_n + M_2\alpha_n h_n\delta_n}{1-\alpha_n h_n^2}$ convergence. In view of Lemma 2.2 and (2.4), we find that $\lim_{n\to\infty} \|x_n - p\|$ exist for all $p \in F$.

Lemma 3.2. *Let E be a uniformly convex Banach space and K be a nonempty closed and convex subset of E. Let $\{S_i\}_{i=1}^N$,$\{T_i\}_{i=1}^N : K \to K$ be $2N$ asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \cap F(S_i) \neq \emptyset$. Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequences in E. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be four real sequences in $[0,1]$ satisfying the following conditions:*

(a) $\alpha_n + \gamma_n \leq 1$ *and* $\beta_n + \delta_n \leq 1$ $,\forall n \geq 1$*;*

(b) $\limsup_{n\to\infty} \alpha_n < 1$*;*

(c) $\sum_{n=1}^{\infty}\gamma_n<\infty, \sum_{n=1}^{\infty}\delta_n<\infty, \sum_{n=1}^{\infty}(h_n-1)<\infty$.

Let the sequences $\{x_n\}$ *and* $\{y_n\}$ *be defined by (1.9). Then*

$$\lim_{n\to\infty}\|x_n-S_{i(n)}^{k(n)}x_n\|=0,$$

$$\lim_{n\to\infty}\|x_n-T_{i(n)}^{k(n)}x_n\|=0.$$

Proof. For all $p\in F$, it follows from Lemma 3.1 that $\lim_{n\to\infty}\|x_n-p\|$ exists. Let $\lim_{n\to\infty}\|x_n-p\|=d$ for some $d\geq 0$. It follows from (3.2) that

$$\|y_n-p\|\leq h_n\|x_n-p\|+M_2\delta_n.$$

Taking $\limsup_{n\to\infty}$ in both sides, we obtain

$$\limsup_{n\to\infty}\|y_n-p\|\leq\limsup_{n\to\infty}\|x_n-p\|=\lim_{n\to\infty}\|x_n-p\|=d,$$

and

$$\limsup_{n\to\infty}\|T_{i(n)}^{k(n)}y_n-p\|\leq\limsup_{n\to\infty}h_n\|y_n-p\|=\lim_{n\to\infty}\|y_n-p\|=d. \tag{3.5}$$

Notice that

$$\|T_{i(n)}^{k(n)}y_n-p+\gamma_n(u_n-x_{n-1})\|\leq\|T_{i(n)}^{k(n)}y_n-p\|+\gamma_n\|u_n-x_{n-1}\|.$$

It follows that

$$\limsup_{n\to\infty}\|T_{i(n)}^{k(n)}y_n-p+\gamma_n(u_n-x_{n-1})\|\leq d,$$

and

$$\|x_{n-1}-p+\gamma_n(u_n-x_{n-1})\|\leq\|x_{n-1}-p\|+\gamma_n\|u_n-x_{n-1}\|.$$

These imply that

$$\limsup_{n\to\infty}\|x_{n-1}-p+\gamma_n(u_n-x_{n-1})\|\leq d. \tag{3.6}$$

On the other hand, we have

$$
\begin{aligned}
d &= \lim_{n\to\infty} \|x_n - p\| \\
&= \lim_{n\to\infty} \|(1-\alpha_n-\gamma_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)} y_n + \gamma_n u_n - p\| \\
&= \lim_{n\to\infty} \|\alpha_n T_{i(n)}^{k(n)} y_n + (1-\alpha_n)x_{n-1} - \gamma_n x_{n-1} + \gamma_n u_n - (1-\alpha_n)p - \alpha_n p\| \\
&= \lim_{n\to\infty} \|\alpha_n T_{i(n)}^{k(n)} y_n - \alpha_n p + \alpha_n\gamma_n u_n - \alpha_n\gamma_n x_{n-1} + (1-\alpha_n)x_{n-1} \\
&- (1-\alpha_n)p - \gamma_n x_{n-1} + \gamma_n u_n - \alpha_n\gamma_n u_n + \alpha_n\gamma_n x_{n-1}\| \\
&= \lim_{n\to\infty} \|\alpha_n(T_{i(n)}^{k(n)} y_n - p + \gamma_n(u_n - x_{n-1})) + (1-\alpha_n)(x_{n-1} - p + \gamma_n(u_n - x_{n-1}))\|.
\end{aligned} \tag{3.7}
$$

From (3.5),(3.6) and (3.7), we find from Lemma 2.3 that

$$\lim_{n\to\infty} \|T_{i(n)}^{k(n)} y_n - x_{n-1}\| = 0, \tag{3.8}$$

Notice that

$$
\begin{aligned}
\|x_n - T_{i(n)}^{k(n)} y_n\| &\le \|x_n - x_{n-1}\| + \|x_{n-1} - T_{i(n)}^{k(n)} y_n\| \\
&\le (1+\alpha_n)\|x_{n-1} - T_{i(n)}^{k(n)} y_n\| + \gamma_n\|u_n - x_{n-1}\|.
\end{aligned}
$$

It follows from (3.8) that

$$\lim_{n\to\infty} \|x_n - T_{i(n)}^{k(n)} y_n\| = 0. \tag{3.9}$$

Notice that

$$\|x_n - p\| \le \|x_n - T_{i(n)}^{k(n)} y_n\| + \|T_{i(n)}^{k(n)} y_n - p\| \le \|x_n - T_{i(n)}^{k(n)} y_n\| + h_n\|y_n - p\|,$$

It follows that

$$d = \lim_{n\to\infty} \|x_n - p\| \le \liminf_{n\to\infty} \|y_n - p\|,$$

In view of

$$d \le \liminf_{n\to\infty} \|y_n - p\| \le \limsup_{n\to\infty} \|y_n - p\| \le d,$$

We find that $\lim_{n\to\infty} \|y_n - p\| = d$. Since $\lim_{n\to\infty} \|x_n - p\| = d$, we see that

$$\limsup_{n\to\infty} \|S_{i(n)}^{k(n)} x_n - p\| \le \limsup_{n\to\infty} h_n\|x_n - p\| = d. \tag{3.10}$$

Notice that

$$\|S_{i(n)}^{k(n)}x_n - p + \delta_n(v_n - x_n)\| \le \|S_{i(n)}^{k(n)}x_n - p\| + \delta_n\|v_n - x_n\|.$$

It follows that

$$\limsup_{n\to\infty} \|S_{i(n)}^{k(n)}x_n - p + \delta_n(v_n - x_n)\| \le d, \tag{3.11}$$

and $\|x_n - p + \delta_n(v_n - x_n)\| \le \|x_n - p\| + \delta_n\|v_n - x_n\|$, which implies that

$$\limsup_{n\to\infty} \|x_n - p + \delta_n(v_n - x_n)\| \le d. \tag{3.12}$$

On the other hand, we have

$$\begin{aligned} d &= \lim_{n\to\infty} \|y_n - p\| \\ &= \lim_{n\to\infty} \|(1-\beta_n-\delta_n)x_n + \beta_n S_{i(n)}^{k(n)}x_n + \delta_n v_n - p\| \\ &= \lim_{n\to\infty} \|\beta_n S_{i(n)}^{k(n)}x_n + (1-\beta_n)x_n - \delta_n x_n + \delta_n v_n - (1-\beta_n)p - \beta_n p\| \\ &= \lim_{n\to\infty} \|\beta_n S_{i(n)}^{k(n)}x_n - \beta_n p + \beta_n\delta_n v_n - \beta_n\delta_n x_n + (1-\beta_n)x_n \\ &- (1-\beta_n)p - \delta_n x_n + \delta_n v_n - \beta_n\delta_n v_n + \beta_n\delta_n x_n\| \\ &= \lim_{n\to\infty} \|\beta_n(S_{i(n)}^{k(n)}x_n - p + \delta_n(v_n - x_n)) + (1-\beta_n)(x_n - p + \delta_n(v_n - x_n))\|. \end{aligned} \tag{3.13}$$

In view of (3.11),(3.12), and (3.13), we obtain from Lemma2.3 that

$$\lim_{n\to\infty} \|x_n - S_{i(n)}^{k(n)}x_n\| = 0. \tag{3.14}$$

Notice that

$$\begin{aligned} \|T_{i(n)}^{k(n)}x_n - x_n\| &\le \|T_{i(n)}^{k(n)}x_n - T_{i(n)}^{k(n)}y_n\| + \|T_{i(n)}^{k(n)}y_n - x_n\| \\ &\le h_n\|x_n - y_n\| + \|T_{i(n)}^{k(n)}y_n - x_n\| \\ &\le h_n\beta_n\|x_n - S_{i(n)}^{k(n)}x_n\| + h_n\delta_n\|v_n - x_n\| + \|T_{i(n)}^{k(n)}y_n - x_n\|. \end{aligned}$$

It follows from (3.9), and (3.14) that

$$\lim_{n\to\infty} \|x_n - T_{i(n)}^{k(n)}x_n\| = 0. \tag{3.15}$$

This completes the proof.

Corollary 3.3. *Under the same hypotheses of lemma 2.2, we also have the following:*

$$\lim_{n\to\infty}\|y_n-x_n\|=0, \lim_{n\to\infty}\|x_{n-1}-T_{i(n)}^{k(n)}x_n\|=0,$$

$$\lim_{n\to\infty}\|x_n-x_{n-1}\|=0, \lim_{n\to\infty}\|T_{i(n)}^{k(n)}x_n-S_{i(n)}^{k(n)}x_n\|=0.$$

Proof. Notice that

$$\|y_n-x_n\|=\|\beta_n(S_{i(n)}^{k(n)}x_n-x_n)+\delta_n(v_n-x_n)\|\le\beta_n\|S_{i(n)}^{k(n)}x_n-x_n\|+\delta_n\|v_n-x_n\|$$

In view of (3.14), we see from the restriction (b) that

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{3.16}$$

Notice that

$$\begin{aligned}\|x_{n-1}-T_{i(n)}^{k(n)}x_n\|&\le\|x_{n-1}-T_{i(n)}^{k(n)}y_n\|+\|T_{i(n)}^{k(n)}y_n-T_{i(n)}^{k(n)}x_n\|\\&\le\|x_{n-1}-T_{i(n)}^{k(n)}y_n\|+h_n\|y_n-x_n\|.\end{aligned}$$

It follows from (3.8), and (3.16) that

$$\lim_{n\to\infty}\|x_{n-1}-T_{i(n)}^{k(n)}x_n\|=0.$$

Notice that

$$\|x_n-x_{n-1}\|\le\alpha_n\|T_{i(n)}^{k(n)}y_n-x_{n-1}\|+\gamma_n\|u_n-x_{n-1}\|.$$

It follows that

$$\lim_{n\to\infty}\|x_n-x_{n-1}\|=0. \tag{3.17}$$

Notice that

$$\|T_{i(n)}^{k(n)}x_n-S_{i(n)}^{k(n)}x_n\|\le\|T_{i(n)}^{k(n)}x_n-x_n\|+\|x_n-S_{i(n)}^{k(n)}x_n\|.$$

It follows from (3.14), and (3.15) that

$$\lim_{n\to\infty}\|T_{i(n)}^{k(n)}x_n-S_{i(n)}^{k(n)}x_n\|=0.$$

Remark 3.4. *By corollary 3.3, we can derive*

$$\lim_{n\to\infty}\|x_{n+j}-x_n\|=0, \forall j\in N. \tag{3.18}$$

Corollary 3.5. *Under the same hypotheses of Lemma 3.2, we have the following:*

$$\lim_{n\to\infty}\|x_n - T_l x_n\| = 0, \lim_{n\to\infty}\|x_n - S_l x_n\| = 0,$$

$$\lim_{n\to\infty}\|S_l x_n - T_l x_n\| = 0, \forall l = 1, 2, \cdots N.$$

Proof. Notice that

$$\begin{aligned}\|x_{n-1} - T_{i(n)}x_n\| &\le \|x_{n-1} - T_{i(n)}^{k(n)}x_n\| + \|T_{i(n)}^{k(n)}x_n - T_{i(n)}x_n\| \\ &\le \|x_{n-1} - T_{i(n)}^{k(n)}x_n\| + h_1\|T_{i(n)}^{k(n)-1}x_n - x_n\| \\ &\le \|x_{n-1} - T_{i(n)}^{k(n)}x_n\| + h_1(\|T_{i(n)}^{k(n)-1}x_n - T_{i(n-N)}^{k(n)-1}x_{n-N}\| \\ &+ \|T_{i(n-N)}^{k(n)-1}x_{n-N} - x_{(n-N)-1}\| + \|x_{(n-N)-1} - x_n\|).\end{aligned} \tag{3.19}$$

It follows that

$$\|T_{i(n)}^{k(n)-1}x_n - T_{i(n-N)}^{k(n)-1}x_{n-N}\| \le L\|x_n - x_{n-N}\|. \tag{3.20}$$

Notice that

$$\|T_{i(n-N)}^{k(n)-1}x_{n-N} - x_{(n-N)-1}\| = \|T_{i(n-N)}^{k(n-N)}x_{n-N} - x_{(n-N)-1}\|. \tag{3.21}$$

Substituting (3.20), and (3.21) into (3.19), we arrive at

$$\begin{aligned}\|x_{n-1} - T_{i(n)}x_n\| \le \|x_{n-1} - T_{i(n)}^{k(n)}x_n\| + L^2\|x_n - x_{n-N}\| + L\|T_{i(n-N)}^{k(n-N)}x_{n-N} - x_{(n-N)-1}\| \\ + L\|x_{n-(N+1)} - x_n\|.\end{aligned}$$

This finds that

$$\lim_{n\to\infty}\|x_{n-1} - T_{i(n)}x_n\| = 0.$$

In view of

$$\|x_n - T_{i(n)}x_n\| \le \|x_n - x_{n-1}\| + \|x_{n-1} - T_{i(n)}x_n\|,$$

we obtain that

$$\lim_{n\to\infty}\|x_n - T_{i(n)}x_n\| = 0. \tag{3.22}$$

In particular

$$\|x_{k_N+1} - T_1 x_{k_N+1}\| \to 0, k \to +\infty;$$

$$\|x_{k_N+2} - T_2 x_{k_N+2}\| \to 0, k \to +\infty;$$

$$\cdots$$

$$\|x_{k_N+N} - T_N x_{k_N+N}\| \to 0, k \to +\infty.$$

Moreover for $l, j = 1, 2, \cdots N$,

$$\begin{aligned}\|x_{k_N+j} - T_l x_{k_N+j}\| &\leq \|x_{k_N+j} - x_{k_N+l}\| + \|x_{k_N+l} - T_l x_{k_N+l}\| \\ &\quad + L\|x_{k_N+l} - x_{k_N+j}\| \to 0, k \to +\infty,\end{aligned}$$

and this is equivalent to

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \forall l = 1, 2, \cdots N. \tag{3.23}$$

Similarly, by using the same argument as in the proof above, we have

$$\lim_{n\to\infty} \|x_n - S_l x_n\| = 0, \forall l = 1, 2, \cdots N. \tag{3.24}$$

Since

$$\|S_l x_n - T_l x_n\| \leq \|S_l x_n - x_n\| + \|T_l x_n - x_n\|,$$

we find from (3.23), and (3.24) that

$$\lim_{n\to\infty} \|S_l x_n - T_l x_n\| = 0, \forall l = 1, 2, \cdots N. \tag{3.25}$$

Theorem 3.6. *Let E be a real uniformly convex Banach space which satisfies Opial's condition and K be a nonempty closed and convex subset of E. Let $\{S_i\}_{i=1}^N$,$\{T_i\}_{i=1}^N : K \to K$ be $2N$ asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \cap F(S_i) \neq \emptyset$. Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequence in K. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be four real sequences in $[0, 1]$ satisfying the following conditions:*

(a) $\alpha_n + \gamma_n \leq 1$ *and* $\beta_n + \delta_n \leq 1$, $\forall n \geq 1$*;*

(b) $\limsup_{n\to\infty} \alpha_n < 1$*;*

(c) $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \delta_n < \infty, \sum_{n=1}^{\infty}(h_n - 1) < \infty$*;*

Let the sequence $\{x_n\}$ be defined by (1.9). Then the sequence $\{x_n\}$ weakly converges to a common fixed point of $\{T_1, T_2, \cdots, T_N, S_1, S_2, \cdots, S_N\}$ in K.

Proof. It follows from Corollary 3.5 that

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \lim_{n\to\infty} \|x_n - S_l x_n\| = 0, \forall l = 1, 2, \cdots, N.$$

Since E is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \to q$ weakly as $n \to \infty$, without loss of generality. By Lemma 2.4 that T_i and S_i are demiclosed at 0, so that $q \in F = \bigcap_{i=1}^N F(T_i) \cap F(S_i)$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v respectively, By demiclosedness principle, $u, v \in F = \bigcap_{i=1}^N F(T_i) \cap F(S_i)$. By Lemma 3.1, we see that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|v_n - v\|$ exist. It follows from Lemma 2.5 that $u = v$, therefore, $\{x_n\}$ converges weakly to a common fixed point of $\{T_1, T_2, \cdots, T_N, S_1, S_2, \cdots, S_N\}$in E.

Theorem 3.7. *Let E be a real uniformly convex Banach space, K be a nonempty closed and convex subset of E. Let $\{S_i\}_{i=1}^N$,$\{T_i\}_{i=1}^N : K \to K$ be $2N$ asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \cap F(S_i) \neq \emptyset$. Assume that there exists at least a $T_i (i \in I)$ which is semi-compact. Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequence in K. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be four real sequences in $[0, 1]$ satisfying the following conditions:*

(a) $\alpha_n + \gamma_n \leq 1$ *and* $\beta_n + \delta_n \leq 1$, $\forall n \geq 1$;

(b) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(c) $\limsup_{n\to\infty} \beta_n < 1$;

(d) $\sum_{n=1}^\infty \gamma_n < \infty, \sum_{n=1}^\infty \delta_n < \infty, \sum_{n=1}^\infty (h_n - 1) < \infty$.

Then the sequence $\{x_n\}$ defined by (1.9) converges strongly to a common fixed point of $\{T_1, T_2, \cdots, T_N, S_1, S_2, \cdots, S_N\}$in E.

Proof. In view of the assumptions, we see that there exists a semi-compact mapping $T_i \in \{T_1, T_2, \cdots T_N\}$, without loss generality, we assume that T_l is semi-compact. In view of (3.22), we assures that

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0. \tag{3.26}$$

Hence there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to x^*$. It follows that

$$\|T_l x^* - x^*\| \le \|T_l x^* - T_l x_{n_i}\| + \|T_l x_{n_i} - x_{n_i}\| + \|x_{n_i} - x^*\|$$
$$\le (1+L)\|x_{n_i} - x^*\| + \|T_l x_{n_i} - x_{n_i}\|.$$

It follows from (3.26) that $\|T_l x^* - x^*\| = 0$ for any $l \in I$. Notice that

$$\|S_l x^* - x^*\| \le \|S_l x^* - S_l x_{n_i}\| + \|S_l x_{n_i} - T_l x_{n_i}\| + \|T_l x_{n_i} - x_{n_i}\| + \|x_{n_i} - x^*\|$$
$$\le (1+L)\|x_{n_i} - x^*\| + \|S_l x_{n_i} - T_l x_{n_i}\| + \|T_l x_{n_i} - x_{n_i}\|.$$

It follows from (3.25), and (3.26) that $\|S_l x^* - x^*\| = 0$ for any $l \in I$. This implies that $x^* \in F = \bigcap_{i=1}^N F(T_i) \cap F(S_i)$. By Lemma 3.1, we see that $\lim_{n\to\infty} \|x_n - x^*\|$ exists. It follows from $x_{n_i} \to x^*$ that $\lim_{n\to\infty} \|x_n - x^*\| = 0$. Hence, the iterative sequence $\{x_n\}$ defined by (1.9)converges strongly to a common fixed point of $\{T_1, T_2, \cdots, T_N, S_1, S_2, \cdots, S_N\}$in E. This completes the proof.

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