COMMON COUPLED FIXED POINTS OF SOME GENERALISED T-CONTRACTIONS IN RECTANGULAR B-METRIC SPACE AND APPLICATION

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Abstract. Common coupled fixed point theorems for a pair of generalised T-contraction mappings are proved in a rectangular b-metric space which generalize and improve some recent results due to Ramesh and Pitchamani [13] and Gu [2] and some references there in. We have given an application of our main result in establishing the existence and convergence of solution of a system of non linear integral equations under some weaker conditions, which has been properly verified using suitable example.

Keywords: coupled fixed points; rectangular b-metric space; T-contraction; weakly compatible mappings.

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1. INTRODUCTION AND PRELIMINARIES

In 2015 George et al [14] introduced rectangular b-metric space (in short RbMS) as a generalization of usual metric space, b-metric space and rectangular metric space. In recent years many
fixed point theorems and their applications have been proved in b-metric space, $RbMS$ and other similar generalised metric spaces (see [1], [4],[5],[6], [7], [8],[9],[10],[11],[12],[15],[16],[17], [18],[19],[20], [21],[22], [23]). Some very recent results on common coupled fixed points can be seen in Gu [2] and Ramesh and Pitchamani [13]. In [2] the author has discussed coupled fixed point theorems for mappings defined on a set with two rectangular b-metrics $r_{b1}$ and $r_{b2}$ where $r_{b2} \leq r_{b1}$. Moreover in the proof of Theorem 2.1 in [2], the author shows that $r_{b1}(gx_n, gx_{n+p}) + r_{b1}(gy_n, gy_{n+p}) \leq \frac{sk^p(1+k)}{1-sk^2} \cdot \delta_0 + s^{m-1}k^{n+2m-2}\max\{\delta_0, \delta_0^s\}, 1-sk^2 \neq 0$ and on the basis of this the author claims that sequences $< gx_n >$ and $< gy_n >$ are Cauchy sequences. Note that here $p = 2m$ or $2m + 1$ and hence the author’s claim does not seems to be proper. In the present note we have given coupled fixed point results for a pair of generalised Reich type $T$ contraction mappings in a $RbMS$. From our main theorem, we deduce a corrected and improved version of Theorem 2.1 of Gu [2]. At the same time we have also obtained an improved and generalised version of the results of Ramesh and Pitchamani [13]. In recent years fixed point theory has been successfully applied in establishing the existence of solution of non linear integral equations (see [13], [3]). We have applied our result in establishing convergence criteria for a unique solution of a system of non linear integral equations. We have used some weaker conditions as compared to those existing in literature.

**Definition 1.1.** [14] Let $M$ be a non empty set. Suppose that the mapping $d_r : M \times M \to \mathbb{R}$ satisfies:

(RbM1) $d_r(x, y) \geq 0$ and $d_r(x, y) = 0$ if and only if $x = y$

(RbM2) $d_r(x, y) = d_r(y, x)$

(RbM3) $d_r(x, y) \leq s[d_r(x, u) + d_r(u, v) + d_r(v, y)]$ for some $s \geq 1$, all $x, y, \in M$ and all distinct points $u, v \in M - \{x, y\}$

Then $(M, d_r)$ is a rectangular b-metric space with coefficient $s$ (in short $RbMS(s)$).

**Definition 1.2.** [14] In the $RbMS (M, d_r)$ the sequence $< x_n >$

(a) converges to $x \in M$ if and only if $d_r(x_n, x) \to 0$ as $n \to \infty$.

(b) is a Cauchy sequence if and only if $d_r(x_n, x_{n+p}) \to 0$ as $n \to \infty$ for all $p > 0$.  


**Remark 1.3.** From Example 2.5 in [14] the following facts are easily observed:

i) In a RbMS open balls may not be an open set.

ii) In a RbMS convergent sequences may not be a Cauchy sequence.

iii) RbMS is not necessarily Hausdorff.

iv) Rectangular b-metric $d$ is not necessarily continuous.

**2. Main Results**

Our main theorems are as follows:

**Theorem 2.1.** Let $(X,d_r)$ be a RbMS, $T: X \to X$ be a one to one mapping, $S: X \times X \to X$ and $g: X \to X$ be mappings such that $S(X \times X) \subset g(X)$, $Tg(X)$ is complete. If there exist real numbers $\lambda, \mu, \nu$ with $0 \leq \lambda < 1$, $0 \leq \mu, \nu \leq 1$, minimum $\{\lambda \mu, \lambda \nu\} < \frac{1}{8}$ such that for all $u,v,w,z \in X$

$$d_r(TS(u,v),TS(w,z)) \leq \lambda \max\{d_r(Tgu,Tgw),d_r(Tgv,Tgz),\mu d_r(Tgu,TS(u,v)),$$

$$\mu d_r(Tgv,TS(v,u)),\nu d_r(Tgw,TS(w,z)),\nu d_r(Tgz,TS(z,w))\}$$

then

(i) $S$ and $g$ has a coupled coincident point.

(ii) A unique common coupled fixed point for $S$ and $g$ will exist provided $S$ and $g$ are weakly compatible.

(iii) If in addition $T$ is sequentially continuous and convergent, then for some arbitrary $(u_0,v_0) \in X \times X$, the iterative sequences $<gu_n>,<gv_n>$ defined by $gu_n = S(u_{n-1},v_{n-1})$ and $gv_n = S(v_{n-1},u_{n-1})$ converges to the unique common coupled fixed point of $S$ and $g$.

**Proof:** (i) We shall start the proof by showing that the sequences $<Tgu_n>$ and $<Tgv_n>$ are Cauchy sequences, where $<gu_n>$ and $<gv_n>$ are as mentioned in the hypothesis.
By (2.1), we have

\[ d_r(T g_{u_n}, T g_{u_{n+1}}) = d_r(T S(u_{n-1}, v_{n-1}), T S(u_n, v_n)) \]

\[ \leq \lambda \max \{d_r(T g_{u_{n-1}}, T g_u), d_r(T g_{v_{n-1}}, T g_v), \mu d_r(T g_{u_{n-1}}, T S(u_{n-1}, v_{n-1})), \mu d_r(T g_{v_{n-1}}, T S(v_{n-1}, u_{n-1})), \nu d_r(T g_u, T S(u_n, v_n)), \nu d_r(T g_v, T S(v_n, u_n)) \} \]

\[ \leq \lambda \max \{d_r(T g_{u_{n-1}}, T g_u), d_r(T g_{v_{n-1}}, T g_v), d_r(T g_{u_{n-1}}, T g_u), d_r(T g_{v_{n-1}}, T g_v) \} \]

(2.2) \[ d_r(T g_{v_{n-1}}, T g_v), d_r(T g_u, T g_{u_{n+1}}), d_r(T g_v, T g_{v_{n+1}}) \]

Similarly we get

\[ d_r(T g_v, T g_{v+1}) \leq \lambda \max \{d_r(T g_{v_{n-1}}, T g_v), d_r(T g_{u_{n-1}}, T g_u), d_r(T g_{v_{n-1}}, T g_v) \} \]

(2.3) \[ d_r(T g_{u_{n-1}}, T g_u), d_r(T g_v, T g_{v_{n+1}}), d_r(T g_u, T g_{u_{n+1}}) \]

Let \( K_n = \max \{d_r(T g_u, T g_{u+1}), d_r(T g_v, T g_{v+1})\} \). By (2.2) and (2.3), we get

(2.4) \[ K_n \leq \lambda \max \{d_r(T g_{v_{n-1}}, T g_v), d_r(T g_{u_{n-1}}, T g_u), d_r(T g_{v_{n-1}}, T g_{v_{n+1}}), d_r(T g_u, T g_{u+1}) \} \]

If

\[ \max \{d_r(T g_{v_{n-1}}, T g_v), d_r(T g_{u_{n-1}}, T g_u), d_r(T g_v, T g_{v_{n+1}}), d_r(T g_u, T g_{u+1}) \} = d_r(T g_v, T g_{v_{n+1}}) \text{ or } d_r(T g_u, T g_{u_{n+1}}), \]

then (2.4) will yield a contradiction. Thus we have

\[ \max \{d_r(T g_{v_{n-1}}, T g_v), d_r(T g_{u_{n-1}}, T g_u), d_r(T g_v, T g_{v_{n+1}}), d_r(T g_u, T g_{u+1}) \} = \max \{d_r(T g_{v_{n-1}}, T g_v), d_r(T g_{u_{n-1}}, T g_u) \}, \]

and then (2.4) gives

(2.5) \[ K_n \leq \lambda \max \{d_r(T g_{v_{n-1}}, T g_v), d_r(T g_{u_{n-1}}, T g_u) \} = \lambda K_{n-1} \leq \lambda^2 K_{n-2} \leq \cdots \leq \lambda^n K_0 \]
For any \( m, n \in N \), we have

\[
d_r(Tgu_m, Tgu_n) = d_r(TS(u_{m-1}, v_{m-1}), TS(u_{n-1}, v_{n-1})\]
\[
\leq \lambda \max\{d_r(Tgu_{m-1}, Tgu_{n-1}), d_r(Tgv_{m-1}, Tgv_{n-1}), \mu d_r(Tgu_{m-1}, TS(u_{m-1}, v_{m-1})),
\mu d_r(Tgv_{m-1}, TS(v_{m-1}, u_{m-1})), v d_r(Tgu_{n-1}, TS(u_{n-1}, v_{n-1})), v d_r(Tgv_{n-1}, TS(v_{n-1}, u_{n-1}))\}\]
\[
\leq \lambda \max\{d_r(Tgu_{m-1}, Tgu_{n-1}), d_r(Tgv_{m-1}, Tgv_{n-1}), d_r(Tgu_{m-1}, Tgu_{m}),
\quad d_r(Tgv_{m-1}, Tgv_{m}), d_r(Tgu_{n-1}, Tgu_{n}), d_r(Tgv_{n-1}, Tgv_{n})\}\]

Then by using 2.5 we get

\[
(2.6) \quad d_r(Tgu_m, Tgu_n) \leq \lambda \max\{d_r(Tgu_{m-1}, Tgu_{n-1}), d_r(Tgv_{m-1}, Tgv_{n-1}), (\lambda^{m-1} + \lambda^{n-1})K_0\}\]

Similarly we have

\[
(2.7) \quad d_r(Tgv_m, Tgv_n) \leq \lambda \max\{d_r(Tgu_{m-1}, Tgu_{n-1}), d_r(Tgv_{m-1}, Tgv_{n-1}), (\lambda^{m-1} + \lambda^{n-1})K_0\}\]

Let \( K_{m,n} = \max\{d_r(Tgu_m, Tgu_n), d_r(Tgv_m, Tgv_n)\} \). By (2.6) and (2.7), we get

\[
(2.8) \quad K_{m,n} \leq \lambda \max\{d_r(Tgu_{m-1}, Tgu_{n-1}), d_r(Tgv_{m-1}, Tgv_{n-1}), (\lambda^{m-1} + \lambda^{n-1})K_0\}\]

If,

\[
\max\{d_r(Tgu_{m-1}, Tgu_{n-1}), d_r(Tgv_{m-1}, Tgv_{n-1}), (\lambda^{m-1} + \lambda^{n-1})K_0\} = (\lambda^{m-1} + \lambda^{n-1})K_0\]

then (2.8) gives

\[
\max\{d_r(Tgu_m, Tgu_n), d_r(Tgv_m, Tgv_n)\} \leq (\lambda^m + \lambda^n)K_0\]

and since \( 0 < \lambda < 1 \), we conclude that \( < Tgu_n > \) and \( < Tgv_n > \) are Cauchy sequences. Now if

\[
\max\{d_r(Tgu_{m-1}, Tgu_{n-1}), d_r(Tgv_{m-1}, Tgv_{n-1}), (\lambda^{m-1} + \lambda^{n-1})K_0\} \neq (\lambda^{m-1} + \lambda^{n-1})K_0\]

then (2.8) gives

\[
(2.9) \quad K_{m,n} \leq \lambda \max\{d_r(Tgu_{m-1}, Tgu_{n-1}), d_r(Tgv_{m-1}, Tgv_{n-1})\}\]

\[
(2.10) \quad \leq \lambda K_{m-1,n-1} \leq \lambda^2 K_{m-2,n-2} \leq \cdots \lambda^r K_{m-r,n-r}\]
for any positive integer \( r \leq \min\{m,n\} \). Since \( 0 < \lambda < 1 \), we can find a positive integer \( q_0 \), such that \( 0 < \lambda^{q_0} < \frac{1}{s} \). Now from 2.9 we have

\[
\begin{align*}
(2.11) & \quad K_{m,m+q_0} \leq \lambda^m K_{0,q_0} \\
(2.12) & \quad K_{n+q_0,n} \leq \lambda^n K_{q_0,0} \\
(2.13) & \quad K_{m+q_0,n+q_0} \leq \lambda^{q_0} K_{m,n}
\end{align*}
\]

Using condition \((RbM3)\) of a rectangular b-metric and the above inequalities 2.11, 2.12 and 2.13, we have

\[
K_{m,n} \leq s[K_{m,m+q_0} + K_{m+q_0,n+q_0} + K_{n+q_0,n}] \\
\quad \leq s(\lambda^m + \lambda^n) 1 - s\lambda^{q_0} K_{0,q_0}
\]

Since \( 0 < \lambda < 1 \), again we conclude that \( < T g u_n > \) and \( < T g v_n > \) are Cauchy sequences.

Since \((T g(X), d)\) is complete, we can find \( w_{x_0}, w_{y_0} \in X \) such that

\[
\lim_{n \to \infty} T g u_n = T g w_{x_0} \text{ and } \lim_{n \to \infty} T g v_n = T g w_{y_0}.
\]

Therefore

\[
\begin{align*}
d_r(T S(w_{x_0},w_{y_0}), T g w_{x_0}) & \leq s[d_r(T S(w_{x_0},w_{y_0}), T S(u_n,v_n)) + d_r(T S(u_n,v_n), T S(u_{n+1},v_{n+1})) + d_r(T S(u_{n+1},v_{n+1}), T g w_{x_0})] \\
& \leq s[\lambda \max\{d_r(T g w_{x_0}, T g u_n), d_r(T g w_{y_0}, T g v_n), \mu d_r(T g w_{x_0}, T S(w_{x_0},w_{y_0}))\} \\
& \quad + \lambda \max\{d_r(T g u_n, T g u_{n+1}), d_r(T g v_n, T g v_{n+1}), \mu d_r(T g u_n, T S(u_n,v_n))\} \\
& \quad + \lambda \max\{d_r(T g v_n, T S(v_n,u_n), v d_r(T g u_{n+1}, T S(u_{n+1},v_{n+1})), v d_r(T g v_{n+1}, T S(v_{n+1},u_{n+1}))\} \\
& \quad + d_r(T g u_{n+2}, T g w_{x_0})]
\end{align*}
\]
Using 2.16 and 2.17 along with the condition minimum (2.17)

other coupled coincident point of

Proceeding on the same lines as above we also have

(ii) Suppose $S$ and $g$ are weakly compatible. First we will show that if $(w_{x_0}^*, w_{y_0}^*)$ is another coupled coincident point of $S$ and $g$ then $gw_{x_0}^* = gw_{x_0}$ and $gw_{y_0}^* = gw_{y_0}$, or in other words
the point of coupled coincidence of $S$ and $g$ is unique. By 2.2 we have

$$d_r(T gw^*_{x_0}, T gw_{y_0}) = d_r(TS(w^*_{x_0}, w^*_{y_0}), TS(w_{x_0}, w_{y_0}))$$

$$\leq \lambda \max\{d_r(T gw^*_{x_0}, T gw_{x_0}), d_r(T gw^*_{y_0}, T gw_{y_0}), \mu d_r(T gw^*_{x_0}, TS(w_{x_0}, w_{y_0})), \nu d_r(T gw_{y_0}, TS(w_{y_0}, w_{x_0}))\}$$

Similarly we have

$$d_r(T gw^*_{y_0}, T gw_{y_0}) \leq \lambda \max\{d_r(T gw^*_{x_0}, T gw_{x_0}), d_r(T gw^*_{y_0}, T gw_{y_0})\}$$

Thus from the above two inequalities, we get

$$\max\{d_r(T gw^*_{x_0}, T gw_{x_0}), d_r(T gw^*_{y_0}, T gw_{y_0}) \leq \lambda \max\{d_r(T gw^*_{x_0}, T gw_{x_0}), d_r(T gw^*_{y_0}, T gw_{y_0})\}$$

which implies that, $T gw^*_{x_0} = T gw_{x_0}$ and $T gw^*_{y_0} = T gw_{y_0}$. Since $T$ is one to one we get $gw^*_{x_0} = gw_{x_0}$ and $gw^*_{y_0} = gw_{y_0}$, that is the point of coupled coincidence of $S$ and $g$ is unique. Since $S$ and $g$ are weakly compatible and since $(w_{x_0}, w_{y_0})$ is a coupled coincident point of $S$ and $g$, we have

$$ggw_{x_0} = gS(w_{x_0}, w_{y_0}) = S(gw_{x_0}, gw_{y_0})$$

and

$$ggw_{y_0} = gS(w_{y_0}, w_{x_0}) = S(gw_{y_0}, gw_{x_0})$$

which shows that $(gw_{x_0}, gw_{y_0})$ is a coupled coincident point of $S$ and $g$. By the uniqueness of the point of coupled coincidence we get $ggw_{x_0} = gw_{x_0}$ and $ggw_{y_0} = gw_{y_0}$ and thus $(gw_{x_0}, gw_{y_0})$ is a common coupled fixed point of $S$ and $g$. Uniqueness of the coupled fixed point follows easily from 2.2.

(iii) Now suppose $T$ is sequentially convergent and continuous. Then since $\lim_{n \to \infty} Tu_n = T gw_{x_0}$ and $\lim_{n \to \infty} T gv_n = T gw_{y_0}$, using sequential convergence of $T$, we see that $< gu_n >$ and $< gv_n >$ are convergent and thus there exist $u_0$ and $v_0$ in $X$ such that $\lim_{n \to \infty} gu_n = u_0$ and $\lim_{n \to \infty} gv_n = v_0$. Now since $T$ is sequentially continuous we get $\lim_{n \to \infty} T gu_n = Tu_0$ and $\lim_{n \to \infty} T gv_n = Tv_0$. Therefore $T gw_{x_0} = Tu_0$ and $T gw_{y_0} = Tv_0$. Since
\( T \) is one to one, we get \( gw_{x_0} = u_0 \) and \( gw_{y_0} = v_0 \), that is \( < gu_n >, < gv_n > \) converges to \((gw_{x_0}, gw_{y_0})\) which is the common coupled fixed point of \( S \) and \( g \).

**Theorem 2.2.** Theorem 2.1 with condition 2.1 replaced with the following:

\[
d_r(TS(u,v), TS(w,z)) + d_r(TS(v,u), TS(z,w)) \leq \lambda \max \{d_r(TS(u,Tgw) + d_r(Tgv, Tgz),
\]

(2.18) \( \mu (d_r(Tgu, TS(u,v)) + d_r(Tgv, TS(v,u)), v(Tgw, TS(w,z)) + d_r(Tgz, TS(z,w))) \}

**Proof:** Putting \( K'_n = d_r(Tgu, Tgu_{n+1}) + d_r(Tgv, Tgv_{n+1}) \) and \( K'_{m,n} = d_r(Tgu, Tgu_n) + d_r(Tgv, Tgv_n) \), and then proceeding exactly as in the proof of Theorem 2.1, we can show that

\[
K'_{m,n} \leq s[K'_{m,m+q_0} + K'_{m+q_0,n+q_0} + mK'_{n+q_0,n}]
\]

and so \( d_r(Tgu, Tgu_n) \leq \frac{s(\lambda^m + \lambda^n)}{1-s(\lambda^q)} K'_{0,q_0} \) and \( d_r(Tgv, Tgv_n) \leq \frac{s(\lambda^m + \lambda^n)}{1-s(\lambda^q)} K'_{0,q_0} \). Thus \( < Tgu_n > \) and \( < Tgv_n > \) are Cauchy sequences. Again proceeding as in the proof of Theorem 2.1 and taking into consideration the fact that \( d_r(TS(w_{x_0}, w_{y_0}), TS(u_n, v_n)) \leq d_r(TS(w_{x_0}, w_{y_0}), TS(u_n, v_n)) + d_r(TS(w_{x_0}, w_{y_0}), TS(u_{n+1}, v_{n+1})) \) and \( d_r(TS(u_n, v_n), TS(u_{n+1}, v_{n+1})) \), we get

\[
d_r(TS(w_{x_0}, w_{y_0}), Tgw_{x_0}) + d_r(TS(w_{y_0}, w_{x_0}), Tgw_{y_0}) \leq s[d_r(TS(w_{x_0}, w_{y_0}), TS(u_n, v_n))
\]

\[
+ d_r(TS(w_{y_0}, w_{x_0}), TS(v_n, u_n))
\]

\[
+ d_r(TS(u_n, v_n), TS(u_{n+1}, v_{n+1})) + d_r(TS(v_n, u_n), TS(v_{n+1}, u_{n+1}))
\]

\[
+ d_r(TS(u_{n+1}, v_{n+1}), Tgw_{x_0}) + d_r(TS(v_{n+1}, u_{n+1}), Tgw_{y_0})
\]

\[
\leq s[\lambda \max \{d_r(Tgw_{x_0}, Tgu), d_r(Tgw_{y_0}, Tgv), d_r(Tgu, TS(u_n, v_n))\} +
\]

\[
\lambda \max \{d_r(Tgu, TS(u_n, v_n)), d_r(Tgv, TS(v_n, u_n))\}
\]

\[
+ \lambda \max \{d_r(Tgu, Tgu_{n+1}) + d_r(Tgv, Tgv_{n+1}), d_r(Tgu, TS(u_n, v_n)) +
\]

\[
d_r(Tgv, TS(v_n, u_n)), d_r(Tgu, TS(u_{n+1}, v_{n+1})), d_r(Tgv, TS(v_{n+1}, u_{n+1}))\}
\]

\[
+ d_r(Tgu_{n+2}, Tgw_{x_0}) + d_r(Tgv_{n+2}, Tgw_{y_0})\]
There exist $\beta$

\[ \text{Theorem 2.3. Theorem 2.1 with condition 2.1 replaced with the following:} \]

Our next result is a corrected and improved version of Theorem 2.1 of Gu [2]. Rest of the proof follows on the same lines as in Theorem 2.1.

Similarly we can show that

\[ d_r(TS(w_{x_0}, w_{y_0}), TS(w_{x_0}, w_{y_0})) + d_r(TS(w_{y_0}, w_{x_0}), TS(w_{y_0}, w_{x_0})) \]

as $n \to \infty$, we get

\[ d_r(TS(w_{x_0}, w_{y_0}), TS(w_{x_0}, w_{y_0})) + d_r(TS(w_{y_0}, w_{x_0}), TS(w_{y_0}, w_{x_0})) \]

Our next result is a corrected and improved version of Theorem 2.1 of Gu [2].

**Theorem 2.3.** Theorem 2.1 with condition 2.1 replaced with the following:

There exist $\beta_1, \beta_2, \beta_3$ in the interval $[0, 1)$, such that $\beta_1 + \beta_2 + \beta_3 < 1$, minimum\{ $\beta_2, \beta_3$ \} $< \frac{1}{s}$ and for all $u, v, w, z \in X$

\[ d_r(TS(u, v), TS(w, z)) + d_r(TS(v, u), TS(z, w)) \leq \beta_1(d_r(Tgu, Tw) + d_r(Tgv, Tgz)) + \]

\[ \beta_2(d_r(Tgu, TS(u, v)) + d_r(Tgv, TS(v, u))) + \beta_3(d_r(Tgw, TS(w, z)) + d_r(Tgz, TS(z, w))) \]

**Proof:** Proceeding on the same line and with the same notations as in the proof of Theorems 2.1 and 2.2, we can show that

\[ K_k' \leq \lambda K_{k-1}' \leq \lambda^2 K_{k-2} \leq \cdots \leq \lambda^n K_0' \]
where $\lambda' = \frac{a_1 + a_2}{a_3} < 1$. Now

$$d_r(Tgu_m, Tgu_n) + d_r(Tgv_m, Tgv_n)$$

$$= d_r(TS(u_{m-1}, v_{m-1}), TS(u_{n-1}, v_{n-1})) + d_r(TS(v_{m-1}, u_{m-1}), TS(v_{n-1}, u_{n-1}))$$

$$\leq \beta_1(d_r(Tgu_m, Tgu_n) + d_r(Tgv_m, Tgv_n)) + \beta_2(d_r(Tgu_m, TS(u_{m-1}, v_{m-1})))$$

$$+ d_r(Tgv_{m-1}, TS(v_{m-1}, u_{m-1})) + \beta_3(d_r(Tgu_{m-1}, TS(u_{n-1}, v_{n-1}))) + d_r(Tgv_{n-1}, TS(v_{n-1}, u_{n-1}))$$

$$\leq \beta_1(d_r(Tgu_m, Tgu_n) + d_r(Tgv_m, Tgv_n)) + \beta_2\lambda'^{m-1}K_0 + \beta_3\lambda'^{n-1}K_0$$

Thus we have

$$K'_{m,n} \leq \beta K'_{m-1,n-1} + \beta^{m-1}K_0' + \beta^{n-1}K_0'$$

$$\leq \beta^r K'_{m-r,n-r} + r(\beta^{m-1} + \beta^{n-1})K_0'$$

where $\beta = \text{Max}\{\beta_1, \beta_2, \beta_3, \lambda'\}$. Note that $\lim_{n \to \infty} \beta^n \to 0$ and so we can find natural number $q_0$ satisfying $0 < \beta^{q_0} < \frac{1}{s}$. Then we have

$$K'_{m,m+q_0} \leq \beta^q K_{0,q_0} + m(\beta^m + \beta^{m+q_0})K_0'$$

(2.21)

$$K'_{n+q_0,n} \leq \beta^n K_{q_0,0} + n(\beta^{n+q_0} + \beta^n)K_0'$$

(2.22)

$$K'_{m+q_0,m+q_0} \leq \beta^{q_0} K_{m,n} + q_0(\beta^{m+q_0} + \beta^{n+q_0})K_0'$$

(2.23)

Now using 2.21,2.22 and 2.23 we get

$$K'_{m,n} \leq s[K'_{m+q_0,m+q_0,n+q_0,n+q_0} + K'_{n+q_0,n}]$$

$$\leq s\frac{(\beta^m + \beta^n)K_0'}{1 - s\beta^{q_0}}$$

$$+ s[\beta^m(m + (m + q_0)\beta^{q_0}) + \beta^n(n + (n + q_0)\beta^{q_0})]K_0'$$

As $m,n \to \infty$, $K'_{m,n} \to 0$ and so $<Tgu_n>$ and $<Tgv_n>$ are Cauchy sequences. Rest of the proof follows on the same line as in proof of Theorem 2.2, by taking into consideration the fact that $\text{minimum}\{\beta_2, \beta_3\} < \frac{1}{s}$.
The next result can be proved in a similar way as in Theorem 2.1 and 2.3 and so we omit the proof.

**Theorem 2.4.** Theorem 2.1 with condition 2.1 replaced with the following: There exist \( \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6 \) in the interval \( \{0,1\} \), such that \( \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 < 1 \), minimum \{\beta_3 + \beta_4, \beta_5 + \beta_6\} < \frac{1}{3} \) and for all \( u, v, w, z \in X \)

\[
d_r(TS(u,v),TS(w,z)) \leq \beta_1 d_r(Tgu,Tgw) + \beta_2 d_r(Tgv,Tgz) + \\
\beta_3 d_r(Tgu,TS(u,v)) + \beta_4 d_r(Tgv,TS(v,u)) + \beta_5 d_r(Tgz,TS(w,z)) + \beta_6 d_r(Tgw,TS(z,w))
\]

Taking \( T \) to be the identity mapping in Theorems 2.1, 2.2, 2.3 and 2.4 we have the following respective corollaries:

**Corollary 2.5.** Let \((X,d)\) be a RbMS(s), \( S: X \times X \to X \) and \( g: X \to X \) be mappings such that \( S(X \times X) \subseteq g(X) \) and \( g(X) \) is complete. Suppose there exist real numbers \( \lambda, \mu, \nu \) with \( 0 < \lambda < 1, 0 \leq \mu, \nu \leq 1 \), minimum \{\lambda \mu, \lambda \nu\} < \frac{1}{3} \) such that for all \( u, v, w, z \in X \) the following holds:

\[
d_r(S(u,v),S(w,z)) \leq \lambda \max\{d_r(gu,gw),d_r(gv,gz),\mu d_r(gu,S(u,v)),\nu d_r(gv,S(v,u))\}
\]

(2.25)\( v d_r(gw,S(w,z)), v d_r(gz,S(z,w))\})

Then \( S \) and \( g \) has a coupled coincident point. Further if \( S \) and \( g \) are weakly compatible then there exist a unique common coupled fixed point for \( S \) and \( g \). Moreover for some arbitrary \((u_0,v_0) \in X \times X\), the iterative sequences \((gu_n, < gv_n >)\) defined by \( gu_n = S(u_{n-1}, v_{n-1}) \) and \( gv_n = S(v_{n-1}, u_{n-1}) \) converges to the unique common coupled fixed point.

**Corollary 2.6.** Corollary 2.5 with condition 2.25 replaced with the following:

\[
d_r(TS(u,v),TS(w,z)) + d_r(TS(v,u),TS(z,w)) \leq \lambda \max\{d_r(gu,gw),d_r(gv,gz),\mu d_r(gu,S(u,v)),\nu d_r(gv,S(v,u))\}
\]

(2.26)\( \mu (d_r(gu,S(u,v)) + d_r(gv,S(v,u))), \nu (d_r(gw,TS(w,z)) + d_r(gz,TS(z,w)))\})

**Corollary 2.7.** Corollary 2.5 with condition 2.25 replaced with the following: There exist \( \beta_1, \beta_2, \beta_3 \) in the interval \( \{0,1\} \), such that \( \beta_1 + \beta_2 + \beta_3 < 1 \), minimum \{\beta_2, \beta_3\} < \frac{1}{3} \) and for all
u, v, w, z ∈ X

\[ d_r(S(u, v), S(w, z)) + d_r(S(v, u), S(z, w)) \leq \beta_1(d_r(gu, gw) + d_r(gv, gz)) + \]

\[ \beta_2(d_r(gu, S(u, v)) + d_r(gv, S(v, u)) + \beta_3(d_r(gw, S(w, z)) + d_r(gz, S(z, w))) \]

(2.27)

**Corollary 2.8.** Corollary 2.5 with condition 2.25 replaced with the following: There exist \( \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6 \) in the interval \([0,1)\), such that \( \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 < 1 \), minimum\( \{\beta_3 + \beta_4, \beta_5 + \beta_6\} < \frac{1}{5} \) and for all \( u, v, w, z \in X \)

\[ d_r(S(u, v), S(w, z)) \leq \beta_1d_r(gu, gw) + \beta_2d_r(gv, gz) + \]

(2.28) \( \beta_3d_r(gu, S(u, v)) + \beta_4d_r(gv, S(v, u)) + \beta_5d_r(gw, S(w, z)) + \beta_6d_r(gz, S(z, w)) \)

**Remark 2.9.** Since every \( b \)-metric space is a rectangular \( b \)-metric space, we note that Theorem 2.1 is a substantial generalisation of Theorem 2.2 of Ramesh and Pitchamani [13]. In fact we donot require continuity and sub sequential convergence of the function \( T \).

**Remark 2.10.** Note that condition 2.1 of Gu [2] implies 2.27 and hence Corollary 2.7 gives an improved version of Theorem 2.1 of Gu [2].

**Example 2.11.** Let \( X = [0, 1] \), \( d(x, y) = |x - y| \).

\[ Tx = \begin{cases} 
  x^2, & \text{if } x \in [0, \frac{1}{2}] \\
  \frac{x^2}{2}, & \text{if } x \in (\frac{1}{2}, 1] 
\end{cases} \]

\[ gx = \begin{cases} 
  \frac{x^2}{2}, & \text{if } x \in [0, \frac{1}{2}] \\
  x^2, & \text{if } x \in (\frac{1}{2}, 1] 
\end{cases} \]

\[ S(x, y) = \sqrt{\frac{x^{16} + y^{16}}{8}}. \]

Then \( T, S \) and \( g \) satisfies conditions of Theorem 2.1 and \((0, 0)\) is the unique common coupled fixed point of \( S \) and \( g \). Note that \( T \) is not continuous.
3. An Application to Integral Equation

In this section, we apply Theorem 2.1 to study the existence and uniqueness of solutions of a system of nonlinear integral equations.

Let \( X = C[0,A] \) be the space of all continuous real valued functions defined on \([0,A], A > 0\). We consider the following system of nonlinear integral equations, for \( t \in [0,A] \)

\[
\begin{align*}
x(t) &= \int_0^A G(t,r)f(t,x(r),y(r))dr + K(t) \\
y(t) &= \int_0^A G(t,r)f(t,y(r),x(r))dr + K(t)
\end{align*}
\]

(3.1)

where \( f : [0,A] \times R \times R \to R \) and \( G : [0,A] \times [0,A] \to R \) and \( K \in C([0,A]). \) Now suppose \( F : X \times X \to X \) be given by

\[
F(x(t),y(t)) = \int_0^A G(t,r)f(t,x(r),y(r))dr + K(t).
\]

\[
F(y(t),x(t)) = \int_0^A G(t,r)f(t,y(r),x(r))dr + K(t).
\]

Then the system of nonlinear integral equations 3.1 is equivalent to the coupled fixed point problem \( F(x,y) = x, F(y,x) = y. \)

**Theorem 3.1.** Suppose that the following hold:

(i) \( G : [0,A] \times [0,A] \to R \) and \( f : [0,A] \times R \times R \to R \) are continuous functions.

(ii) \( k \in C([0,A]). \)

(iii) For all \( x, y, u, v \in X \) and \( t \in [0,A], \) we can find a function \( g : X \to X \) and real numbers \( s \geq 1, \lambda, \mu, \nu \) with \( 0 \leq \lambda < 1, 0 \leq \mu, \nu \leq 1, \) minimum \( \{\lambda \mu, \lambda \nu\} < \frac{1}{3^s} \) satisfying

(iiiia):

\[
| f(t,x(r),y(r)) - f(t,u(r),v(r)) |^s 
\]

\[
\leq \lambda \max\{ | g(x(r)) - g(u(r)) |^s, | g(y(r)) - g(v(r)) |^s, \\
\mu | g(x(r)) - F(x(r),y(r)) |^s, \mu | g(y(r)) - F(y(r),x(r)) |^s, \\
\nu | g(u(r)) - F(u(r),v(r)) |^s, \nu | g(v(r)) - F(v(r),u(r)) |^s \}.
\]
and

\[(iiiib): \quad F(g(x(t)), g(y(t))) = g(F(x(t), y(t))) \text{ whenever } F(x(t), y(t)) = g(x(t)) \text{ and } F(y(t), x(t)) = g(y(t)).\]

(iv) \[\sup_{t \in [0, A]} \int_0^A |g(t, r)|^r \, dr \leq \frac{1}{\lambda^r}.

Then 3.1 has a unique solution in \( C[0, A] \). Moreover, for some arbitrary \( x_0(t), y_0(t) \) in \( X \), the sequence \( \{< g_{x_n}(t) >, < g_{y_n}(t) >\} \) defined by

\[
g_{x_n}(t) = \int_0^A G(t, r) f(t, x_{n-1}(r), y_{n-1}(r)) \, dr + K(t)
\]

(3.2)

\[
g_{y_n}(t) = \int_0^A G(t, r) f(t, y_{n-1}(r), x_{n-1}(r)) \, dr + K(t)
\]

converges to the unique solution.

**Proof:** Define \( d_r : X \times X \to R \) such that for all \( x, y \in X \),

\[
d_r(x, y) = \sup_{t \in [0, A]} |x(t) - y(t)|^r
\]

(3.3)

Clearly \( d_r \) is a \( RbMS(3^{r-1}) \).

For some \( r \in [0, A] \), we have

\[
d_r(F(x, y), F(u, v)) = |F(x, y)(t) - F(u, v)(t)|^r
\]

\[
= \left| \left[ \int_0^A G(t, r) f(t, x(r), y(r)) \, dr + g(t) \right] - \left[ \int_0^A G(t, r) f(t, u(r), v(r)) \, dr + g(t) \right] \right|^r
\]

\[
\leq \int_0^A |G(t, r)|^r |f(t, x(r), y(r)) - f(t, u(r), v(r))|^r \, dr
\]

\[
\leq \left( \int_0^A |G(t, r)|^r \, dr \right) \lambda^r \max\{ |g(x(r)) - g(u(r))|^r, |g(y(r)) - g(v(r))|^r, |\mu| |g(x(r)) - F(x(r), y(r))|^r, |\mu| |g(y(r)) - F(y(r), x(r))|^r, |\nu| |g(u(r)) - F(u(r), v(r))|^r, |\nu| |g(v(r)) - F(v(r), u(r))|^r \}
\]

\[
\leq \left( \int_0^A |G(t, r)|^r \, dr \right) \lambda^r \max\{ d_r(x, u), d_r(y, v), \mu d_r(g(x), F(x, y)), \mu d_r(g(y), F(y, x)), \nu d_r(g(u), F(u, v)), \nu d_r(g(v), F(v, u)) \}
\]

\[
\leq \lambda \max\{ d_r(x, u), d_r(y, v), \mu d_r(g(x), F(x, y)), \mu d_r(g(y), F(y, x)), \nu d_r(g(u), F(u, v)), \nu d_r(g(v), F(v, u)) \}
\]

\[
\nu d_r(g(u), F(u, v)), \nu d_r(g(v), F(v, u))\}
Thus we have
\[
d_r(F(x,y), F(u,v)) = \sup_{t \in [0,A]} |F(x,y)(t) - F(u,v)(t)|^s
\]
\[
\leq \lambda \max\{d_r(x,u), d_r(y,v), \mu d_r(g(x), F(x,y)), \mu d_r(g(y), F(y,x)),
\nu d_r(u), F(u,v)), \nu d_r(v), F(v,u))\}
\]

This shows that contractive condition of Theorem 2.1 holds. Therefore, by Theorem 2.1 \(F\) has a unique coupled fixed point \((x', y') \in C([0,A] \times C([0,A])\) which is the unique solution of 3.1 and the sequence \(\{<g \cdot x_n(t)>, <g \cdot y_n(t)>, \}\) defined by 3.2 converges to the unique solution of the system of integral equations 3.1.

**Remark 3.2.** Condition (iv) of Theorem 3.1 above is weaker than the corresponding conditions used in similar theorems of [13] and [3].

**Example 3.3.** Let \(X = C[0,1]\) be the space of all continuous real valued functions defined on \([0,1]\) and define \(d_3 : X \times X \to R\) such that for all \(x, y \in X,\)
\[
d_3(x,y) = \sup_{t \in [0,1]} |x(t) - y(t)|^2
\]

Clearly \(d_3\) is a rectangular \(b\)-metric with coefficient 3. Now consider the functions \(f : [0,1] \times R \times R \to R\) given by \(f(t,x,y) = t^2 + \frac{9}{20}x + \frac{8}{20}y, G : [0,1] \times [0,1] \to R\) given by \(G(t,r) = \frac{\sqrt{45(t+r)}}{10}, K \in C([0,1])\) given by \(K(t) = t.\) Then the system of non linear integral equations 3.1 becomes
\[
x(t) = t + \int_0^1 \frac{\sqrt{45(t+r)}}{10} (t^2 + \frac{9}{20}x(r) + \frac{8}{20}y(r))dr
\]
\[
y(t) = t + \int_0^1 \frac{\sqrt{45(t+r)}}{10} (t^2 + \frac{9}{20}y(r) + \frac{8}{20}x(r))dr
\]

Then
\[
| f(t,x,y) - f(t,u,v) |^2 = | \frac{9}{20}(x-u) + \frac{8}{20}(y-v) |^2
\]
\[
\leq | Max\{\frac{9}{10}(x-u), \frac{8}{10}(y-v)\} |^2
\]
\[
\leq \frac{81}{100} Max\{|x-u|^2, |y-v|^2\}
\]
Also

\[ \sup_{t \in [0,1]} \int_0^1 |G(t, r)|^2 \, dr = \int_0^1 \frac{45}{100}(t + r)^2 \, dr = 1.125 \]

We see that all conditions of Theorem 3.1 are satisfied, with \( \lambda = \frac{81}{100}, \mu = 0, \nu = 0, s = 2 \) and \( g = I_X \) (Identity mapping). Hence Theorem 3.1 ensures a unique solution of the system of non linear integral equations 3.5. Now for \( x_0(t) = 1 \) and \( y_0(t) = 0 \), we construct the sequence \( \{< x_n(t) >, < y_n(t) >\} \), given by

\[
\begin{align*}
  x_n(t) &= t + \int_0^1 \frac{\sqrt{45}(t + r)}{10} \left( t^2 + \frac{9}{20} x_{n-1}(r) + \frac{8}{20} y_{n-1}(r) \right) dr \\
  y_n(t) &= t + \int_0^1 \frac{\sqrt{45}(t + r)}{10} \left( t^2 + \frac{9}{20} y_{n-1}(r) + \frac{8}{20} x_{n-1}(r) \right) dr
\end{align*}
\]

(3.6)

Using MATLAB we see that above sequence converges to \( \{0.6708t^3 + 0.3354t^2 + 2.2339t + 0.7677, 0.6708t^3 + 0.3354t^2 + 2.2339t + 0.7677\} \) and this is the unique solution of the system of non linear integral equations 3.5. The convergence table is as given below.
The author(s) declare that there is no conflict of interests.

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### CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

### Remark 3.4

In example 3.3 above we see that \( \sup_{t \in [0,1]} \int_0^1 | G(t,r) |^2 dr = \int_0^1 \frac{45}{100} (t + r)^2 dr = 1.125 > 1 \) and thus condition (v) of Theorem 3.1 of [13] and condition (30) of Theorem 3.1 of [3] is not satisfied.
REFERENCES


