COMMON FIXED POINT THEOREM IN Menger SPACES USING T-NORM T OF HADŽIĆ-TYPE

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Abstract. In this paper, we prove a common fixed point theorem using t-norm T of Hadžić-type (H-type). In fact our result is a generalization of the result of Choudhury and Das [1] under more general condition, that answer to the open problem of Choudhury and Das [1].

Keywords: Menger spaces; φ - contraction; weakly compatible mappings; t-norm T of Hadžić-type.


1. Introduction

In 1922, Banach proved an important result which is the mile stone in the fixed point theory and its applications. A new class of fixed point problems in metric spaces was addressed by Khan et al. [4]. They proved fixed point theorem for mappings satisfying certain inequalities involving the altering distances function.

In 1942, Menger [5] introduced the notion of probabilistic metric space or statistical metric space, which is in fact, a generalization of metric space. The idea in probabilistic metric space is to associate a distribution function with a point pairs, say (p,q), denoted by F(p,q;t) where t > 0 and identify this function as the probability that distance between p and q is less than t. Sehgal and Reid A.T.Bharucha [12] initiated the study of contraction mapping theorems in PM-spaces. Subsequently, several contraction mapping theorems for different variants of

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commuting and compatible mappings have been proved in PM-spaces. Various aspects of this theory have been elaborately discussed in the book of Hadžić and Pap [3].

Recently Choudhury et. al. [1] extended the idea of altering distances in probabilistic metric spaces and proved a contraction principle in Menger spaces using t-norm $T_M$ given by $T_M(a, b) = \min\{a, b\}$ and put an open problem that whether contraction Principle is valid for any other choice of the t-norm.

Now in this paper, we prove a common fixed point theorem using t-norm $T$ of Hadžić-type (H-type for short) that answer to the open problem of Choudhury and Das [1].

2. Preliminaries

First, we recall that a real valued function defined on the set of real numbers is known as a distribution function if it is non-decreasing, left continuous and $\inf f(x) = 0$, $\sup f(x) = 1$.

An example of a distribution function is the Heavy side function $H(x)$, defined by

$H(x) = 0$ if $x \leq 0$ and $H(x) = 1$ if $x > 0$.

**Definition 2.1.**[3] A mapping $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous t-norm if the following conditions are satisfied:

- $T(a, 1) = a$ for every $a \in [0, 1]$;
- $T(a, b) = T(b, a)$ for every $a, b \in [0, 1]$;
- $T(T(a, b), c) = T(a, T(b, c))$;
- $T(a, b) \leq T(c, d)$, for $a \leq c, b \leq d$.

Basic examples of t-norm are the Lukasiewicz t-norm $T_L$, $T_L(a, b) = \max(a + b - 1, 0)$, t-norm $T_P$, $T_P(a, b) = ab$, and t-norm $T_M$, $T_M(a, b) = \min\{a, b\}$, $T_D(x, y) = \begin{cases} \min(x, y) \text{ if } \max(x, y) = 1, \\ 0 \text{ otherwise.} \end{cases}$

**Definition 2.2.**[3] A Menger space is a triplet $(X, F, T)$, where $X$ is a non-empty set, $F$ is a function defined on $X \times X$ to $L_+$ (set of all distribution functions) which satisfies the following conditions:

(i) $F_{xy}(0) = 0$,

(ii) $F_{xy}(s) = 1$ for all $s > 0$ iff $x = y$,

(iii) $F_{xy}(s) = F_{yx}(s)$,

(iv) $F_{xy}(u + v) \geq T(F_{xz}(u), F_{zy}(v))$ for all $u, v \geq 0$ and $x, y, z \in X$ where $T$ is a t-norm.

For a given metric space $(X, d)$ with usual metric $d$, one can put $F_{xy}(t) = H(t - d(x, y))$, where $H$ is defined as:
\[ H(s) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s \leq 0 \end{cases} \]

and t-norm \( T \) is defined as \( T(a, b) = \min\{a, b\} \).

If \((X, F, T)\) is a Menger space with continuous t-norm then the topology induced by the family \( \{S_{\epsilon, \lambda}(p) : p \in X, \epsilon > 0, \lambda > 0\} \) is called the \((\epsilon, \lambda)\) - topology, where \( S_{\epsilon, \lambda}(p) = \{q \in X : F_{pq}(\epsilon) > 1 - \lambda\} \) is called the \((\epsilon, \lambda)\) - neighborhood of \( p \).

A sequence \( \{x_n\} \subset X \) is said to be

(i) converge to some point \( x \in X \) in the \((\epsilon, \lambda)\) - topology if and only if given \( \epsilon > 0, \lambda > 0 \) we can find a positive integer \( N_{\epsilon, \lambda} \) such that, for all \( n > N_{\epsilon, \lambda} \), \( F_{x_n x}(\epsilon) \geq 1 - \lambda \).

(ii) a Cauchy sequence in \( X \) if given \( \epsilon > 0, \lambda > 0 \) there exists a positive integer \( N_{\epsilon, \lambda} \) such that \( F_{x_n x_m}(\epsilon) \geq 1 - \lambda \) for all \( m, n > N_{\epsilon, \lambda} \).

A Menger space \((X, F, T)\) is said to be complete if every Cauchy sequence is convergent.

In 1979, Hadžić [2] introduced a special class of t-norms (called as a Hadžić-type norm) as follows:

**Definition 2.3.** [2] Let \( T \) be a t-norm and let \( T_n : [0, 1] \to [0, 1] \) \((n \in \mathbb{N})\) be defined in the following way,

\[ T_1(x) = T(x, x), \quad T_{n+1}(x) = T(T_n(x), x) \quad (n \in \mathbb{N}, \ x \in [0, 1]). \]

We say that the t-norm \( T \) is of H-type if \( T \) is continuous and the family \( \{T_n(x), n \in \mathbb{N}\} \) is equicontinuous at \( x = 1 \).

The family \( \{T_n(x), n \in \mathbb{N}\} \) is equicontinuous at \( x = 1 \), if for every \( \lambda \in (0, 1) \) there exists \( \delta(\lambda) \in (0, 1) \) such that the following implication holds:

\[ x > 1 - \delta(\lambda) \text{ implies } T_n(x) > 1 - \lambda \text{ for all } n \in \mathbb{N}. \]

A trivial example of t-norm of H-type is \( T = T_M \) \((T_M(a, b) = \min\{a, b\})\).

**Remark 2.4.** Every t-norm \( T_M \) is of Hadžić-type but converse need not be true, see [3].

There is a nice characterization of continuous t-norm \( T \) of H-type[8] as given below:

(i) If there exists a strictly increasing sequence \( \{b_n\}_{n \in \mathbb{N}} \) in \([0, 1]\) such that \( \lim_{n \to \infty} b_n = 1 \) and \( T(b_n, b_n) = b_n \forall n \in \mathbb{N} \), then \( T \) is of Hadžić-type.

(ii) If \( T \) is continuous and \( T \) is of Hadžić-type, then there exists a sequence \( \{b_n\}_{n \in \mathbb{N}} \) as in (i).

**Definition 2.5.** [3] If \( T \) is a t-norm and \((x_1, x_2, ..., x_n) \in [0, 1]^n \) \((n \in \mathbb{N})\), then \( T_{i=1}^n x_i \) is defined recurrently by 1, if \( n = 0 \) and \( T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n) \) for all \( n \geq 1 \). If \( \{x_i\}_{i \in \mathbb{N}} \) is a sequence of numbers from \([0, 1]\), then \( T_{i=1}^\infty x_i \) is defined as \( \lim_{n \to \infty} T_{i=1}^n x_i \) (this limit always exists) and \( T_{i=1}^\infty x_i \)
as $T^\infty_{i=1} x_{n+i}$. In fixed point theory in probabilistic metric spaces there are of particular interest t-norms $T$ and sequences $\{x_n\} \subseteq [0,1]$ such that $\lim_{n \to \infty} x_n = 1$ and $\lim_{n \to \infty} T^\infty_{i=1} x_{n+i} = 1$.

In 1972, Sehgal and Bharucha-Reid [12] introduced the idea of contraction in PM space.

**Definition 2.6.** Probabilistic $q$-contraction [3] Let $(X, F)$ be a probabilistic metric space. A mapping $f : X \to X$ is a probabilistic $q$-contraction ($q \in (0, 1)$) if $F_{f u f v}(x) \geq F_{u v}(x/q)$ for every $u, v \in X$ and every $x \in R$.

The following Theorem was proved by Sehgal and Bharucha-Reid [12].

**Theorem 2.7.** Let $(X, F_{T_M})$ be a complete Menger space where $T_M(a, b) = \min\{a, b\}$ and $f : X \to X$ is a probabilistic $q$-contraction. Then there exist a unique fixed point $x$ of the mapping $f$ and $x = \lim_{n \to \infty} f^n p$ for every $p \in X$.

In 1984, Khan et al. [4] introduced a new category of contractive fixed point problems using a control function (altering distance function) that alters the distance between two points in a metric space.

**Definition 2.8.** [4] An altering distance function is a function $\psi : [0,\infty) \to [0,\infty)$ such that

(i) which is monotone increasing and continuous and

(ii) $\psi(t) = 0$ if and only if $t = 0$.

Khan et al. [4] proved the following result using altering distance function.

**Theorem 2.9.** [4] Let $(X, d)$ be a complete metric space and $\psi$ be an altering distance function. Let $f : X \to X$ be a self mapping which satisfies the following inequality:

$\psi(d(fx, fy)) \leq c\psi(d(x, y))$, for all $x, y \in X$ and for some $0 < c < 1$.

Then $f$ has a unique fixed point.

In fact, Khan et al. [4] proved a more general fixed point theorem (Theorem 2 in [4]) of which the above result is a corollary. This result was further generalized in a different direction by various authors. One can refer to [7], [9] and [10].

Recently, Choudhury et al. [1] extended the idea of altering distance function in Menger spaces and obtained fixed point results for self-mapping using $\phi$ function.

**Definition 2.10.** A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to satisfy the condition $(\phi)$ if it satisfies the following conditions:

(i) $\phi(t) = 0$ if and only if $t = 0$,

(ii) $\phi(t)$ is increasing and $\phi(t) \to \infty$ as $t \to \infty$,

(iii) $\phi$ is left continuous in $(0, \infty)$,
(iv) \( \varphi \) is continuous at 0,
(v) \( \varphi \) is superadditive, that is, \( \varphi(x + y) \geq \varphi(x) + \varphi(y) \), for all \( x, y \geq 0 \).

**Definition 2.11.** Let \((X, F, T)\) be a Menger space. A self map \( f : X \to X \) is said to be \( \varphi \)-contractive if

\[ (*) \quad F_{fx} \geq T \left( \varphi \left( \frac{t}{c} \right) \right), \]

where \( 0 < c < 1, x, y \in X \) and \( t > 0 \) and the function \( \varphi \) satisfy the condition \( (\varphi) \).

**Definition 2.12.** Two maps \( f \) and \( g \) are said to be weakly compatible if they commute at their coincidence points.

**Example 2.13.** Let \( X = [0, 1] \) be equipped with the usual metric \( d(x, y) = |x - y| \).

Define \( f, g : [0, 1] \to [0, 1] \) by

\[
\begin{align*}
  f(x) &= \begin{cases} 
  0 & \text{if } x = 0, \\
  0.15 & \text{if } x > 0 
  \end{cases} \\
  g(x) &= \begin{cases} 
  0 & \text{if } x = 0, \\
  0.35 & \text{if } x > 0 
  \end{cases}
\end{align*}
\]

Then, 0 is a coincidence point and \( fg \) 0 = \( gf \) 0, showing that \( f, g \) are weakly compatible maps on \( [0, 1] \).

**Proposition 2.14.** Let \( (x_n, n \in \mathbb{N}) \) be a sequence of numbers in \([0, 1]\) such that \( \lim_{n \to \infty} x_n = 1 \) and the t-norm T is of H-type, then

\[
\lim_{n \to \infty} T_{\{1\}} = \lim_{n \to \infty} T_{\{\infty\}} = 1.
\]

Throughout this paper, \((X, F, T)\) will denote a Menger space which satisfies the condition \( \lim_{t \to \infty} F_{xy}(t) = 1 \) for all \( x, y \in X \) and \( t > 0 \).

3. **Main Result**

Recently, Choudhury et al. [1] proved the following fixed point theorem using continuous t-norm \( T_M \), which is strongest t-norm.

**Theorem 3.1.** Let \((X, F, T_M)\) be a Menger space with continuous t-norm \( T_M \) and \( f : X \to X \) be \( \varphi \)-contractive satisfying \( (*) \). Then \( f \) has a fixed point.

Now we prove our main result for a pair of weakly compatible maps using continuous t-norm \( T \) of H-type.

**Theorem 3.2.** Let \((X, F, T)\) be a complete Menger space with continuous t-norm \( T \) of H-type and let \( f, g \) be two self-mappings on \( X \) satisfy the following inequality:

\[
\begin{align*}
  (3.1) & \quad f(X) \subseteq g(X), \\
  (3.2) & \quad \text{any one of } f(X) \text{ and } g(X) \text{ is complete},
\end{align*}
\]
(3.3) $F_{f \times f \times y}(\varphi(t)) \geq F_{g \times g \times y}(\varphi(\frac{t}{c}))$, where $0 < c < 1$, $x, y \in X$ and $t > 0$ and the function $\varphi$ satisfy the condition $(\varphi)$.

For any $x_0 \in X$, the sequence $\{y_n\}$ in $X$ be constructed as follows: $y_n = f_{x_n} = g_{x_{n+1}}$, $n = 0, 1, 2, 3, \ldots$ and for $\mu \in (c, 1)$ the following condition holds:

$$\lim_{n \to \infty} T^{(\frac{1}{\mu})} F_{y_0 y_1} = 1.$$  

Then $f$ and $g$ have a unique common fixed point provided $f$ and $g$ are weakly compatible on $X$.

**Proof:** In view of the properties of $(\varphi)$-function, for $u > 0$ we can find a positive number $r$ such that $u > \varphi(r)$. For $u > 0$, we have

$$F_{y_n y_{n+1}}(u) \geq F_{f_{x_n} f_{x_{n+1}}} (\varphi(r))$$

$$\geq F_{g_{x_n} g_{x_{n+1}}} (\varphi(\frac{r}{c}))$$

$$= F_{y_{n-1} y_n} (\varphi(\frac{r}{c}))$$

$$= F_{f_{x_{n-1}} f_{x_n}} (\varphi(\frac{r}{c}))$$

$$\geq F_{g_{x_{n-1}} g_{x_n}} (\varphi(\frac{r}{c^2}))$$

$$= F_{y_{n-2} y_{n-1}} (\varphi(\frac{r}{c^2}))$$

$$\ldots$$

$$\geq F_{y_0 y_1} (\varphi(\frac{r}{c^n})).$$

Therefore,

$$F_{y_n y_{n+1}} (u) \geq F_{y_0 y_1} (\varphi(\frac{r}{c^n})).$$

Proceeding limit as $n \to \infty$, we have $\lim_{n \to \infty} F_{y_n y_{n+1}} (u) = 1$.

We claim that the sequence $\{y_n\}$ is a Cauchy sequence.

Let, $\sigma = \frac{c}{\mu}$, where $\mu \in (c, 1)$ and $c \in (0, 1)$, then $0 < \sigma < 1$, therefore the series $\sum_{i=1}^{\infty} \sigma^i$ is convergent and there exists $m_0 \in N$ such that $\sum_{i=m_0}^{\infty} \sigma^i < 1$. Now for every $m > m_0$ and for every $s \in N$ and in view of $(\varphi)$,

$$u > \varphi(r) > \varphi \left( r \sum_{i=m_0}^{\infty} \sigma^i \right) > \varphi \left( r \sum_{i=m_0}^{m+s} \sigma^i \right)$$

which implies that

$$F_{y_{m+s+1} y_0} (u) > F_{y_{m+s+1} y_{m+s+1}} (\varphi(r))$$

$$\geq F_{y_{m+s+1} y_{m+s+1}} (\varphi \left( r \sum_{i=m}^{m+s} \sigma^i \right))$$

$$\geq T \left( T \ldots T \left( F_{y_{m+s+1} y_{m+s+1}} (\varphi \left( r \sigma^{m+s} \right), F_{y_{m+s} y_{m+s-1}} (\varphi \left( r \sigma^{m+s-1} \right) \right) \right) s-times$$
\[ \ldots, F_{y_{m+1}y_m} \varphi(r \sigma^m)) \]
\[ \geq T(T \ldots T(F_{y_0y_1} \varphi(r \sigma^{m+s})) \ldots, F_{y_0y_1} \varphi(r \sigma^m))) \]
\[ \geq T_{i=m}^{m+s} F_{y_0y_1} \varphi(\frac{r}{\mu^i}) \]
\[ = T_{i=m}^{\infty} F_{y_0y_1} \varphi(\frac{r}{\mu^i}). \]

It is obvious that,
\[ \lim_{n \to \infty} T_{i=n}^{\infty} F_{y_0y_1} \varphi(\frac{1}{\mu^i}) = 1, \] implies that \( \lim_{n \to \infty} T_{i=n}^{\infty} F_{y_0y_1} \varphi(\frac{1}{\mu^i}) = 1, \) and this implies that,
\[ \lim_{n \to \infty} T_{i=n}^{\infty} F_{y_0y_1} \varphi(\frac{r}{\mu^i}) = 1, \] for every \( r > 0. \)

Now for every \( u > 0, \) there exists \( r > 0 \) such that \( u > \varphi(r) > 0, \) there exist \( m_t(\varphi(r), \lambda) \) such that
\[ F_{y_{m+s+1}y_m}(u) > 1 - \lambda, \] for every \( m \geq m_t(\varphi(r), \lambda) \) and every \( s \in \mathbb{N}. \)

This means that the sequence \( \{y_n\} \) is Cauchy sequence. Since either \( f(X) \) or \( g(X) \) is complete, for definiteness assume that \( g(X) \) is complete subspace of \( X \) then the subsequence of \( \{y_n\} \) must get a limit in \( g(X). \) Call it be \( z. \) Let \( p \in g^{-1}z. \) Then \( g \circ p = z \) as \( \{y_n\} \) is a Cauchy sequence containing a convergent subsequence, therefore the sequence \( \{y_n\} \) also convergent implying thereby the convergence of subsequence of the convergent sequence.

Which gives,
\[ \lim_{n \to \infty} y_n = \lim_{n \to \infty} g x_n = \lim_{n \to \infty} f x_n = z. \]

Now we claim that \( fp = z. \)

From the property of \( (\varphi) \), it follows that given \( \varepsilon > 0, \) we can find \( \varepsilon_1 > 0 \) such that \( \varepsilon > \varphi(\varepsilon_1) > 0. \)

Then for all \( n = 0, 1, 2, 3, \ldots, \)
\[ F_{fpz}(\varepsilon) \geq T(F_{fpyn} \varphi(\varepsilon(1)), F_{ynz}(\varepsilon - \varphi(\varepsilon(1)))) \]
\[ = T(F_{fpfx_n} \varphi(\varepsilon(1)), F_{ynz}(\varepsilon - \varphi(\varepsilon(1)))) \]
\[ \geq T(F_{gpgx_n}(\varphi(\frac{\varepsilon_1}{c})), F_{ynz}(\varepsilon - \varphi(\varepsilon(1)))) \]
\[ = T(F_{zyn^{-1}}(\varphi(\frac{\varepsilon_1}{c})), F_{ynz}(\varepsilon - \varphi(\varepsilon(1))). \]

Since \( T \) is continuous, taking limit as \( n \to \infty \) in the above inequality, we have for all \( \varepsilon > 0, \) \( F_{fpz}(\varepsilon) = 1, i.e., fp = z, \) we get \( fp = gp = z, \) since \( f \) and \( g \) are weakly compatible therefore we have \( fg \circ p = gf \circ p, i.e., fz = gz. \)

We claim that \( fz = z, \) from (3.3), we have
\[ F_{fzz}(\varphi(t)) = F_{fzfp}(\varphi(t)) \geq F_{gzz}(\varphi(\frac{t}{c})) = F_{fzfp}(\varphi(\frac{t}{c})) \geq F_{gzz}(\varphi(\frac{t}{c}). \)
Proceeding as above, for any \( t > 0 \), \( F_{fz}(\varphi(t)) \geq F_{fz}(\varphi(\frac{\epsilon_1}{cn})) \rightarrow 1 \) as \( n \rightarrow \infty \), which gives \( fz = z = gz \). Thus \( z \) is a common fixed point of \( f \) and \( g \).

**Uniqueness.**

If possible let \( w \) and \( v \) be two fixed points of \( f \) and \( g \), then in view of \( (\varphi) \) for given \( \epsilon > 0 \), we can find \( \epsilon_1 > 0 \) such that \( \epsilon > \varphi(\epsilon_1) > 0 \). Then one can see that 

\[
F_{wv}(\epsilon) = F_{wfv}(\epsilon)
\geq F_{wfv}(\varphi(\epsilon_1))
\geq F_{gwgv}(\varphi(\frac{\epsilon_1}{c}))
= F_{fwfv}(\varphi(\frac{\epsilon_1}{c}))
\geq F_{gwgv}(\varphi(\frac{\epsilon_1}{c^2}))
= F_{wv}(\varphi(\frac{\epsilon_1}{c^2})).
\]

Proceeding as above, for any \( \epsilon > 0 \), \( F_{wv}(\epsilon) \geq F_{wv}(\varphi(\frac{\epsilon_1}{cn})) \rightarrow 1 \) as \( n \rightarrow \infty \), which gives \( w = v \).

Next we give the following example to validate our result

**Example 3.3.** Let \( X \{a, b, c, d\} \), \( T_M \) is the \( t \)-norm and \( F \) be defined as

\[
F_{ab}(t) = F_{ac}(t) = F_{ad}(t) = \begin{cases} 
0 & \text{if } t \leq 0, \\
0.4 & \text{if } 0 < t < 4, \\
1 & \text{if } t \geq 4.
\end{cases}
\]

Then \( X, F, T_M \) is a complete Menger space.

If we define \( f, g : X \rightarrow X \) as follows:

\( f(a) = d, f(b) = c, f(c) = c, f(d) = d, \) and \( g(a) = d, g(b) = c, g(c) = c, g(d) = c, \) where \( \varphi(t) = t \) and \( c \) is the unique common fixed point of \( f \) and \( g \), then the mappings \( f \) and \( g \) satisfy all the conditions of the Theorem 3.2.

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