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CARISTI TYPE FIXED POINT THEOREMS IN GENERALIZED METRIC SPACES

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Abstract. In this paper we give a generalized version of some new fixed point theorems of mappings satisfying Caristi type conditions recently introduced by E.Karapinar et al. And we obtain some extension of these results in the settings of JS metric spaces. Some examples are given to validate our main results.

Keywords: fixed point; JS metric; b-metric; Caristi type mapping.

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1. Introduction

Fixed point theory is one of the most studied research axes in mathematics. It has provided a powerful tool to prove existence of solutions for numerous problems in different branches of science. The publication of the Banach contraction principle [1], has motivated and inspired researchers from different fields in science which have developed the fixed point results and prove their interest in applications to solve various scientific problems such as transport theory, biomathematics, economics,etc.

The generalization of the Banach principle followed different directions; but one of the most important direction is the generalization of the contraction (see for example [2], [3]). Banach

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and Caristi contractions are among the most used results in applications. Let us recall the following principle

Theorem 1. [1] *Let (X,d) be a complete metric space and $f : X \to X$ be a mapping. Suppose that there exists $q \in [0;1[$ such that*

$$d(fx, fy) \leq qd(x,y)$$

for all $x, y \in X$. Then f has a unique fixed point in X.

Ćirić [2] introduced the notion of quasi-contraction and obtained one of the most well-known generalization of the Banach contraction principle.

Theorem 2. *Let (X,d) be a complete metric space and $f : X \to X$ be a mapping. Assume that there exists $q \in [0;1[$ such that*

$$d(fx, fy) \leq qN(x;y)$$

for all $x, y \in X$; where $N(x,y) = \max\{d(x,y), d(fx,x), d(fy,y), d(fx,y), d(x,fy)\}$.
Then f has a unique fixed point in X.

In 1976, Caristi[3] proved the following fixed point theorem, which is a generalization of the Banach contraction principle.

Theorem 3. *Let (X,d) be a complete metric space and $\varphi : X \to [0,+\infty)$ be a lower semicontinuous and bounded below function. If f satisfies*

$$d(x, fx) \leq \varphi(x) - \varphi(fx)$$

for each $x \in X$, then it has a fixed point in X.

2. Preliminaries

Definition 1. *Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0,\infty)$ is said to be a b-metric if and only if for all $x, y, z \in X$, the following conditions are satisfied:*

(1) $d(x,y) = 0$ *if and only if* $x = y$;
(2) $d(x,y) = d(y,x)$;

(3) $d(x,z) \leq s[d(x,y)+d(y,z)]$.

(X,d,s) is called a b-metric space with coefficient s.

In 2019, E.Karapinar et al. [4], a new fixed point theorem in the setting of b-metric space has been given, by mixing Banach and Caristi-type contraction they proved the following theorem:

Theorem 4. *Let (X,d) be a complete b-metric space and $T : X \longrightarrow X$ be a self-mapping on X. Assume that there exists a function $\varphi : X \longrightarrow \mathbb{R}$ such that*
(i) φ is bounded from below.
(ii) $d(x,Tx) > 0$ implies $d(Tx,Ty) \leq (\varphi(x)-\varphi(Tx))d(x,y)$.
Then, T has at least one fixed point in X.

Using a similar idea, E.Karapinar et al. [5] proved a fixed point theorem by combining Ćirić and Caristi-type contraction. This Theorem can be stated as follows.

Theorem 5. *Let (X,d) be a complete metric space and $T : X \longrightarrow X$ be a map. Assume that there exists a function $\varphi : X \longrightarrow [0,\infty[$ with :*

$$d(x,Tx) > 0 \text{ implies } d(Tx,Ty) \leq (\varphi(x)-\varphi(Tx))N(x,y)$$

for all $x,y \in X$; where $N(x,y)$ as defined above. Then T has at least one fixed point in X.

In 2015 M.Jleli and B.Samet [7] introduced a new generalized space which covers standard metric spaces, b-metric spaces, dislocated metric spaces and modular spaces.
For every $x \in X$, let us define the set $C(D,X,x) = \left\{ \{x_n\} \subset X : \lim\limits_{n\to+\infty} D(x_n,x) = 0 \right\}$.

Definition 2. *D is called a JS metric on X if it satisfies the following conditions:*
(D_1) for every $(x,y) \in X \times X$, we have $D(x,y) = 0 \Longrightarrow x = y$;
(D_2) for every $(x,y) \in X \times X$, we have $D(x,y) = D(y,x)$;
(D_3) there exists $C > 0$ such that if $(x,y) \in X \times X$ and $\{x_n\} \in C(D,X,x)$, then $D(x,y) \leq C \limsup\limits_{n\longrightarrow+\infty} D(x_n,y)$.
The pair (X,D) is called a JS metric space.

We mention that convergent sequences and Cauchy sequences can be introduced in a similar manner as in metric spaces.

Definition 3. *Let (X,D) be a JS metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. We say that $\{x_n\}$ converges to x if $\{x_n\} \in C(D,X,x)$ (i.e $\lim_{n\to\infty} D(x_n,x) = 0$).*

Definition 4. *Let (X,D) be a JS metric space. Let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is a Cauchy sequence if $\lim\limits_{m,n\to\infty} D(x_n,x_{n+m}) = 0$.*

Definition 5. *Let (X,D) be a JS metric space. It is said to be complete if every Cauchy sequence in X is convergent to some element in X.*

3. Main Results

In the following, we give a simple proof of Banach-Caristi type fixed point theorem in a b-metric space. And, we give a strong version of Theorem 1 in [4]. We will show that the iterate sequence $\{T^n x\}$ turn to be stationary for every $x \in X$ and without the need of completeness assumption.

Lemma 1 ([8]). *If λ is a real number such that $0 < \lambda < 1$ and let $\{b_n\}$ be a sequence of positives reals numbers such that $\lim\limits_{n\longrightarrow+\infty} b_n = 0$. Then, for any sequence of positives numbers $\{a_n\}$ satisfying $a_{n+1} \leq \lambda a_n + b_n$ for all $n \in \mathbb{N}$, we have $\lim\limits_{n\longrightarrow+\infty} a_n = 0$.*

We can deduce the following result:

Lemma 2. *Let $\{b_n\}$ and $\{\lambda_n\}$ be two sequences of positives reals numbers such that $\lim\limits_{n\to+\infty} b_n = 0$ and $\limsup\limits_{n\to+\infty} \lambda_n = \lambda \in]0,1[$. Then, for any sequence of positives numbers $\{a_n\}$ satisfying $a_{n+1} \leq \lambda_n a_n + b_n$ for all $n \in \mathbb{N}$, we have $\lim\limits_{n\to+\infty} a_n = 0$.*

Proof. Since $\limsup\limits_{n\to+\infty} \lambda_n = \lambda < 1$, there exists an integer $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have $\lambda_n \leq \lambda$. And then, for every $n \geq n_0$ we have $a_{n+1} \leq \lambda a_n + b_n$. By lemma1, we obtain $\lim\limits_{n\to+\infty} a_n = 0$.

□

Theorem 6. *Let (X,d,s) be a b-metric space and T be a self mapping on X. Assume that there exists a function $\varphi : X \longrightarrow \mathbb{R}$ such that :*

(i) φ is bounded from below.

(ii) $d(x,Tx) > 0$ *implies* $d(Tx,Ty) \leq (\varphi(x) - \varphi(Tx))d(x,y)$ *for each* $x,y \in X$.

Then for each $x \in X$*, there exists* $p \in \mathbb{N}$ *such that* $T^p x \in Fix(T)$.

Proof. Let $x \in X$, and $\{x_n\}$ the sequence defined by $x_0 = x$ and $x_{n+1} = Tx_n$, for all $n \in \mathbb{N}$.
Assume that $d(x_{n+1},x_n) > 0$, for every $n \in \mathbb{N}$.
By (ii) we obtain

$$d(x_{n+1},x_n) \leq (\varphi(x_n) - \varphi(x_{n+1}))d(x_n,x_{n-1})$$

for every $n \in \mathbb{N}$. It follows that $\{\varphi(x_n)\}$ is a decreasing sequence which is bounded below by (i), then it converges to some $r \in \mathbb{R}$. And hence $\varphi(x_n) - \varphi(x_{n+1}) \xrightarrow[n\to+\infty]{} 0$.
Since $d(x_n,Tx_n) > 0$, we have

$$d(x_{n+1},Ty) \leq (\varphi(x_n) - \varphi(x_{n+1}))d(x_n,y)$$

for every $y \in X$. Thus by relaxed triangular inequality we get

$$d(x_{n+1},Ty) \leq s(\varphi(x_n) - \varphi(x_{n+1}))d(x_n,Ty) + s(\varphi(x_n) - \varphi(x_{n+1}))d(Ty,y)$$

And by Lemma1, we obtain $\lim\limits_{n\longrightarrow+\infty} d(x_n,Ty) = 0$. This means that T is a constant mapping on X; which is a contradiction. And then, we conclude that for each $x \in X$ there exists $p \in \mathbb{N}$ such that $d(x_p,x_{p+1}) = 0$ that is $T^p x \in Fix(T)$. □

Example 1. *Let* $X = \{0,1,2\}$ *endowed with the following metric:*
$d(0,1) = 1, d(2,0) = 1, d(1,2) = \frac{3}{2}$ *and* $d(a,a) = 0$*, for all* $a \in X$*,* $d(a,b) = d(b,a)$*, for all* $a,b \in X$*. Define* T *by* $T(0) = 0, T(1) = 2, T(2) = 0$.
Define $\varphi : X \longrightarrow [0;+\infty[$ *as* $\varphi(2) = 2, \varphi(0) = 0, \varphi(1) = 4$*. Thus for all* $x \in X$ *such that* $d(x,Tx) > 0$*, we have*

$$d(T1,T2) \leq (\varphi(1) - \varphi(T(1)))d(2,1)$$

$$d(T2,T1) \leq (\varphi(2) - \varphi(T(2)))d(2,1),$$

$$d(T1,T0) \leq (\varphi(1) - \varphi(T(1)))d(1,0),$$

$$d(T2,T0) \leq (\varphi(2) - \varphi(T(2)))d(2,0).$$

Thus the mapping T *satisfies all conditions. And we have* $T0 = 0, T^2 1 = 0$ *and* $T2 = 0$

Example 2. *Let $X = \{1,2,4,5\}$ endowed with the euclidian metric:*
$d(x,y) = |x-y|$ for all $x,y \in X$. Let $T(2) = T(1) = 1, T(5) = T(4) = 4$.
Define $\varphi : X \longrightarrow [0;+\infty[$ as $\varphi(2) = \varphi(5) = 2, \varphi(1) = \varphi(4) = 0$. Thus for all $y \in X$ and $x \in \{2;5\}$ we have

$$d(Tx,Ty) \leq (\varphi(x) - \varphi(Tx))d(x,y)$$

The mapping T satisfies all conditions, also has two fixed points and we have $T5 = T4 = 4$, and $T2 = T1 = 1$

Example 3. *Let $X = \mathbb{N}$ equipped with the euclidian metric: $d(x,y) = |x-y|$ for all $x,y \in X$. Let $T : \mathbb{N} \longrightarrow \mathbb{N}$, defined by $T(n) = n-1$ and $T(0) = 0$, we define $\varphi : X \longrightarrow [0;+\infty[$ by $\varphi(n) = n$. For all $n \in \mathbb{N}$ and $m \in \mathbb{N}$ we have*

$$d(Tn,Tm) \leq (\varphi(n) - \varphi(Tn))d(n,m)$$

The mapping T satisfies all conditions and has 0 as a unique fixed point, moreover we have for every $n \in \mathbb{N}^*$ $T^n n = 0$ and $T0 = 0$.

Example 4. *Let $X = \left\{\frac{1}{n}/n \in \mathbb{N}^*\right\}$ endowed with the b-metric defined by: $d(x,y) = (x-y)^2$ for all $x,y \in X$, here the coefficient $s = 2$. We notice that (X,d) is not complete.*
Let $T : X \longrightarrow X$, defined by $T(\frac{1}{n}) = \frac{1}{n-1}$ and $T(1) = 1$ and $\varphi : X \longrightarrow [0;+\infty[$ defined by $\varphi(\frac{1}{n}) = 4n$ for all $n \in \mathbb{N}^*$ and $m \in \mathbb{N}$ we have

$$d(T\frac{1}{n}, T\frac{1}{m}) \leq (\varphi(\frac{1}{n}) - \varphi(T\frac{1}{n}))d(\frac{1}{n}, \frac{1}{m})$$

The mapping T satisfies all conditions and has 1 as a unique fixed point and we have for every $n \in \mathbb{N}^*$ $T^{n-1}(\frac{1}{n}) = 1$ and $T1 = 1$.

In the sequel, we give an extension of Caristi fixed point theorem in the setting of JS metric spaces.
For $x \in X$, we put $\delta(D,T,x) = \sup\{D(T^n x, T^m x); n,m \in \mathbb{N}\}$.

Theorem 7. *Let (X,D) be a complete JS metric space and $T : X \longrightarrow X$ be a map. Suppose that there exists $x \in X$ such that $\delta(D,T,x) < \infty$ and a function $\varphi : X \longrightarrow \mathbb{R}$ with :*
(i) φ is bounded from below,

(ii) for each $x,y \in X$, $D(x,Tx) > 0$ implies $D(Tx,Ty) \leq (\varphi(x) - \varphi(Tx))D(x,y)$

Then there exists $p \in \mathbb{N}$ such that $T^p x \in Fix(T)$.

Proof. Let $\{x_n\}$ be the sequence defined by $x_0 = x$ and $x_{n+1} = Tx_n$.

Suppose that

$$\text{for every } n \in \mathbb{N} \text{ we have } d(x_{n+1},x_n) > 0 \tag{1}$$

We proceed by the same way as in the proof of Theorem 6 we get

$$\varphi(x_n) - \varphi(x_{n+1}) \xrightarrow[n\to+\infty]{} 0 \tag{2}$$

Thus for $\lambda \in]0,1[$, there is an integer $p \in \mathbb{N}$ such that for every $n \geq p$, we have $\varphi(x_n) - \varphi(x_{n+1}) \leq \lambda$.

Now for $n,m \in \mathbb{N}$ such that $n > m \geq p$, we have

$$D(x_n,x_m) \leq (\varphi(x_{n-1}) - \varphi(x_n))D(x_{n-1},x_{m-1}) \tag{3}$$

Hence we get

$$D(x_n,x_m) \leq \lambda D(x_{n-1},x_{m-1}) \text{ for every } n > m \geq p$$

And by induction we obtain

$$D(x_n,x_m) \leq \lambda^{m-p} D(x_{n-m+p},x_p) \text{ for every } n > m \geq p$$

We have

$$D(x_{n-m+p},x_p) \leq \delta(D,T,x_0)$$

Then

$$D(x_n,x_m) \leq \lambda^{m-p}\delta(D,T,x_0)$$

We get $D(x_n,x_m) \xrightarrow[n,m\to+\infty]{} 0$ consequently $\{x_n\}$ is a Cauchy sequence.

Since (X,D) is complete, there exists $\omega \in X$ such that $\lim_{n\to+\infty} D(x_n,\omega) = 0$.

Using (ii) we have for $n,m \in \mathbb{N}$

$$\begin{aligned} D(x_n,x_m) &\leq (\varphi(x_{n-1}) - \varphi(x_n))D(x_n,x_m) \\ &\leq (\varphi(x_{n-1}) - \varphi(x_n))\delta(D,T,x) \end{aligned}$$

Letting n tend to infinity we get $D(x_n,x_m) \xrightarrow[n\to+\infty]{} 0$.
By the uniqueness of the limit we get $x_m = \omega$ fro evey $m \in \mathbb{N}$, which is a contradiction with (1).
Consequently there exists $p \in \mathbb{N}$ such that $d(x_{p+1},x_p) = 0$; that is $T^p x \in Fix(T)$. □

In the following, we give a generalization of the Ćirić-Caristi type fixed point theorem in b-metric space.

Lemma 3. [6] *Let (X,d,s) be a b-metric space and let $\{x_n\}$ be a sequence in X. Assume that there exists $\lambda \in [0,1)$ satisfying $d(x_{n+1},x_n) \leq \lambda d(x_n,x_{n-1})$ for any $n \in \mathbb{N}$. Then $\{x_n\}$ is Cauchy.*

Lemma 4. *Let (X,d,s) be a b-metric space and $T : X \longrightarrow X$ be a map. Let $\{x_n\}$ be the sequence defined by $x_0 \in X$ and $x_{n+1} = Tx_n$. If $\{x_n\}$ converges to some $\omega \in X$, then $\limsup\limits_{n\longrightarrow+\infty} N(x_n,y) < \infty$.*

Proof. By definition we have

$$N(x_n,y) = \max\{d(x_n,y), d(x_{n+1},y), d(x_n,Ty), d(x_{n+1},x_n), d(y,Ty)\}$$

For each $n \in \mathbb{N}$ we have

$$d(x_n,y) \leq sd(x_n,\omega) + sd(\omega,y)$$

Then we get $\limsup\limits_{n\longrightarrow+\infty} d(x_n,y) \leq sd(\omega,y)$.
By the same way we have $\limsup\limits_{n\longrightarrow+\infty} d(x_n,Ty) \leq sd(\omega,Ty)$.
We have also $\limsup\limits_{n\longrightarrow+\infty} d(x_n,x_{n+1}) = 0$.
Consequently we get

$$\limsup_{n\longrightarrow+\infty} N(x_n,y) < \max\{sd(\omega,y), sd(\omega,Ty), d(y,Ty)\} < \infty.$$

□

Theorem 8. *Let (X,d,s) be a complete b-metric space and $T : X \longrightarrow X$ be a map. If there is a function $\varphi : X \to [0,+\infty)$, such that :*

$$d(x,Tx) > 0 \text{ implies } d(Tx,Ty) \leq (\varphi(x) - \varphi(Tx))N(x,y), \text{ for all } y \in X \tag{4}$$

Then for each $x \in X$ there exists $p \in \mathbb{N}$ such that $T^p x \in Fix(T)$.

Proof. Let $x \in X$, and $\{x_n\}$ the sequence defined by $x_0 = x$ and $x_{n+1} = Tx_n$, we need to prove that there exists $p \in \mathbb{N}$ such that $T^p x \in Fix(T)$, which is equivalent to $d(x_{p+1}, x_p) = 0$. Suppose that

$$\forall n \in \mathbb{N}, b_n = d(x_{n+1}, x_n) > 0 \tag{5}$$

Then from (4) we have

$$b_{n+1} \leq (\varphi(x_n) - \varphi(x_{n+1}))N(x_{n+1}, x_n) \tag{6}$$

then the sequence $\{\varphi(x_n)\}$ is decreasing bounded from below, so it converges to some $r \in \mathbb{R}$. We have

$$N(x_{n+1}, x_n) = \max\{d(x_{n+1}, x_n), d(x_{n+2}, x_n), d(x_{n+1}, x_{n+2})\}$$

From relaxed triangular inequality we get

$$d(x_{n+1}, x_{n+2}) \leq s[d(x_{n+1}, x_n) + d(x_n, x_{n+2})]$$

Then from (2)

$$b_{n+1} \leq (\varphi(x_n) - \varphi(x_{n+1})) \max\{b_n, b_{n+1}, s(b_n + b_{n+1})\} \tag{7}$$

Denote by I the set defined by

$$I = \{n \in \mathbb{N} / max\{b_n, b_{n+1}, s(b_n + b_{n+1})\} = b_{n+1}\}$$

We claim that I is finite. Suppose that I is infinite, it follows that there is a subsequence $\{b_{n_k}\}$ of $\{b_n\}$ such that $b_{n_k+1} \leq (\varphi(x_{n_k}) - \varphi(x_{n_k+1}))b_{n_k+1}$

From (4) we have

$$1 \leq (\varphi(x_{n_k}) - \varphi(x_{n_k+1})$$

As $k \longrightarrow +\infty$ we get a contradiction.

We conclude that I is finite which implies that for $n_0 = maxI$ we have $\forall n > n_0$ $b_{n+1} \leq (\varphi(x_n) - \varphi(x_{n+1})) \max\{b_n, s(b_n + b_{n+1})\}$.

By simple calculation we get

$$\frac{b_{n+1}}{b_n} \leq \max\left\{\varphi(x_n) - \varphi(x_{n+1}), \frac{s((\varphi(x_n) - \varphi(x_{n+1}))}{1 - s((\varphi(x_n) - \varphi(x_{n+1}))}\right\}$$

We have $\lim_{n\to\infty} \varphi(x_n) - \varphi(x_{n+1}) = 0$ and $\lim_{n\to\infty} \frac{s((\varphi(x_n)-\varphi(x_{n+1}))}{1-s((\varphi(x_n)-\varphi(x_{n+1}))} = 0$.

This means that $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = 0$.

Then for $\lambda \in [0,1[$ there exists $n_1 > n_0$ such that for every $n \geq n_1$ we have $b_{n+1} \leq \lambda b_n$.

Now using Lemma (3) and completeness of the b-metric space (X,d,s), we obtain that the sequence $\{x_n\}$ converges to some $\omega \in X$.

For each $y \in X$ we have

$$d(x_{n+1}, Ty) \leq (\varphi(x_n) - \varphi(x_{n+1}))N(x_n, y)$$

We have $\lim_{n\longrightarrow+\infty} (\varphi(x_n) - \varphi(x_{n+1})) = 0$ and by Lemma (4) $\limsup_{n\longrightarrow+\infty} N(x_n, y) < \infty$.

Hence we get $\lim_{n\longrightarrow+\infty} d(x_{n+1}, Ty)) = 0$ for all $y \in X$ and by the uniqueness of the limit we obtain $Ty = \omega$, this means that T is a constant mapping which is a contradiction with the above assumption (5).

And hence there exists $p \in \mathbb{N}$ such that $T^p x \in Fix(T)$.

□

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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