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EXTENSION OF PHASE-ISOMETRIES BETWEEN THE UNIT SPHERES OF COMPLEX $\mathscr{L}^\infty(\Gamma)$ SPACES

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Abstract. Let Γ be a nonempty index set, and X, Y are complex $\mathscr{L}^\infty(\Gamma)$-type spaces. $f : S_X, S_Y$ will denote their unit spheres. Give a surjective mapping $f : S_X \to S_Y$ satisfying the functional equation

$$\{\|f(x)+f(y)\|,\|f(x)-f(y)\|\} = \{\|x+y\|,\|x-y\|\} \quad (x,y \in S_X)$$

We show that there exists a function $\varepsilon : S_X \to \{-1,1\}$ such that εf is an isometry. Moreover, this isometry is the restriction of a real linear isometry from X to Y.

Keywords: $\mathscr{L}^\infty(\Gamma)$ spaces; Tingley's problem; phase-isometry; Wigner's theorem.

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1. Introduction

The famous Tingley's problem is important on mathematics. In 1987, Tingley raised a question in [10], that is, let X and Y be normed spaces, S_X and S_Y denote their unit spheres. Suppose $f : S_X \to S_Y$ is a surjective isometry, whether f can be extended to a real-linear (bijective) isometry $F : X \to Y$ between the corresponding space? In [10], Tingley give the positive solution in two finite dimensional Banach spaces, which is $f(-x) = -f(x)$ for every x in the unit spheres of the domain spaces. For the Tingley's problem was attracted much attention, someone established results in a wide range of classical Banach spaces, such as detailed presentation (G. D in

[1]), $\ell^p(\Gamma)$ spaces, where $1 \le p \le \infty$ ($G.D[2,3,4]$), $C_0(L)$ spaces (R. Wang [11]), $\mathscr{L}^p(\Omega.\Sigma,\mu)$ spaces, where $1 \le p \le \infty$ and (Ω,Σ,μ) is a σ-finite measure space (D. Tan in [16, 17] and [18]).

Recently, the Tingley's problem on operator algebras' research was started, like compact linear operators on a complex Hilbert spaces (A.M. Peralta and R. Tanaka in [15]), finite dimensional c^*-algebras and finite VonNeumamn algebras (R. Tanaka in [23]), weakly compact JB^*-triples and atomic JBW^*-triples (F.J. Fernández-Polo, A.M. Peralta in [12, 13, 14]). Other important results may be seen in the references.

Wigner's theorem is another important conclusion related to linear isometries, which also plays a fundamental role in quantum mechanics. Wigner's theorem has may forms, Rätz gives a real version in inner product spaces. It is that suppose X and Y are real inner product spaces, define a mapping $f : X \to Y$, then f satisfies

$$|<f(x),f(y)>| = |<x+y>| \quad (x,y \in X).$$

if and only if there exists a phase function ε take value in module one scalar such that $f(x) = \varepsilon(x)U(x)$, $x \in X$, where U is a linear isometry.

In the complex version, the solution can be considered to phase equivalent to a linear or conjugate linear isometry (see [19]). In 2013, G. Maksa and Z. Páles gave a equation of real version in norm spaces of Wigner's theorem [7]

$$\{\|f(x)+f(y)\|,\|f(x)-f(y)\|\} = \{\|x+y\|,\|x-y\|\} \quad (x,y \in X) \qquad (1)$$

Meanwhile, they asked the following question: whether the result remains positive solution when $f : X \to Y$ of satisfies the equation (1) with X and Y being normed but not necessarily inner product spaces? In the real cases, we have got positive solutions in $\ell^p(\Gamma)$ spaces with $p \ge 1$ and $\mathscr{L}^\infty(\Gamma)$ spaces [20].

Combining with the Tingley's problem and the Wigner's theorem, we begin to consider a question: suppose X and Y are complex Banach spaces, define a mapping $f : S_X \to S_Y$ satisfying the equation (1), where $x,y \in S_X$, is it phase equivalent to an isometry which is just the restriction of a linear isometry from X to Y? The aim of this paper is to answer the question in complex $\mathscr{L}^\infty(\Gamma)$-type spaces. Our most results in this paper comes from [5].

2. RESULTS

Throughout this section, we consider the spaces all over the complex field. Let X and Y be complex Banach spaces, S_X and S_Y will denote their unit spheres respectively. B_X will denote the closed unit ball. Meanwhile, $\mathbb{R}$ will denote the real sets, $\mathbb{C}$ will denote the complex sets and $\mathbb{T}$ will denote the unit sphere of $\mathbb{C}$. In this paper, the symbols Γ, Δ will be used by nonempty sets. For $a, b \in \mathbb{R}$, we write $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

Let Γ will be a nonempty set. The space of all bounded complex-valued functions on an index set Γ equipped with the supremum norm is denoted by $\ell^{\infty}(\Gamma)$ and any of its subspaces containing all $e_{\gamma}'s$ ($\gamma \in \Gamma$) are called $\mathscr{L}^{\infty}(\Gamma)$-type spaces. For example, the space $c_0(\Gamma), c(\Gamma), \ell^{\infty}(\Gamma)$ are $\mathscr{L}^{\infty}(\Gamma)$-type spaces. The $\ell^{\infty}(\Gamma)$-space is

$$\ell^{\infty}(\Gamma) = \{x = \{\xi_{\gamma}\}_{\gamma \in \Gamma} : \ \|x\| = \sup_{\gamma \in \Gamma} |\xi_{\gamma}| < \infty, \ \xi_{\gamma} \in \mathbb{C}, \ \gamma \in \Gamma\}.$$

For arbitrary $x = \{x_{\gamma}\}_{\gamma \in \Gamma} \in \mathscr{L}^{\infty}(\Gamma)$, we write $x = \{x_{\gamma}\}$, and omit the subscripts $\gamma \in \Gamma$ for simplicity of notation. We use Γ_x to express the support of x , i.e.,

$$\Gamma_x = \{\gamma \in \Gamma : x_{\gamma} \neq 0\}.$$

When working with $\mathscr{L}^{\infty}(\Gamma)$ one has to be particulary careful with the meaning of the notations. The e_{γ} is the vector in $\mathscr{L}^{\infty}(\Gamma)$ having 1 at the γ-th entry and otherwise 0. Given $x \in \mathscr{L}^{\infty}(\Gamma)$, we denote the γ-th function value of x by $x_{\gamma} \in \mathbb{C}$. The canonical notion of (algebraic) orthogonality in $\mathscr{L}^{\infty}(\Gamma)$ reads as follows: $x, y \in \mathscr{L}^{\infty}(\Gamma)$ are said to be *orthogonal or disjoint* if $xy = 0$, or equivalently $\Gamma_x \cap \Gamma_y = \emptyset$. The star of x with respect to $S_{\mathscr{L}^{\infty}(\Gamma)}$ is defined by

$$St(x) = \{y : y \in S_{\mathscr{L}^{\infty}(\Gamma)}, \|y + x\| = 2\}.$$

Before proving the main Theorem, we will give some Lemmas.

Lemma 2.1. *Let X and Y be complex Banach spaces. Suppose that $f : S_X \to S_Y$ is a surjective phase-isometry. Then $f(-x) = -f(x)$ for each $x \in S_X$.*

Proof: Fix y in S_X and let $f(y) = -f(x)$, since f is phase-isometry mapping, we have

$$\{\|x + y\|, \|x - y\|\} = \{\|f(x) + f(y)\|, \|f(x) - f(y)\|\} = \{2, 0\},$$

which implies $y \in \{x, -x\}$.

If $y = x$, then $f(x) = -f(x)$, which means $f(x) = 0$, leads to contradiction.

So the only positive solution is $y = -x$. The proof is completed. $\square$

Our next Lemma gives a characterization of norm-one element in $\mathscr{L}^\infty(\Gamma)$ with a single support.

Lemma 2.2. *Let x be a norm-one element in $\mathscr{L}^\infty(\Gamma)$. Then Γ_x is a singleton if and only if the inequality $\|y - x\| \leq 1$ holds for all $y \in St(x)$.*

The idea of the next Lemma comes from [5], whose proof is similar.

Lemma 2.3. *Let $X = \mathscr{L}^\infty(\Gamma)$ and $Y = \mathscr{L}^\infty(\Delta)$. Suppose that $f : S_X \to S_Y$ is a surjective phase-isometry. Then for each $\gamma_0 \in \Gamma$ and $\alpha \in \mathbb{T}$, we have $\Delta_{f(\alpha e_{\gamma_0})} = \Delta_{f(e_{\gamma_0})}$ is a singleton. Moreover, one the following statements holds:*
(1) *$f(\alpha e_{\gamma_0}) = \pm \alpha f(e_{\gamma_0})$ for every $\alpha \in \mathbb{T}$;*
(2) *$f(\alpha e_{\gamma_0}) = \pm \bar{\alpha} f(e_{\gamma_0})$ for every $\alpha \in \mathbb{T}$.*

Proof: We fix $\gamma_0 \in \Gamma$, $\alpha \in \mathbb{T}$. Let us take $x \in S_X$ such that $f(x) \in St(f(\alpha e_{\gamma_0}))$. Since f is a phase-isometry,

$$\|x + \alpha e_{\gamma_0}\| \vee \|x - \alpha e_{\gamma_0}\| = \|f(x) + f(\alpha e_{\gamma_0})\| \vee \|f(x) - f(\alpha e_{\gamma_0})\| = 2,$$

which shows that $x \in \pm St(\alpha e_{\gamma_0})$.

It follows from Lemma 2.2 that

$$\|f(x) - f(\alpha e_{\gamma_0})\| = \|x + \alpha e_{\gamma_0}\| \wedge \|x - \alpha e_{\gamma_0}\| \leq 1,$$

and so $\Delta_{f(\alpha e_{\gamma_0})}$ is a singleton. Clearly,

$$4 = \|\alpha e_{\gamma_0} + e_{\gamma_0}\|^2 + \|\alpha e_{\gamma_0} - e_{\gamma_0}\|^2 = \|f(\alpha e_{\gamma_0}) + f(e_{\gamma_0})\|^2 + \|f(\alpha e_{\gamma_0}) - f(e_{\gamma_0})\|^2,$$

which assures that $\Delta_{f(\alpha e_{\gamma_0})} = \Delta_{f(e_{\gamma_0})}$ is a singleton. Suppose that $f(\alpha e_{\gamma_0}) = \beta f(e_{\gamma_0})$ for some $\beta \in \mathbb{T}$. Then it follows from

$$\begin{aligned} \{|\alpha + 1|, |\alpha - 1|\} &= \{\|\alpha e_{\gamma_0} + e_{\gamma_0}\|, \|\alpha e_{\gamma_0} - e_{\gamma_0}\|\} \\ &= \{\|f(\alpha e_{\gamma_0}) + f(e_{\gamma_0})\|, \|f(\alpha e_{\gamma_0}) - f(e_{\gamma_0})\|\} \\ &= \{|\beta + 1|, |\beta - 1|\} \end{aligned}$$

that $\beta \in \{\pm\alpha, \pm\overline{\alpha}\}$.

We have shown above that $f(ie_{\gamma_0}) = \pm i f(e_{\gamma_0})$ and $f(-e_{\gamma_0}) = -f(e_{\gamma_0})$ (by Lemma 2.3). Let us assume that $f(\alpha e_{\gamma_0}) = \pm\alpha f(e_{\gamma_0})$ and $f(\beta e_{\gamma_0}) = \pm\overline{\beta} f(e_{\gamma_0})$ for some $\alpha, \beta \in \mathbb{T} \setminus \{\pm 1, \pm i\}$. By the assumptions we have

$$\begin{aligned} & 2 + 2|Re(\alpha\overline{\beta})| = \|\alpha e_{\gamma_0} + \beta e_{\gamma_0}\|^2 \vee \|\alpha e_{\gamma_0} - \beta e_{\gamma_0}\|^2 \\ = \ & \|f(\alpha e_{\gamma_0}) + f(\beta e_{\gamma_0})\|^2 \vee \|f(\alpha e_{\gamma_0}) - f(\beta e_{\gamma_0})\|^2 \\ = \ & |\alpha + \overline{\beta}|^2 \vee |\alpha - \overline{\beta}|^2 = 2 + 2|Re(\alpha\beta)|, \end{aligned}$$

equivalently

$$|Re(\alpha)Re(\beta) + Im(\alpha)Im(\beta)| = |Re(\alpha)Re(\beta) - Im(\alpha)Im(\beta)|$$

which is impossible because $\alpha, \beta \in \mathbb{T} \setminus \{\pm 1, \pm i\}$. It follows that $f(\alpha e_{\gamma_0}) = \pm\alpha f(e_{\gamma_0})$ for all $\alpha \in \mathbb{T}$, or $f(\alpha e_{\gamma_0}) = \pm\overline{\alpha} f(e_{\gamma_0})$ for all $\alpha \in \mathbb{T}$. □

The next result describes the behaviour of surjective phase-isometries on complex $\mathscr{L}^\infty(\Gamma)$-type spaces.

Proposition 2.4. *Let $X = \mathscr{L}^\infty(\Gamma)$ and $Y = \mathscr{L}^\infty(\Delta)$. Suppose that $f : S_X \to S_Y$ is a surjective phase-isometry. Then there exists a bijection $\sigma : \Gamma \to \Delta$ such that for every $x = \{x_\gamma\} \in S_X$, we have $f(x) = \{y_{\sigma(\gamma)}\} \in S_Y$, where $\frac{y_{\sigma(\gamma)}}{|y_{\sigma(\gamma)}|} e_{\sigma(\gamma)} = \pm f(\frac{x_\gamma}{|x_\gamma|} e_\gamma)$ for every $\gamma \in \Gamma_x$ and $y_{\sigma(\gamma)} = 0$, $\gamma \notin \Gamma_x$.*

Proof: We can define a mapping $\sigma : \Gamma \to \Delta$ by Lemma 2.3 (2) that

$$f(e_\gamma) = \alpha_\gamma e_{\sigma(\gamma)}, \quad \alpha_\gamma \in \mathbb{T}, \ \forall \gamma \in \Gamma.$$

First, We shall show that σ is bijective. Let us take $\gamma_1, \gamma_2 \in \Gamma$ and write $f(e_{\gamma_1}) = \alpha_{\gamma_1} e_{\sigma(\gamma_1)}$ and $f(e_{\gamma_2}) = \alpha_{\gamma_2} e_{\sigma(\gamma_2)}$ with $\alpha_{\gamma_1}, \alpha_{\gamma_2} \in \mathbb{T}$. If $\gamma_1 \neq \gamma_2$, then

$$\begin{aligned} & \|f(e_{\gamma_1}) + f(e_{\gamma_2})\|^2 + \|f(e_{\gamma_1}) - f(e_{\gamma_2})\|^2 \\ = \ & \|e_{\gamma_1} + e_{\gamma_2}\|^2 + \|e_{\gamma_1} - e_{\gamma_2}\|^2 \\ = \ & 1 + 1 = 2 \end{aligned}$$

This implies that $\sigma(\gamma_1) \neq \sigma(\gamma_2)$, and thus σ is injective. Next, we would consider that σ is surjective. Indeed, given $\delta \in \Delta$, by applying Lemma 2.3 (2) to f^{-1}, we can find some $\gamma \in \Gamma$ and $\alpha \in \mathbb{T}$ such that $f(\alpha e_\gamma) = e_\delta$. Therefore, σ is a surjective mapping.

Set

$$\Gamma_1 := \{\gamma \in \Gamma : f(\alpha e_\gamma) = \pm\alpha f(e_\gamma), \forall \alpha \in \mathbb{T}\}$$

$$\Gamma_2 := \{\gamma \in \Gamma : f(\alpha e_\gamma) = \pm\overline{\alpha} f(e_\gamma), \forall \alpha \in \mathbb{T}\}.$$

From Lemma 2.3(b), we know that $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. Fix $\gamma \in \Gamma_x \cap \Gamma_1$, the proof of the case of $\gamma \in \Gamma_x \cap \Gamma_2$ holds is same to it. We have shown that

$$f(\alpha e_\gamma) = \pm\alpha f(e_\gamma) = \pm\alpha\alpha_\gamma e_{\sigma(\gamma)}$$

for some $\alpha_\gamma \in \mathbb{T}$, and so $f(\frac{x_\gamma}{|x_\gamma|}e_\gamma) = se_{\sigma(\gamma)}$, where $s = \pm\frac{x_\gamma}{|x_\gamma|}\alpha_\gamma$. What's more, for every $x = \{x_\gamma\} \in S_X$, we have $f(x) = \{y_{\sigma(\gamma)}\} \in S_Y$. Therefore,

$$\begin{aligned} & |x_\gamma| + 1 = \|x + \frac{x_\gamma}{|x_\gamma|}e_\gamma\| \vee \|x - \frac{x_\gamma}{|x_\gamma|}e_\gamma\| \\ = \ & \|f(x) + f(\frac{x_\gamma}{|x_\gamma|}e_\gamma)\| \vee \|f(x) - f(\frac{x_\gamma}{|x_\gamma|}e_\gamma)\| \\ = \ & |y_{\sigma(\gamma)} + s| \vee |y_{\sigma(\gamma)} - s| \leq |y_{\sigma(\gamma)}| + 1, \end{aligned}$$

which shows that $|x_\gamma| \leq |y_{\sigma(\gamma)}|$. By applying the same argument to f^{-1}, we can obtain $|y_{\sigma(\gamma)}| \leq |x_\gamma|$, and so $|x_\gamma| = |y_{\sigma(\gamma)}|$. So the previous inequality can become an equality

$$|y_{\sigma(\gamma)} + s| \vee |y_{\sigma(\gamma)} - s| = |y_{\sigma(\gamma)}| + |s| = |y_{\sigma(\gamma)}| + 1,$$

and so $y_{\sigma(\gamma)} = \pm\frac{s}{|s|}|x_\gamma| = \pm x_\gamma\alpha_\gamma$ for every $\gamma \in \Gamma_x \cap \Gamma_1$. It is easily to see when $\gamma \in \Gamma_x \cap \Gamma_2$, $s = \pm\frac{\overline{x_\gamma}}{|x_\gamma|}\alpha_\gamma$ and $y_{\sigma(\gamma)} = \pm\frac{s}{|s|}|x_\gamma| = \pm\overline{x_\gamma}\alpha_\gamma$. The above argument also shows that $y_{\sigma(\gamma')} = 0$ for every $\gamma' \in \Gamma \setminus \Gamma_x$. The proof is completed. □

For every $x = \{x_\gamma\} \in \mathscr{L}^\infty(\Gamma)$, define a mapping $\tau : \mathscr{L}^\infty(\Gamma) \to \mathscr{L}^\infty(\Gamma)$.

$$\tau(x)(\gamma) = \begin{cases} \frac{x_\gamma}{|x_\gamma|} & \text{if } \gamma \in \Gamma_x; \\ 0 & \text{if } \gamma \in \Gamma \setminus \Gamma_x. \end{cases}$$

Then we have

$$\tau(x+y)=\tau(x)+\tau(y) \text{ and } \tau(\alpha x)=\alpha\tau(x)$$

for arbitrary two nonzero orthogonal vectors $x,y\in\mathscr{L}^\infty(\Gamma)$ and $\alpha\in\mathbb{T}$. It is obviously that $x=y$ if and only if $\tau(x)=\tau(y)$ and $x_\gamma=\pm y_\gamma$ for each $\gamma\in\Gamma$, where $x,y\in\mathscr{L}^\infty(\Gamma)$ and x, y nonempty.

The following result will be used to prove a property of f.

Lemma 2.5. *Let $X=\mathscr{L}^\infty(\Gamma)$ and $Y=\mathscr{L}^\infty(\Delta)$. Suppose that $f:S_X\to S_Y$ is a surjective phase-isometry. Then $\tau\circ f(x)=\pm f\circ\tau(x)$ for every $x\in S_X$.*

Proof: Proposition 2.4 implies that $\sigma:\Gamma\to\Delta$ is bijectiv. For every $\gamma\in\Gamma$ and $x\in S_X$, we can suppose $f(e_\gamma)=\alpha_\gamma e_{\sigma(\gamma)}$ with $\alpha_\gamma\in\mathbb{T}$. Also, we can get

$$f(x)_{\sigma(\gamma')}=f\circ\tau(x)_{\sigma(\gamma')}=0,\qquad x\in S_X$$

for every $\gamma'\in\Gamma\setminus\Gamma_x$. Let us fix $\gamma\in\Gamma_x$. For f is a phase-isometry mapping, we can get

$$\begin{aligned}&\|f(x)+f\circ\tau(x)\|\wedge\|f(x)-f\circ\tau(x)\|\\ =\ &\|x+\tau(x)\|\wedge\|x-\tau(x)\|\\ =\ &1-\inf_{\gamma\in\Gamma_x}\{|x_\gamma|\}.\end{aligned}$$

By Proposition 2.4, for every $x=\{x_\gamma\}\in S_X$, $f(x)=\{y_{\sigma(\gamma)}\}\in S_Y$, we have $|y_{\sigma(\gamma)}|=|x_\gamma|$. Combining with the Proposition 2.4 and the property of τ, we can get

$$\tau\circ f(x)=\pm f\circ\tau(x).$$

□

Lemma 2.6. *Let $X=\mathscr{L}^\infty(\Gamma)$ and $Y=\mathscr{L}^\infty(\Delta)$. Suppose that $f:S_X\to S_Y$ is a surjective phase-isometry. Then for every $x,y\in S_X$ with $\Gamma_x\cap\Gamma_y=\varnothing$, and two positive real numbers a, b with $ax+by\in S_X$, there exist two real numbers α and β with $|\alpha|=|\beta|=1$ such that*

$$f(ax+by)=a\alpha(ax,by)f(x)+b\beta(ax,by)f(y),$$

Proof: By Proposition 2.4 and the properties of τ, we should only prove that there exist $\alpha, \beta \in \{-1, 1\}$ such that

$$\tau \circ f(ax+by) = \alpha\tau \circ af(x) + \beta\tau \circ bf(y) = \alpha\tau \circ f(x) + \beta\tau \circ f(y).$$

for $\alpha, \beta \in \{-1, 1\}$.

Meanwhile, Lemma 2.5 implies that the equality is equivalent to

$$f \circ \tau(ax+by) = f \circ (\tau(x) + \tau(y)) = \alpha(x,y) f \circ \tau(x) + \beta(x,y) f \circ \tau(y),$$

where $\alpha(x,y), \beta(x,y) \in \{-1, 1\}$.

Let $\sigma : \Gamma \to \Delta$ be the bijection from Proposition 2.4. We can write

$$f \circ \tau(x) = \{w_{\sigma(\gamma)}\},\ f \circ \tau(y) = \{v_{\sigma(\gamma)}\},\ f \circ \tau(x+y) = \{w'_{\sigma(\gamma)} + v'_{\sigma(\gamma)}\},$$

where $w_{\sigma(\gamma)} = \pm w'_{\sigma(\gamma)} \in \mathbb{T}$ for every $\gamma \in \Gamma_x$ and $v_{\sigma(\gamma)} = \pm v'_{\sigma(\gamma)} \in \mathbb{T}$ for every $\gamma \in \Gamma_y$ respectively. Thus

$$\|f \circ \tau(x+y) + f \circ \tau(x)\| \wedge \|f \circ \tau(x+y) - f \circ \tau(x)\| = \|\tau(x+y) + \tau(x)\| \wedge \|\tau(x+y) - \tau(x)\| = 1.$$

It follows that $\{w'_{\sigma(\gamma)}\} = \pm f \circ \tau(x)$, and similarly $\{v'_{\sigma(\gamma)}\} = \pm f \circ \tau(y)$. This shows that

$$f \circ \tau(x+y) = \alpha(x,y) f \circ \tau(x) + \beta(x,y) f \circ \tau(y)$$

for some $\alpha(x,y), \beta(x,y) \in \{-1, 1\}$, which completes the proof. □

Lemma 2.7. *Let $X = \mathscr{L}^\infty(\Gamma)$ and $Y = \mathscr{L}^\infty(\Delta)$. Suppose that $f : S_X \to S_Y$ is a surjective phase-isometry. For every $x, y \in S_X$ with $\Gamma_x \cap \Gamma_y = \varnothing$, we write $f(x+y) = \alpha(x,y) f(x) + \beta(x,y) f(y)$, where $\alpha(x,y), \beta(x,y) \in \{-1, 1\}$. Then*

$$\alpha(x,y)\beta(x,y) = \alpha(-x,y)\beta(-x,y) = \alpha(x,-y)\beta(x,-y).$$

Proof: For this conclusion, we only need to check

$$\alpha(x,y)\beta(x,y) = \alpha(-x,y)\beta(-x,y).$$

From Lemma 2.5, we have known

$$\tau \circ f(x) = \pm f \circ \tau(x),$$

where $x \in S_X$. Therefore,

$$\begin{aligned} \tau\circ f(x+y) &= \alpha(x,y)\tau\circ f(x)+\beta(x,y)\tau\circ f(y), & \alpha(x,y),\beta(x,y)\in\{-1,1\},\\ \tau\circ f(-x+y) &= \alpha(-x,y)\tau\circ f(-x)+\beta(-x,y)\tau\circ f(y), & \alpha(-x,y),\beta(-x,y)\in\{-1,1\}. \end{aligned}$$

Combining with Lemma 2.1 and Lemma 2.5, we can get

$$\begin{aligned} 2 &= \|\tau(x+y)+\tau(-x+y)\|\wedge\|\tau(x+y)-\tau(-x+y)\|\\ &= \|f\circ\tau(x+y)+f\circ\tau(-x+y)\|\wedge\|f\circ\tau(x+y)-f\circ\tau(-x+y)\|\\ &= \|\tau\circ f(x+y)+\tau\circ f(-x+y)\|\wedge\|\tau\circ f(x+y)-\tau\circ f(-x+y)\|\\ &= \wedge\{\|\beta(x,y)\tau\circ f(x+y)\pm\beta(-x,y)\tau\circ f(-x+y)\|\}\\ &= \|\alpha(x,y)\beta(x,y)\tau\circ f(x)-\alpha(-x,y)\beta(-x,y)\tau\circ f(-x)\|\\ &= |\alpha(x,y)\beta(x,y)+\alpha(-x,y)\beta(-x,y)|, \end{aligned}$$

which shows that $\alpha(x,y)\beta(x,y)=\alpha(-x,y)\beta(-x,y)$. The proof is completed. □

Define a mapping F, which is the natural extension of f from X to Y. For arbitrary $x\in X$, defined by

$$F(x)=\begin{cases}\|x\|f(\frac{x}{\|x\|}), & \text{if } x\neq 0,\\ 0, & \text{if } x=0.\end{cases}$$

Theorem 2.8. *Let $X=\mathscr{L}^\infty(\Gamma)$ and $Y=\mathscr{L}^\infty(\Delta)$, suppose that $f: S_X\to S_Y$ is a surjective phase isometry. Then its extension mapping which on the whole space is phase equivalent to a real linear isometry.*

Proof: In order to complete the proof, we should prove that F, the extension of f, is phase equivalent to a real linear isometry. Lemma 2.3 implies that for every $\gamma_0\in\Gamma$ and all $\alpha\in\mathbb{T}$, we have $f(\alpha e_{\gamma_0})=\pm\alpha f(e_{\gamma_0})$ for all $\alpha\in\mathbb{T}$ or $f(\alpha e_{\gamma_0})=\pm\bar{\alpha} f(e_{\gamma_0})$ for all $\alpha\in\mathbb{T}$. We shall only prove the case in which $f(\alpha e_{\gamma_0})=\pm\alpha f(e_{\gamma_0})$ for all $\alpha\in\mathbb{T}$, the other statement is very similar. Set

$$Z:=\{x\in X: x\cdot e_{\gamma_0}=0\}\quad\text{and}\quad W:=\{y\in Y: y\cdot f(e_{\gamma_0})=0\}.$$

Clearly,

$$X = Z \oplus_\infty \mathbb{C}e_{\gamma_0} \quad \text{and} \quad Y = W \oplus_\infty \mathbb{C}f(e_{\gamma_0}).$$

We can also define the unit spheres of Z and W are

$$S_Z := \{x \in S_X : x \cdot e_{\gamma_0} = 0\} \quad \text{and} \quad S_W := \{y \in S_Y : y \cdot f(e_{\gamma_0}) = 0\}.$$

It is easily to see

$$S_X = \{az + be_{\gamma_0} : z \in S_Z, a \in \mathbb{R}, b \in \mathbb{C}, |a| \vee |b| = 1\}$$

and

$$S_Y = \{af(z) + bf(e_{\gamma_0}) : f(z) \in S_W, a \in \mathbb{R}, b \in \mathbb{C}, |a| \vee |b| = 1\}.$$

By Proposition 2.4, the restricted mapping $f|_{S_Z} : S_Z \to S_W$ is a surjective phase-isometry. Lemma 2.6 implies that

$$f(z + e_{\gamma_0}) = \alpha(z, e_{\gamma_0})f(z) + \beta(z, e_{\gamma_0})f(e_{\gamma_0}),\ \alpha(z, e_{\gamma_0}), \beta(z, e_{\gamma_0}) \in \{-1, 1\}$$

for each $z \in S_Z$. Define a mapping $g : S_Z \to S_W$ given by

$$g(z) = \alpha(z, e_{\gamma_0})\beta(z, e_{\gamma_0})f(z)$$

for each $z \in S_Z$. It is easily seen that $g(z) = \pm f(z)$ for each $z \in S_Z$. Applying Lemma 2.7 we have

$$\alpha(z, e_{\gamma_0})\beta(z, e_{\gamma_0}) = \alpha(-z, e_{\gamma_0})\beta(-z, e_{\gamma_0}),\ (z \in S_Z).$$

This shows that $g(-z) = -g(z)$, and so g is surjective. We will prove that g is a surjective isometry. Given $z_1, z_2 \in S_Z$, we can write

$$f(z_1 + e_{\gamma_0}) = \alpha(z_1, e_{\gamma_0})f(z_1) + \beta(z_1, e_{\gamma_0})f(e_{\gamma_0}),\ \ \alpha(z_1, e_{\gamma_0}), \beta(z_1, e_{\gamma_0}) \in \{-1, 1\},$$

$$f(z_2 + e_{\gamma_0}) = \alpha(z_2, e_{\gamma_0})f(z_2) + \beta(z_2, e_{\gamma_0})f(e_{\gamma_0}),\ \ \alpha(z_2, e_{\gamma_0}), \beta(z_2, e_{\gamma_0}) \in \{-1, 1\},$$

then

$$\begin{aligned}\|g(z_1)-g(z_2)\| &= \|z_1+z_2+2e_{\gamma_0}\|\wedge\|z_1-z_2\|\\ &= \|f(z_1+e_{\gamma_0})+f(z_2+e_{\gamma_0})\|\wedge\|f(z_1+e_{\gamma_0})-f(z_2+e_{\gamma_0})\|\\ &= \wedge\|\beta(z_1,e_{\gamma_0})f(z_1+e_{\gamma_0})\pm\beta(z_2,e_{\gamma_0})f(z_2+e_{\gamma_0})\|\\ &= \|\alpha(z_1,e_{\gamma_0})\beta(z_1,e_{\gamma_0})f(z_1)-\alpha(z_2,e_{\gamma_0})\beta(z_2,e_{\gamma_0})f(z_2)\|\\ &= \|z_1-z_2\|,\end{aligned}$$

which shows that g is an isometry.

Give a mapping $G : Z \to W$

$$G(z_0)=\alpha(\frac{z_0}{\|z_0\|},e_{\gamma_0})\beta(\frac{z_0}{\|z_0\|},e_{\gamma_0})F(z_0),$$

where $z_0 \in Z$. Since g is a surjective isometry, by [21,Theorem 1.1], G, the extension of g, is a real linear isometry.

Define a mapping $\widetilde{f} : S_X \to S_Y$, given by

$$\widetilde{f}(az+be_{\gamma_0})=ag(z)+bf(e_{\gamma_0}),$$

where $z \in S_Z$, $a \in \mathbb{R}$, $b \in \mathbb{C}$, $|a|\vee|b|=1$. We will show $\widetilde{f}(x)$ is a surjective isometry. We first prove $\widetilde{f}(x)$ is an isometry.

Assume $x_1 = a_1z_1 + b_1e_{\gamma_0}$, $x_2 = a_2z_2 + b_2e_{\gamma_0}$, where $x_1,x_2 \in S_X$, $z_1,z_2 \in S_Z$, $|a_1|\vee|b_1| = 1$, $|a_2|\vee|b_2| = 1$, $a_1,a_2 \in \mathbb{R}$, $b_1,b_2 \in \mathbb{C}$. Then

$$\begin{aligned}&\|\widetilde{f}(x_1)-\widetilde{f}(x_2)\|\\ =\ &\|a_1g(z_1)-a_2g(z_2)\|\vee|b_1-b_2|\\ =\ &\|G(a_1z_1)-G(a_2z_2)\|\vee|b_1-b_2|\\ =\ &\|a_1z_1-a_2z_2\|\vee|b_1-b_2|\\ =\ &\|x_1-x_2\|,\end{aligned}$$

Then we will prove $\widetilde{f}(x)$ is surjective. It remains to prove that $f(x) = \pm\widetilde{f}(x)$ for every $x \in S_X$. Given $z \in S_Z$, by Lemma 2.6, we have

$$\widetilde{f}(az+be_{\gamma_0}) = a\alpha(z,e_{\gamma_0})\beta(z,e_{\gamma_0})f(z)+bf(e_{\gamma_0}),\ \alpha(z,e_{\gamma_0}),\beta(z,e_{\gamma_0}) \in \{-1,1\},$$
$$f(az+be_{\gamma_0}) = a\alpha(az,be_{\gamma_0})f(z)+b\beta(az,be_{\gamma_0})f(e_{\gamma_0}),\ \ \alpha(az,be_{\gamma_0}),\beta(az,be_{\gamma_0}) \in \{-1,1\},$$

where $a \in \mathbb{R}$, $b \in \mathbb{C}$, $|a| \vee |b| = 1$ and $z \in S_Z$.

Next we want to know that

$$\alpha(az,be_{\gamma_0})\beta(az,be_{\gamma_0}) = \alpha(z,e_{\gamma_0})\beta(z,e_{\gamma_0}).$$

We need two steps to finish this conclusion.

We first to show $\alpha(az,be_{\gamma_0})\beta(az,be_{\gamma_0}) = \alpha(z,\frac{b}{|b|}e_{\gamma_0})\beta(z,\frac{b}{|b|}e_{\gamma_0})$.

$$\begin{aligned}
& \{|1+a| \vee |b|+1, |1-a| \vee 1-|b|\} \\
= \ & \{\|(az+be_{\gamma_0})+(z+\frac{b}{|b|}e_{\gamma_0})\|, \|az+be_{\gamma_0}-(z+\frac{b}{|b|}e_{\gamma_0})\|\} \\
= \ & \{\|f(az+be_{\gamma_0})+f(z+\frac{b}{|b|}e_{\gamma_0})\|, \|f(az+be_{\gamma_0})-f(z+\frac{b}{|b|}e_{\gamma_0})\|\} \\
= \ & \{\|\beta(az,be_{\gamma_0})f(az+be_{\gamma_0}) \pm \beta(z,\frac{b}{|b|}e_{\gamma_0})f(z+\frac{b}{|b|}e_{\gamma_0})\|\} \\
= \ & \{\|(a\alpha(az,be_{\gamma_0})\beta(az,be_{\gamma_0})f(z)+bf(e_{\gamma_0})) \pm (\alpha(z,\frac{b}{|b|}e_{\gamma_0})\beta(z,\frac{b}{|b|}e_{\gamma_0})f(z)+\frac{b}{|b|}f(e_{\gamma_0}))\|\} \\
= \ & \{|a\alpha(az,be_{\gamma_0})\beta(az,be_{\gamma_0})+\alpha(z,\frac{b}{|b|}e_{\gamma_0})\beta(z,\frac{b}{|b|}e_{\gamma_0})| \vee 1+|b|, \\
& |a\alpha(az,be_{\gamma_0})\beta(az,be_{\gamma_0})-\alpha(z,\frac{b}{|b|}e_{\gamma_0})\beta(z,\frac{b}{|b|}e_{\gamma_0})| \vee 1-|b|\}
\end{aligned}$$

which shows $\alpha(az,be_{\gamma_0})\beta(az,be_{\gamma_0}) = \alpha(z,\frac{b}{|b|}e_{\gamma_0})\beta(z,\frac{b}{|b|}e_{\gamma_0})$.

Next we will show $\alpha(z,e_{\gamma_0})\beta(z,e_{\gamma_0}) = \alpha(z,\frac{b}{|b|}e_{\gamma_0})\beta(z,\frac{b}{|b|}e_{\gamma_0})$. If $|\frac{b}{|b|}+1| \neq |\frac{b}{|b|}-1|$ or $b \neq it$ for $t \in \mathbb{R}$, $|t| \leq 1$, then we get the desired equation

$$
\begin{aligned}
&\{2, |1-\frac{b}{|b|}|\} = \{\|(z+e_{\gamma_0})+(z+\frac{b}{|b|}e_{\gamma_0})\|, \|z+e_{\gamma_0}-(z+\frac{b}{|b|}e_{\gamma_0})\|\} \\
= \ & \{\|f(z+e_{\gamma_0})+f(z+\frac{b}{|b|}e_{\gamma_0})\|, \|f(z+e_{\gamma_0})-f(z+\frac{b}{|b|}e_{\gamma_0})\|\} \\
= \ & \{\|\beta(z,e_{\gamma_0})f(z+e_{\gamma_0}) \pm \beta(z,\frac{b}{|b|}e_{\gamma_0})f(z+\frac{b}{|b|}e_{\gamma_0})\|\} \\
= \ & \{\|(\alpha(z,e_{\gamma_0})\beta(z,e_{\gamma_0})f(z)+f(e_{\gamma_0})) \pm (\alpha(z,\frac{b}{|b|}e_{\gamma_0})\beta(z,\frac{b}{|b|}e_{\gamma_0})f(z)+\frac{b}{|b|}f(e_{\gamma_0}))\|\} \\
= \ & \{|\alpha(z,e_{\gamma_0})\beta(z,e_{\gamma_0})+\alpha(z,\frac{b}{|b|}e_{\gamma_0})\beta(z,\frac{b}{|b|}e_{\gamma_0})| \vee |1+\frac{b}{|b|}|, \\
& |a\alpha(z,e_{\gamma_0})\beta(z,e_{\gamma_0})-\alpha(z,\frac{b}{|b|}e_{\gamma_0})\beta(z,\frac{b}{|b|}e_{\gamma_0})| \vee |1-\frac{b}{|b|}|\},
\end{aligned}
$$

which shows $\alpha(z,e_{\gamma_0})\beta(z,e_{\gamma_0}) = \alpha(z,\frac{b}{|b|}e_{\gamma_0})\beta(z,\frac{b}{|b|}e_{\gamma_0})$.

Now assume that $b = it$ for $t \in \mathbb{R}$, $|t| \leq 1$. Choose $\theta \in \mathbb{T} \setminus \{\pm 1, \pm i\}$. Following a similar argument as above, we get

$$
\begin{aligned}
&\{2, |\frac{b}{|b|}-\theta|\} = \{\|(z+\frac{b}{|b|}e_{\gamma_0})+(z+\theta e_{\gamma_0})\|, \|\frac{b}{|b|}e_{\gamma_0}-\theta e_{\gamma_0}\|\} \\
= \ & \{\|f(z+\frac{b}{|b|}e_{\gamma_0})+f(z+\theta e_{\gamma_0})\|, \|f(z+\frac{b}{|b|}e_{\gamma_0})-f(z+\theta e_{\gamma_0})\|\} \\
= \ & \{\|\beta(z,\frac{b}{|b|}e_{\gamma_0})f(z+\frac{b}{|b|}e_{\gamma_0}) \pm \beta(z,\theta e_{\gamma_0})f(z+\theta e_{\gamma_0})\|\} \\
= \ & \{|\alpha(z,\frac{b}{|b|}e_{\gamma_0})\beta(z,\frac{b}{|b|}e_{\gamma_0})+\alpha(z,\theta e_{\gamma_0})\beta(z,\theta e_{\gamma_0})| \vee |\frac{b}{|b|}+\theta|, \\
& |\alpha(z,\frac{b}{|b|}e_{\gamma_0})\beta(z,\frac{b}{|b|}e_{\gamma_0})-\alpha(z,\theta e_{\gamma_0})\beta(z,\theta e_{\gamma_0})| \vee |\frac{b}{|b|}-\theta|\}.
\end{aligned}
$$

Since $|\frac{b}{|b|}-\theta| \neq |\frac{b}{|b|}+\theta|$, we obtain

$$\alpha(z,\frac{b}{|b|}e_{\gamma_0})\beta(z,\frac{b}{|b|}e_{\gamma_0}) = \alpha(z,\theta e_{\gamma_0})\beta(z,\theta e_{\gamma_0}).$$

Thus we get $\alpha(z,e_{\gamma_0})\beta(z,e_{\gamma_0}) = \alpha(az,be_{\gamma_0})\beta(az,be_{\gamma_0})$, which shows $f(x) = \pm\widetilde{f}(x)$ for every $x \in S_X$.

What's more,

$$\begin{aligned} \widetilde{f}(-x) &= \widetilde{f}(-az-be_{\gamma_0}) \\ &= ag(-z)+bf(-e_{\gamma_0}) \\ &= -ag(z)-bf(e_{\gamma_0}) \\ &= -\widetilde{f}(x), \end{aligned}$$

which shows $\widetilde{f}(-x) = -\widetilde{f}(x)$ for every $x \in S_X$. Thus $\widetilde{f}(x)$ is a surjective isometry.

By [21,Theorem 1.1], we have known $\widetilde{F}(x)$, the extension of $\widetilde{f}(x)$ is a real linear isometry, and $F(x)$ is phase equivalent to $\widetilde{F}(x)$, the proof is completed. □

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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