A UNIQUE COMMON FIXED POINT THEOREM IN CONE METRIC SPACES

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Abstract: In this paper we prove a unique common fixed point theorem in cone metric spaces which generalize and extend metric space into cone metric spaces. Our result generalizes and extends some recent results.

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1. Introduction

In 2007 Huang and Zhang [5] have generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying different contractive conditions. Subsequently , Abbas and Jungck [1] and Abbas and Rhoades [2] have studied common fixed point theorems in cone metric spaces (see also [3,4] and the references mentioned therein).

In this paper we extend the fixed point theorem of S.L.Singh et .al. [8] in metric space into cone metric spaces.

Throughout this paper, $E$ is a real Banach space, $N = \{1, 2, 3, \ldots\}$ the set of all natural numbers. For the mappings $f, g: X \to X$, let $C(f, g)$ denotes set of coincidence
points of $f, g$, that is $C(f, g) = \{ z \in X : fz = gz \}$.

2. Preliminaries

We recall some definitions of cone metric spaces and some of their properties [5].

**Definition 1.1.** Let $E$ be a real Banach Space and $P$ a subset of $E$. The set $P$ is called a cone if and only if:

(a) $P$ is closed, nonempty and $P \neq \{0\}$;
(b) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P$ implies $ax + by \in P$;
(c) $x \in P$ and $-x \in P$ implies $x = 0$.

**Definition 1.2.** Let $P$ be a cone in a Banach Space $E$, define partial ordering $\leq$ on $E$ with respect to $P$ by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate $x \leq y$ but $x \neq y$ while $X < y$ will stand for $y - x \in \text{Int } P$, where $\text{Int } P$ denotes the interior of the set $P$. This Cone $P$ is called an order cone.

**Definition 1.3.** Let $E$ be a Banach Space and $P \subseteq E$ be an order cone. The order cone $P$ is called normal if there exists $L > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq L \|y\|.$$

The least positive number $L$ satisfying the above inequality is called the normal constant of $P$.

**Definition 1.4.** Let $X$ be a nonempty set of $E$. Suppose that the map $d : X \times X \rightarrow E$ satisfies:

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d2) \(d(x, y) = d(y, x)\) for all \(x, y \in X\);

(d3) \(d(x, y) \leq d(x, z) + d(z, y)\) for all \(x, y, z \in X\).

Then \(d\) is called a cone metric on \(X\) and \((X, d)\) is called a cone metric space.

It is obvious that the cone metric spaces generalize metric spaces.

**Example 1.1.** ([5]). Let \(E = \mathbb{R}^2\), \(P = \{ (x, y) \in E \text{ such that } x, y \geq 0 \} \subseteq \mathbb{R}^2\), \(X = \mathbb{R}\) and \(d: X \times X \rightarrow E\) such that \(d(x, y) = (|x - y|, \alpha|x - y|)\), where \(\alpha \geq 0\) is a constant. Then \((X, d)\) is a cone metric space.

**Definition 1.5.** Let \((X, d)\) be a cone metric space. We say that \(\{x_n\}\) is

(a) a Cauchy sequence if for every \(c\) in \(E\) with \(0 << c\), there is \(N\) such that

for all \(n, m > N\), \(d(x_n, x_m) << c\);

(b) a convergent sequence if for any \(0 << c\), there is \(N\) such that for all \(n > N\), \(d(x_n, x) << c\), for some fixed \(x \in X\).

A cone metric space \(X\) is said to be complete if every Cauchy sequence in \(X\) is convergent in \(X\).

**Lemma 1.1.** ([5]). Let \((X, d)\) be a cone metric space, and let \(P\) be a normal cone with normal constant \(L\). Let \(\{x_n\}\) be a sequence in \(X\). Then

(i). \(\{x_n\}\) converges to \(x\) if and only if \(d(x_n, x) \rightarrow 0\) \((n \rightarrow \infty)\).

(ii). \(\{x_n\}\) is a Cauchy sequence if and only if \(d(x_n, x_m) \rightarrow 0\) \((n, m \rightarrow \infty)\).

**Definition 1.6.** ([8]). Let \(f, g: X \rightarrow X\). Then the pair \((f, g)\) is said to be

(IT)-Commuting at \(z \in X\) if \(f(g(z)) = g(f(z))\) with \(f(z) = g(z)\).
3. Main results

In this section we obtain a unique common fixed point theorem in cone metric spaces, which extend a metric space into cone metric spaces.

The following theorem is extend and improves the theorem 2.3. [8]

**Theorem 3.1.** Let \((X, d)\) be a cone metric space \(P\) be an order cone and \(f, g: X\to X\) be self-maps. Let

\((f, g)\) be asymptotically regular at \(x_0 \in X\) and the following conditions are satisfied:

\((C1): f(X) \subseteq g(X);\)

\((C2): d(fx, gy) \leq \varphi(m(x, y))\) for all \(x, y \in X.\)

Where \(m(x, y) = d(gx, gy) + \gamma [d(gx, fx) + d(gy, fy)]\), \(0 \leq \gamma \leq 1\).

If \(f(X)\) or \(g(X)\) is a complete sub space of \(X\). Then

(i). \(C(f, g)\) is non-empty. Further,

(ii). \(f\) and \(g\) have a unique common fixed point provided that \(f\) and \(g\) are (IT)-

Commuting at a point \(u \in C(f, g)\).

**Proof.**

Let \(x_0\) be an arbitrary point in \(X\). Since if \((f, g)\) is asymptotically regular at \(x_0 \in X,\)

Then there exists a sequence \(\{x_n\}\) in \(X\), such that

\(f x_n = g x_{n+1}, \quad n = 0, 1, 2, \ldots \) and

\(\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0.\)

First we shall show that \(\{gx_n\}\) is a Cauchy sequence.

Suppose \(\{gx_n\}\) is not a Cauchy sequence. Then there exists \(\mu > 0\) and increasing sequences \(\{m_k\}\) and \(\{n_k\}\) of positive integers such that \(m_k\) even and \(n_k\) odd and for all \(k\), \(m_k < n_k,\)

\(d(gx_{m_k}, gx_{n_k}) \geq \mu\) and \(d(gx_{m_k}, gx_{n_{k-1}}) < \mu\)

\((2.1.)\)
By the triangle inequality,

\[ d(g_{x_n}, g_{x_n'}) \leq d(g_{x_{n+1}}, g_{x_{n'}}) + d(g_{x_{n+1}}, g_{x_n}) \]. Letting \( k \to \infty \), we get that

\[ \lim_{k \to \infty} d(g_{x_n}, g_{x_n'}) < \mu + 0. \]

(Since, \( \lim_{n \to \infty} d(g_{x_n}, g_{x_{n+1}}) = 0 \), We get \( \lim_{k \to \infty} d(g_{x_{n+1}}, g_{x_n}) = 0 \).

Therefore there exists \( k_0 \) such that

\[ d(g_{x_n}, g_{x_n'}) < \mu \quad \forall \ k \geq k_0 \] . \quad (2.2)

By (2.1) and (2.2), we get that

\[ \mu \leq d(g_{x_n}, g_{x_n'}) < \mu \quad \forall \ k \geq k_0 \].

Implies \( \lim_{k \to \infty} d(f_{x_n}, f_{x_n'}) = \mu \).

By (C2), we have

\[ d(g_{x_{n+1}}, g_{x_{n+1}'}) = d(f_{x_n}, f_{x_n'}) \leq \varphi (m(x_m, x_n)) \]

\[ = \varphi (d(g_{x_n}, g_{x_n'}) + \gamma [d(f_{x_n}, g_{x_n}) + d(g_{x_n}, f_{x_n})]). \]

Letting \( k \to \infty \), we get that

\[ \mu \leq \varphi (\mu) \text{ and as per definition of } \varphi \text{-map, } \varphi (\mu) < \mu. \]

Hence \( \mu \leq \varphi (\mu) < \mu \), a contradiction.

Thus \( \{g_{x_n}\} \) is Cauchy sequence. Suppose \( g(X) \) is a complete sub space of \( X \). Then \( \{g_{x_n}\} \) being contained in \( g(X) \) has a limit in \( g(X) \). Call it \( z \). Let \( u = g^{-1}z \).

Thus \( gu = z \) for some \( u \in X \).

By using (C2), we have

\[ d(fu, f_{x_n}) \leq \varphi (m(u, x_n)) \]

\[ \leq \varphi (d(gu, g_{x_n}) + \gamma [d(fu, gu) + d(f_{x_n}, g_{x_n})]). \]

Letting \( n \to \infty \), we get that
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\[ d(fu, z) \leq \varphi(\gamma[d(fu, z) + d(fu, z)]) < d(fu, z), \quad \text{a contradiction.} \]

Therefore, \( fu = z = gu \).

Thus \( C(f, g) \) is non-empty. This proves (i).

And the pair \( (f, g) \) is (IT) - Commuting at \( u \), then

\[ fgu = gfu \quad \text{and} \quad ffu = fgu = gfu = ggu. \]

In view of (C2) it follows that\n\[
\begin{align*}
  d(fu, ffu) & \leq \varphi(m(u, x_n)) \\
  & \leq \varphi(d(gu, gfu) + \gamma[d(fu, gu) + d(ffu, gfu)]) \\
  & < d(fu, ffu), \quad \text{a contradiction.}
\end{align*}
\]

Therefore, \( ffu = fu \) and \( fgu = ffu = fu = z \).

Therefore, \( f \) and \( g \) have a common fixed point.

Uniqueness, let \( w \) be another fixed point of \( f \) and \( g \).

Consider, \( d(z, w) = d(fz, fw) \leq \varphi(m(z, w)) \)

\[
\begin{align*}
  & = \varphi(d(gz, gw) + \gamma[d(gz, fz) + d(gw, fw)]) \\
  & \leq \varphi(d(z, w) + \gamma[d(z, z) + d(w, w)]) \\
  & \leq \varphi(d(z, w) < d(z, w) \quad \text{(Since \( \varphi \)-map, \( \varphi(\omega) < \omega \),}
  
  \quad \text{a contradiction.}
\end{align*}
\]

Therefore, \( f \) and \( g \) have a unique common fixed point.

REFERENCES


