ČIRIĆ TYPE FIXED POINT THEOREMS UNDER \( c \)-DISTANCE ON NON-NORMAL CONE METRIC SPACES OVER BANACH ALGEBRAS

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Abstract. In this paper, we obtain Ćirić type fixed point theorems for continuous or non-continuous mappings under \( c \)-distance on mapping-orbitally complete cone metric spaces over Banach algebras without normalities.

Keywords: Ćirić type fixed point; \( c \)-distance; cone metric space over Banach algebra; mapping-orbitally complete.

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1. INTRODUCTION AND PRELIMINARIES

Ćirić\(^{[1]}\) introduced and studied the following quasicontraction as one of the most general classes of contrative type mappings:

Let \((X,d)\) is a complete space. \( f : X \rightarrow X \) is said to be a quasicontraction if, for some \( k \in (0,1) \) and for all \( x,y \in X \), one has

\[
d(fx, fy) \leq k \max \{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(fx, y)\}.
\]

He proved that any quasicontraction \( f \) has a unique fixed point on a complete metric space \((X,d)\). Recently, many researchers discussed and obtained various similar results on metric

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spaces, cone metric spaces and cone metric spaces over Banach algebras, for details, see [2-12]. These conclusions goodly generalize and improve Ćirić’s fixed point theorem.

On the other hand, some authors discussed (common) fixed point problems under $c$-distance on cone metric spaces, see [13-19] and others. Especially, Huang et al[20] and Huang et al[21] discussed and obtained fixed point theorems for mappings under $c$-distance on cone metric space over Banach algebras without normalities.

In this paper, we will discuss and obtain Ćirić type fixed point problems for continuous or non-continuous mappings under $c$-distance on mapping-orbitally complete cone metric spaces over Banach algebras without normalities.

Now, we give some known definitions and lemmas:

Let $\mathcal{A}$ always be a Banach algebra, that is, $\mathcal{A}$ is a real Banach space in which an operation of multiplication is defined, subject to the following properties(for all $x, y, z \in \mathcal{A}$, $\alpha \in \mathbb{R}$):

1. $(xy)z = x(yz)$;
2. $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;
3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
4. $\|xy\| \leq \|x\| \|y\|$.

In this paper, we shall assume that a Banach algebra $\mathcal{A}$ has a unit (i.e., a multiplicative identity) $e$ such that $ex = xe = x$ for all $x \in \mathcal{A}$. an element $x \in \mathcal{A}$ is said to be invertible if there is an inverse element $y \in A$ such that $xy = yx = e$. The inverse of $x$ denoted by $x^{-1}$. For more detail, we refer to [22-24].

A subset $P$ of a Banach algebra $\mathcal{A}$ is called a cone if

1. $P$ is nonempty closed and $\{0, e\} \subset P$;
2. $\alpha P + \beta P \subset P$ for all non-negative real numbers $\alpha, \beta$;
3. $P^2 = PP \subset P$;
4. $P \cap (-P) = \{0\}$.

Where $0$ denotes the null of the Banach algebra $\mathcal{A}$.

For a given cone $P \subset \mathcal{A}$, we can define a partial ordering $\leq$ with respect to $P$ by $x \leq y$ if and only if $y - x \in P$. $x < y$ stand for $x \leq y$ and $x \neq y$. While $x \ll y$ sill stand for $y - x \in \text{int} P$, where $\text{int} P$ denotes the interior of $P$. A cone $P$ is called solid if $\text{int} P \neq \emptyset$. 
The cone $P$ is called normal if there is a number $M > 0$ such that for all $x, y \in \mathcal{A}$,

$$0 \leq x \leq y \implies \|x\| \leq M \|y\|.$$ 

The least positive number satisfying the above is called the normal constant of $P$.

Here, we always assume that $P$ is a solid and $\leq$ is the partial ordering with respect to $P$.

**Definition 1.1.** [20, 21] Let $X$ be a non-empty set. Suppose that the mapping $d : X \times X \to \mathcal{A}$ satisfies

1. $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a cone metric on $X$ and $(X, d)$ is called a cone metric space (over a Banach algebra $\mathcal{A}$).

**Remark 1.1.** If $\mathcal{A} = E$ is a Banach space in Definition 1.1, then $(X, d)$ is called a cone metric space.

**Definition 1.2.** [21] Let $(X, d)$ be a cone metric space over a Banach algebra $\mathcal{A}$, $x \in X$ and $\{x_n\}$ a sequence in $X$. Then:

1. $\{x_n\}$ converges to $x$ whenever for each $c \in \mathcal{A}$ with $0 \ll c$ there is a natural number $N$ such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$.
2. $\{x_n\}$ is Cauchy sequence whenever for each $c \in \mathcal{A}$ with $0 \ll c$ there is a natural number $N$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.
3. $(X, d)$ is a complete cone metric space if every Cauchy sequence is convergent.

**Definition 1.3.** [17, 18, 22] Let $P$ be a solid cone in a Banach space $\mathcal{A}$. A sequence $\{u_n\} \subset \mathcal{A}$ is a $c$-sequence if for each $c \gg 0$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for all $n \geq n_0$.

**Definition 1.4.** [20, 21] Let $(X, d)$ be a cone metric space over a Banach algebra. A function $q : X \times X \to \mathcal{A}$ is called a $c$-distance on $X$. If

$(q_1)$ $\theta \leq q(x, y)$ for all $x, y \in X$;

$(q_2)$ $q(x, z) \leq q(x, y) + q(y, z)$ for all $x, y, z \in X$;

$(q_3)$ If a sequence $\{y_n\}$ in $X$ converges to a point $y \in X$, and for any $x \in X$, there exists $u = u_x \in P$ such that $q(x, y_n) \leq u$ holds for each $n \in \mathbb{N}$, then $q(x, y) \leq u$;
For each \( c \in \mathcal{A} \) with \( \theta \ll c \), there exists \( e \in \mathcal{A} \) with \( \theta \ll e \), such that \( q(z, x) \ll e \) and \( q(z, y) \ll e \) implies \( d(x, y) \ll c \).

**Remark 1.2.** [13, 15] Generally, \( q(x, y) \neq (y, x) \) for \( x, y \in X \), and \( q(x, y) = 0 \) is not necessarily equivalent to \( x = y \).

**Definition 1.5.** [12] Let \((X, d)\) be a cone metric space over a Banach algebra \( \mathcal{A} \), \( T : X \to X \) a mapping. For any \( x \in X \) and any positive number \( n \), let

\[
O_T(x, n) = \{x, Tx, T^2x, \ldots, T^nx\}, \quad O_T(x, +\infty) = \{x, Tx, T^2x, \ldots\}.
\]

The set \( O_T(x, +\infty) \) is called the \( T \)-orbit at \( x \). \((X, d)\) is said to be \( T \)-orbitally complete if, every Cauchy sequence in \( O_T(x, +\infty) \) is convergent for every \( x \in X \).

**Lemma 1.1.** [22] Let \( \mathcal{A} \) be a Banach algebra with a unit \( e \), and \( x \in \mathcal{A} \). If the spectral radius \( r(x) \) of \( x \) is less than 1, i.e.,

\[
r(x) = \lim_{n \to \infty} \| x^n \|^{\frac{1}{n}} = \inf_{n \to \infty} \| x^n \|^{\frac{1}{n}} < 1.
\]

Then \( (e - x) \) is invertible. Actually,

\[
(e - x)^{-1} = \sum_{i=0}^{+\infty} x^i.
\]

**Lemma 1.2.** [22] Let \( P \) is a solid cone in a Banach algebra \( \mathcal{A} \) and \( \{u_n\} \) and \( \{v_n\} \) be two \( c \)-sequences in \( \mathcal{A} \). If \( k, l \in P \) are two arbitrarily given vectors, then \( \{ku_n + lv_n\} \) is a \( c \)-sequence in \( \mathcal{A} \).

**Lemma 1.3.** [22] Let \( P \) be a solid cone in Banach algebra \( \mathcal{A} \) and \( u, v, w \in \mathcal{A} \). If \( u \leq v \ll w \), then \( u \ll w \).

**Lemma 1.4.** [11] Let \( P \) be a solid cone in a Banach algebra \( \mathcal{A} \) and \( a, k \in P \) with \( r(k) < 1 \). If \( a \leq ka \), then \( a = 0 \).

**Lemma 1.5.** [12] If \( E \) is a real Banach space with a solid cone \( P \) and if \( \| x_n \| \to 0 \) as \( n \to \infty \), then for any \( 0 \ll c \), there exists \( N \in \mathbb{N} \) such that \( x_n \ll c \) for all \( n > N \).

**Lemma 1.6.** [23] If \( \mathcal{A} \) is a Banach algebra and \( k \in \mathcal{A} \) with \( r(k) < 1 \), then \( \| k^n \| \to 0 \) as \( n \to \infty \).

**Lemma 1.7.** [23] Let \( A \) be a Banach algebra and \( x, y \in \mathcal{A} \). If \( x \) and \( y \) commute, then the following hold:

(i) \( r(xy) \leq r(x)r(y) \);
(ii) \( r(x + y) \leq r(x) + r(y) \);

(iii) \( |r(x) - r(y)| \leq r(x - y) \).

**Lemma 1.8.** [24] Let \((X, d)\) be a cone metric space over a Banach algebra \(\mathcal{A}\), \(\{x_n\} \subset X\) a sequence. If \(\{x_n\}\) is convergent, then the limits of \(\{x_n\}\) is unique.

**Lemma 1.9.** [21]. Let \((X, d)\) be a cone metric space over Banach algebra \(\mathcal{A}\), \(q\) a \(c\)-distance on \(X\). Suppose that \(\{x_n\}\) is a sequences in \(X\) and \(y, z \in X\). If \(\{u_n\}\) and \(\{v_n\}\) are two \(c\)-sequences in \(P\), then the following properties hold:

1. If \(q(x_n, y) \leq u_n \) and \(q(x_n, z) \leq v_n, \forall n \in \mathbb{N}\), then \(y = z\). In particular, if \(q(x, y) = 0\) and \(q(x, z) = 0\), then \(y = z\).

2. If \(q(x_n, x_m) \leq u_n\) for all \(m > n > n_0\), then \(\{x_n\}\) is a Cauchy sequence in \(X\).

## 2. ĆIRIĆ TYPE FIXED POINT THEOREMS UNDER \(c\)-DISTANCE

**Theorem 2.1.** Let \((X, d)\) be a cone metric space over a Banach algebra, \(q\) be a \(c\)-distance on \(X\), \(f : X \to X\) be continuous on \((X, d)\), \(k \in P\) with \(r(k) < 1\). Suppose that for each \(x, y \in X\),

\[
q(fx, fy) \leq kv(x, y),
\]  

(2.1)

where

\[
v(x, y) \in \{q(x, y), q(x, fx), q(y, fy), q(x, fy)\}.
\]  

(2.2)

If \(X\) is \(f\)-orbitally complete, then \(f\) has a unique fixed point \(x^* \in X\) and \(q(x^*, x^*) = 0\).

**Proof.** For any \(x \in X\), Let \(x_n = f^n x\) for all \(n = 1, 2, \cdots\), then \(x_n = fx_{n-1}\) for all \(n = 1, 2, \cdots\). (Here, set \(x_0 = x\)).

First, we will prove that for each \(n \geq 2\) and for all \(i, j\) such that \(1 \leq i < j \leq n\), one has

\[
q(x_i, x_j) \leq k (1 - k)^{-1} q(x_0, x_1).
\]  

(2.3)

If \(n = 2\), then \(i = 1, j = 2\). Hence

\[
q(x_1, x_2) = q(fx_0, fx_1) \leq kv(x_0, x_1),
\]
where
\[ v(x_0, x_1) \]
\[ \in \{ q(x_0, x_1), q(x_0, f x_0), q(x_1, f x_1), q(x_0, f x_1) \} \]
\[ = \{ q(x_0, x_1), q(x_1, x_2), q(x_0, x_2) \}. \]

If \( v(x_0, x_1) = q(x_0, x_1) \), then
\[ q(x_1, x_2) \leq k q(x_0, x_1) \leq k (e - k)^{-1} q(x_0, x_1). \]

If \( v(x_0, x_1) = q(x_1, x_2) \), then
\[ q(x_1, x_2) \leq k q(x_1, x_2) \Rightarrow (e - k) q(x_1, x_2) \leq 0, \]
therefore
\[ q(x_1, x_2) = 0 \leq k (e - k)^{-1} q(x_0, x_1). \]

If \( v(x_0, x_1) = q(x_0, x_2) \), then
\[ q(x_1, x_2) \leq k q(x_0, x_2) \leq k [q(x_0, x_1) + q(x_1, x_2)], \]
hence
\[ q(x_1, x_2) \leq k (e - k)^{-1} q(x_0, x_1). \]

Based on the above discussions, (2.3) is set up for \( n = 2 \).

Assume that (2.3) is true for \( n = m > 2 \), that is,
\[ q(x_i, x_j) \leq k (e - k)^{-1} q(x_0, x_1), \ \text{1} \leq i < j \leq m. \]  \hspace{1cm} (2.5)

Now, we will prove that (2.3) also holds for \( n = m + 1 \). If \( 1 \leq i < j \leq m \), then (2.3) holds by the assumption (i.e., by (2.5)). Thus, without loss of generality, we assume that \( j = m + 1 \) and \( 1 \leq i \leq m \). Denote \( i = i_0 \). By (2.1),
\[ q(x_{i_0}, x_{m+1}) = q(f x_{i_0-1}, f x_m) \leq k v(x_{i_0-1}, x_m), \]  \hspace{1cm} (2.6)
where
\[ v(x_{i_0-1}, x_m) \in \{ q(x_{i_0-1}, x_m), q(x_{i_0-1}, x_{i_0}), q(x_m, x_{m+1}), q(x_{i_0-1}, x_{m+1}) \}. \]  \hspace{1cm} (2.7)

Firstly, we consider that \( i_0 = 1 \).
If \( v(x_{i_0-1}, x_m) = d(x_0, x_m) \), then

\[
q(x_{i_0}, x_{m+1}) \\
\leq k q(x_0, x_m) \\
\leq k [q(x_0, x_1) + q(x_1, x_m)] \\
\leq k [q(x_0, x_1) + k(e - k)^{-1} q(x_0, x_1)] \\
= k(e - k)^{-1} q(x_0, x_1),
\]

and the statement follows.

If \( v(x_{i_0-1}, x_m) = q(x_0, x_1) \), then

\[
q(x_{i_0}, x_{m+1}) \leq k q(x_0, x_1) \leq k(e - k)^{-1} d(x_0, x_1),
\]

and the statement also holds.

If \( v(x_{i_0-1}, x_m) = q(x_m, x_{m+1}) \), then we let \( i_1 = m \) and we have

\[
q(x_{i_0}, x_{m+1}) \leq k q(x_{i_1}, x_{m+1}).
\]

If \( v(x_{i_0-1}, x_m) = q(x_0, x_{m+1}) \), then

\[
q(x_{i_0}, x_{m+1}) \leq k q(x_0, x_{m+1}) \leq k [d(x_0, x_1) + d(x_{i_0}, x_{m+1})],
\]

which implies that

\[
q(x_{i_0}, x_{m+1}) \leq k(e - k)^{-1} d(x_0, x_1),
\]

and the statement also holds.

Secondly, we consider that \( 2 \leq i_0 \leq m \).

If \( v(x_{i_0-1}, x_m) = q(x_{i_0-1}, x_m) \) or \( v(x_{i_0-1}, x_m) = q(x_{i_0-1}, x_{i_0}) \), then by the assumption,

\[
q(x_{i_0}, x_{m+1}) \leq k v(x_{i_0-1}, x_m) \leq k^2 (e - k)^{-1} q(x_0, x_1) \leq k(e - k)^{-1} q(x_0, x_1),
\]

and the statement follows.

If \( v(x_{i_0-1}, x_m) = q(x_m, x_{m+1}) \) or \( v(x_{i_0-1}, x_m) = q(x_{i_0-1}, x_{m+1}) \), then we let \( i_1 = m \) or \( i_1 = i_0 - 1 \geq 1 \), respectively, hence

\[
q(x_{i_0}, x_{m+1}) \leq k v(x_{i_0-1}, x_{m+1}) = k q(x_{i_1}, x_{m+1}).
\]
In conclusion from discussion of both cases, it results that either the proof is complete, that is
\[ q(x_{i_0}, x_{m+1}) \leq k(e - k)^{-1} q(x_0, x_1), \]  
(2.14)
or there exists an integer \( i_1 \) such that
\[ q(x_{i_0}, x_{m+1}) \leq kd(x_{i_1}, x_{m+1}), \quad 1 \leq i_1 \leq m. \]  
(2.15)
As for the latter situation, we continue in a similar way, and come to the result that either
\[ q(x_{i_1}, x_{m+1}) \leq k(e - k)^{-1} q(x_0, x_1), \]  
(2.16)
which implies that
\[ q(x_{i_0}, x_{m+1}) \leq k q(x_{i_1}, x_{m+1}) \leq k^2 (e - k)^{-1} q(x_0, x_1) \leq k (e - k)^{-1} q(x_0, x_1), \]  
(2.17)
and the proof is complete, or there exists integer \( i_2 \) such that
\[ q(x_{i_1}, x_{m+1}) \leq k q(x_{i_2}, x_{m+1}), \exists 1 \leq i_2 \leq m, \]  
(2.18)
which implies that
\[ q(x_{i_0}, x_{m+1}) \leq k^2 q(x_{i_2}, x_{m+1}), \exists 1 \leq i_2 \leq m. \]  
(2.19)
Generally, if the procedure ends by the \( l \)-th step with \( l \leq m - 1 \), that is, there exist \( l + 1 \) integers
\[ i_0, i_1, \cdots, i_l \in \{1, 2, \cdots, m\} \]  
(2.20)
such that
\[ q(x_{i_0}, x_{m+1}) \leq k q(x_{i_1}, x_{m+1}) \leq \cdots \leq k^l q(x_{i_l}, x_{m+1}), \]  
(2.21)
and
\[ q(x_{i_l}, x_{m+1}) \leq k (e - k)^{-1} q(x_0, x_1), \]  
(2.22)
then
\[ q(x_{i_0}, x_{m+1}) \leq k^l q(x_{i_l}, x_{m+1}) \leq k^{l+1} (e - k)^{-1} q(x_0, x_1) \leq k (e - k)^{-1} q(x_0, x_1). \]  
(2.23)
Hence, the proof is complete.

If the procedure continues more than \( m \) steps, then exist \( (m + 1) \) integers
\[ i_0, i_1, \cdots, i_m \in \{1, 2, \cdots, m\} \]  
(2.24)
such that
\[ q(x_{i_0}, x_{m+1}) \leq k q(x_{i_1}, x_{m+1}) \leq \cdots \leq k^m q(x_{i_m}, x_{m+1}), \]  
(2.25)

From (2.24), there must exist integers \( p \) and \( q \) such that
\[ 0 \leq p < q \leq m, \quad i_p = i_q. \]  
(2.26)

Hence by (2.25) and (2.26),
\[ q(x_{i_p}, x_{m+1}) \leq k^{q-p} q(x_{i_q}, x_{m+1}) = k^{q-p} d(x_{i_p}, x_{m+1}), \]  
(2.27)

which implies that
\[ (e - k^{q-p}) q(x_{i_p}, x_{m+1}) \leq 0. \]

Hence \( d(x_{i_q}, x_{m+1}) = 0 \) since \( r(k^{q-p}) \leq (r(k))^{q-p} < 1 \) implies that \((e - k^{q-p})\) is invertible. From (2.25) again,
\[ q(x_{i_0}, x_{m+1}) \leq k^p q(x_{i_p}, x_{m+1}) = 0 \leq k(e - k)^{-1} q(x_0, x_1). \]  
(2.28)

Therefore, by induction, (2.3) holds.

For any \( 1 < m < n \), denote that
\[ C(m, n) = \{ q(x_i, x_j) | m \leq i < j \leq n \}. \]  
(2.29)

From (2.1) and (2.2), for each \( u \in C(m, n) \), there exists \( v \in C(m - 1, n) \) such that
\[ u \leq k v. \]  
(2.30)

Consequently, using (2.3) and (2.30), we obtain that
\[ q(x_m, x_n) \leq k u_1 \leq k^2 u_2 \leq k^{m-1} u_{m-1} \leq k^m (e - k)^{-1} q(x_0, x_1), \]  
(2.31)

where
\[ u_1 \in C(m - 1, n), \quad u_2 \in C(m - 2, n) \cdots u_{m-1} \in C(1, n), \quad u_{m-1} \leq k(e - k)^{-1} q(x_0, x_1). \]  
(2.32)

Since \( r(k) < 1, k^m (e - k)^{-1} q(x_0, x_1) \) is a \( c \)-sequence by Lemma 1.2 and Lemma 1.5 - Lemma 1.6, which implies that \( \{x_n\} \) is a Cauchy sequence by Lemma 1.9 and (2.31). Thus there exists \( x^* \in X \) such that \( x_n \to x^* \) as \( n \to \infty \) by the \( f \)-orbitally completeness of \( X \).
Since \( x_{n+1} = fx_n \) for all \( n \) and \( f \) is continuous about the metric \( d \),

\[
x^* = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} fx_n = fx^*,
\]

that is, \( x^* \) is a fixed point of \( f \). By (2.1) again,

\[
q(x^*, x^*) = q(fx^*, fx^*) \leq kv(x^*, x^*) = kq(x^*, x^*),
\]

hence \( q(x^*, x^*) = 0 \) since \( r(k) < 1 \) implies that \( (e - k) \) is invertible.

If \( y^* \) is also a fixed point of \( f \), then \( fy^* = y^* \) and \( q(y^*, y^*) = 0 \) by the above discussion. By (2.1) again,

\[
q(x^*, y^*) = q(fx^*, fy^*) \leq kv(x^*, y^*),
\]

where

\[
v(x^*, y^*) \in \{q(x^*, y^*), 0\}.
\]

Hence \( q(x^*, y^*) = 0 \) for any one of two cases, therefore \( x^* = y^* \) by Lemma 1.9. So \( f \) has a unique fixed point.

Now, we once give another version of Theorem 2.1 under removing the continuity of \( f \):

**Theorem 2.2.** Let \( (X, d) \) be a cone metric space over Banach algebra, \( q \) be a \( c \)-distance on \( X \), \( f : X \to X \) a mapping, \( k \in P \) with \( r(k) < 1 \). Suppose that for each \( x, y \in X \),

\[
q(fx, fy) \leq ku(x,y), \quad (2.33)
\]

where

\[
u(x,y) \in \{q(x,y), q(x,fx), q(x,fy)\}. \quad (2.34)
\]

If \( X \) is \( f \)-orbitally complete, then \( f \) has a unique fixed point \( x^* \in X \) and \( q(x^*, x^*) = 0 \).

**Proof.** Repeating the proof of Theorem 2.1, we know that there exists a sequence \( \{x_n\} \) in \( X \) (Here, \( \{x_n\} \) satisfies \( x_n = fx_{n-1} \) for all \( n = 1, 2, \ldots \) ) converging to a point \( x^* \in X \). For any \( n \), by (2.33),

\[
q(x_n, fx^*) = q(fx_{n-1}, fx^*) \leq ku(x_{n-1},x^*),
\]

where

\[
u(x_{n-1},x^*) \in \{q(x_{n-1},x^*), q(x_{n-1},x_n), q(x_{n-1}, fx^*)\}.
\]
From (2.31) and Definition 1.4(q3), we have
\[ q(x_m, x^n) \leq k^m (1 - k)^{-1} d(x_0, x_1), \forall m \geq 1. \] (2.35)

If \( u(x_{n-1}, x^n) = q(x_{n-1}, x^n) \), then
\[ q(x_n, fx^n) \leq k q(x_{n-1}, x^n). \] (2.36)

If \( u(x_{n-1}, x^n) = q(x_{n-1}, x_n) \), then
\[ q(x_n, fx^n) \leq k q(x_{n-1}, x_n). \] (2.37)

If \( u(x_{n-1}, x^n) = q(x_{n-1}, fx^n) \), then
\[ q(x_n, fx^n) \leq k q(x_{n-1}, fx^n) \leq k [q(x_{n-1}, x_n) + q(x_n, fx^n)], \]
hence
\[ q(x_n, fx^n) \leq k (e - k)^{-1} q(x_{n-1}, x_n). \] (2.38)

\( \{q(x_m, x_n)\}_{n>m} \) and \( \{q(x_n, x^n)\} \) are both \( c \)-sequences by (2.31) and (2.35) and Lemma 1.5–Lemma 1.6, hence the right sides of inequalities in (2.35)-(2.38) are all \( c \)-sequences. Therefore \( x^n = fx^n \) by Lemma 1.9(1). The rest is similar to the proof of Theorem 2.1.

**CONFICT OF INTERESTS**

The author(s) declare that there is no conflict of interests.

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