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FIXED POINTS AND SOLUTIONS OF NONLINEAR FUNCTIONAL EQUATIONS IN BANACH SPACES

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Abstract. In this paper, we obtain some common fixed point theorems in Banach spaces for two compatible mapping of type (T)/(I) and for weakly biased mappings. Also, we give applications for the solvability of certain non-linear functional equations.

Keywords: Fixed points, compatible maps of type $(T)/(I)$, weakly biased maps, operator equation.

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1. Introduction

The concept of compatible mappings of type (T) (type(I)) introduced by Pathak et al (see [10]).

Definition 1.1. [10] *Let I and T be a mappings from a normed space E into itself. The mappings I and T are said to be compatible mappings of type (T) if*

$$\lim_{n\to\infty} \|ITx_n - Ix_n\| + \lim_{n\to\infty} \|ITx_n - TIx_n\| \leq \lim_{n\to\infty} \|TIx_n - Tx_n\|$$

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whenever $\{x_n\}$ *is a sequence in* E *such that*

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t \text{ for some } t \in E.$$

Definition 1.2. [10] *Let* I *and* T *be a mappings from a normed space* E *into itself. The mappings* I *and* T *are said to be compatible mappings of type (I) if*

$$\lim_{n\to\infty} \|TIx_n - Tx_n\| + \lim_{n\to\infty} \|ITx_n - TIx_n\| \leq \lim_{n\to\infty} \|ITx_n - Ix_n\|$$

whenever $\{x_n\}$ *is a sequence in* E *such that*

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t \text{ for some } t \in E.$$

Definition 1.3. [4, 5] *Let* F *and* G *be self-maps of a metric space* (M, d). *The pair* $\{F, G\}$ *is G-biased if and only if whenever* $\{x_n\}$ *is a sequence in* X *and* $Fx_n, Gx_n \to t \in X$, *then*

$$\alpha d(GFx_n, Gx_n) \leq \alpha d(FGx_n, Fx_n) \quad \text{if } \alpha = \liminf \text{ and if } \alpha = \limsup .$$

Definition 1.4. [4, 5] *Let* F *and* G *be self-maps of* X. *The pair* $\{F, G\}$ *is weakly G-biased if and only if* $Fp = Gp$ *implies*

$$d(GFp, Gp) \leq d(FGp, Fp).$$

Clearly, every biased mappings are weakly biased mappings (see proposition 1.1 in [4]). The paper is organized as follows: In Section 1, we explain some notations, concepts and the results as noted earlier which can be found in [1-13] . In Section 2, we prove a coincidence fixed point theorem for two compatible mappings of type (T)/ (I) in Banach spaces. Finally, we apply both definitions of compatible mappings of type (T)/ (I) and weakly G-baised to obtain solutions of nonlinear functional equations in Banach spaces as in Section 3.

2. A common fixed point theorem

In this section, we obtain a common fixed point theorem in Banach spaces for two compatible mappings of type T. Now, we prove the following result. Our result is more general the correspondence one in [6] and [11].

Theorem 2.1. *Let T and I be two compatible mappings of type $(T)/(I)$ of a Banach space X into itself, satisfying the following conditions:*

$$(1) \qquad (1-k)I(X)+kT(X)\subset I(X), \quad \textit{where } \ 0<k<1,$$

$$(2) \qquad \|Tx-Ty\|^p \leq \Phi(\max\{\|Ix-Iy\|^p, \|Ix-Tx\|^p, \|Iy-Ty\|^p, \|Ix-Ty\|^p, \|Iy-Tx\|^p\})$$

for all $x, y\in X$, where $p>0$, and the function Φ satisfies the following conditions:
(a)$\Phi:[0,\infty)\to[0,\infty)$ is nondecreasing and right continuous
(b) For every $t>0$, $\Phi(t)<t$.
If for some $x_0\in X$, the sequence $\{x_n\}$ defined by

$$(3) \qquad Ix_{n+1}=(1-c_n)Ix_n+c_nTx_n, \ \textit{for all} \quad n\geq 0$$

with (i) $0<c_n\leq 1$ and (ii) $\lim_{n\to\infty}c_n=h>0$ for $n=0,1,2,...$ converges to a point z in X and if I is continuous at z, then T and I have a unique common fixed point. Further, if I is continuous at Tx, then T and I have a unique common fixed point at which T is continuous.

Proof. Let $z\in X$ such that

$$\lim_{n\to\infty}x_n=z.$$

Now since I is continuous at z. Then, we have that $Ix_n\to Iz$ as $n\to\infty$, so from (3) we have

$$Tx_n=\frac{Ix_{n+1}-(1-c_n)Ix_n}{c_n}\to\frac{Iz-(1-h)Iz}{h}=Iz \text{ as } n\to\infty.$$

Now we shall show that $Tz = Iz$. Form (2) we have

$$\|Tx_n - Tz\|^p \leq \Phi(\max\{\|Ix_n - Iz\|^p, \|Ix_n - Tx_n\|^p, \|Iz - Tz\|^p, \|Ix_n - Tz\|^p, \|Iz - Tx_n\|^p\}).$$

Taking the limit as $n \to \infty$ yields

$$\|Iz - Tz\|^p \leq \Phi(\max\{0, 0, \|Iz - Tz\|^p, \|Iz - Tz\|^p, 0\}).$$

If $\|Tz - Iz\|^p > 0$, then one obtains the contradiction

$$\|Iz - Tz\|^p < \|Iz - Tz\|^p.$$

Therefore, $Tz = Iz$ i.e, z is a coincidence point of T and I. Now since T and I are compatible mappings of type (T) if

$$\lim_{n\to\infty} \|ITx_n - Ix_n\| + \lim_{n\to\infty} \|ITx_n - TIx_n\| \leq \lim_{n\to\infty} \|TIx_n - Tx_n\|$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t, \quad \text{for some } t \in X.$$

Hence using (2),

$$(4)\; \|T^2z - Tz\|^p \leq \Phi(\max\{\|ITz - Iz\|^p, \|ITz - T^2z\|^p, \|Iz - Tz\|^p, \|ITz - Tz\|^p, \|Iz - T^2z\|^p\}),$$

or

$$\|T^2z - Tz\|^p \leq \|ITz - TIz\|^p \leq (\|TIz - Iz\| + \|ITz - Iz\|)^p$$

or

$$\|T^2z - Tz\| \leq \|T^2z - Tz\| - \|ITz - Iz\|,$$

which implies that, $\|ITz - Iz\| \leq 0$, and so

$$(5) \qquad ITz = Iz = Tz.$$

Substitute from (5) to (4), we obtain that

$$\begin{aligned}\|T^2z - Tz\|^p &\leq \Phi(\max\{0, \|Iz - T^2z\|^p, 0, 0, \|Iz - T^2z\|^p\}) \\ &\leq \Phi(\max\{0, \|Tz - T^2z\|^p, 0, 0, \|Tz - T^2z\|^p\}) \\ &\leq \Phi(\|T^2z - Tz\|^p) \leq \|T^2z - Tz\|^p\end{aligned}$$

which is a contradiction. Therefore, we obtain that

$$T^2z = Tz = ITz,$$

i.e., Tz is a common fixed point of T and I. Let v be a another common fixed point of T and I. By (2), we have

$$\begin{aligned}\|u - v\|^p &= \|Tu - Tv\|^p \leq \Phi(\max\{\|Iu - Iv\|^p, \|Iu - Tu\|^p, \|Iv - Tv\|^p, \\ &\quad \|Iu - Tv\|^p, \|Iv - Tu\|^p\}), \\ &= \Phi(\max\{\|u - v\|^p, 0, 0, \|u - v\|^p, \|v - u\|^p\})\end{aligned}$$

which implies that $u = v$. This completing the proof of the theorem.

Remark 2.1. Similar result of Theorem 2.1, can be obtained if we used the concept of weakly G-biased mappings instead of compatible mappings of type $(T)/(I)$.

3. Application to operator equations

By using a contraction condition more general than that used in [11] and replacing weakly compatible mappings by compatible mappings of type (T)/ (I), we investigate the solvability of certain nonlinear functional equations in Banach spaces.

Theorem 3.1. *Let $\{f_n\}$ be sequence of elements in a Banach space X. Let ν_n be the unique solution of the equation $u - TIu = f_n$, where $T, I : X \to X$ satisfying the following conditions*

(h_1) T and I are compatible mapping of type (T)

(h_2) $T^2 = I^2 = \mathbf{I}$, where $\mathbf{I}$ denotes the identity mapping,

(h_3) $\|Tx-Ty\|^2 \le q\max\{\|Ix-Iy\|^2, \|Ix-Tx\|^2, \|Iy-Ty\|^2, \|Ix-Ty\|^2, \|Iy-Tx\|^2\}$ *for all* $x, y \in X$, *where* $q \in (0,1)$. *If* $\|f_n\| \to 0$ *as* $n \to \infty$, *then the sequence* $\{\nu_n\}$ *converges to the solution of the equation*

$$u = Tu = Iu.$$

Proof. We will show that $\{\nu_n\}$ is a Cauchy sequence.

$$\|\nu_n-\nu_m\|^2 \le \Big[\|\nu_n-TI\nu_n\|+\|TI\nu_n-TI\nu_m\|+\|TI\nu_m-\nu_m\|\Big]^2$$
$$\le \Big[\|\nu_n-TI\nu_n\|+\|TI\nu_m-\nu_m\|\Big]^2+2\Big[\|\nu_n-TI\nu_n\|+\|TI\nu_m-\nu_m\|\Big]$$
$$\times\Big[\|TI\nu_n-\nu_n\|+\|\nu_n-\nu_m\|+\|\nu_m-TI\nu_m\|\Big]+\|TI\nu_n-TI\nu_m\|^2$$
$$\le \Big[\|\nu_n-TI\nu_n\|+\|TI\nu_m-\nu_m\|\Big]^2+2\Big[\|\nu_n-TI\nu_n\|+\|TI\nu_m-\nu_m\|\Big]$$
$$\times\Big[\|TI\nu_n-\nu_n\|+\|\nu_n-\nu_m\|+\|\nu_m-TI\nu_m\|\Big]$$
$$+q\max\Big\{\|I^2\nu_n-I^2\nu_m\|^2, \|I^2\nu_n-TI\nu_n\|^2, \|I^2\nu_m-TI\nu_m\|^2,$$
$$\|I^2\nu_n-TI\nu_m\|^2, \|I^2\nu_m-TI\nu_n\|^2\Big\}$$
$$\le \Big[\|\nu_n-TI\nu_n\|+\|TI\nu_m-\nu_m\|\Big]^2+2\Big[\|\nu_n-TI\nu_n\|+\|TI\nu_m-\nu_m\|\Big]$$
$$\times\Big[\|TI\nu_n-\nu_n\|+\|\nu_n-\nu_m\|+\|\nu_m-TI\nu_m\|\Big]^2$$
$$+q\max\Big\{\|\nu_n-\nu_m\|^2, \|\nu_n-TI\nu_n\|^2, \|\nu_m-TI\nu_m\|^2, \Big[\|\nu_n-\nu_m\|+\|\nu_m-TI\nu_m\|\Big]^2,$$
$$\Big[\|\nu_m-\nu_n\|+\|\nu_n-TI\nu_n\|\Big]^2\Big\}.$$

On letting $n \to \infty$ and using the hypothesis, we obtain

$$\|\nu_n-\nu_m\|^2 \le q\|\nu_n-\nu_m\|^2,$$

which is a contradiction. It follows therefore that $\{\nu_n\}$ is a Cauchy sequence in X. Hence it converges, say to ν in X. Also

$$\begin{aligned}\|\nu - TI\nu\| &\leq \|\nu-\nu_n\| + \|\nu_n - TI\nu_n\| + \|TI\nu_n - TI\nu\| \\ &\leq \|\nu-\nu_n\| + \|\nu_n - TI\nu_n\| + \{q\max\{\|\nu_n-\nu\|^2, \|\nu_n - TI\nu_n\|^2, \|\nu - TI\nu\|^2, \\ &\quad [\|\nu_n-\nu\| + \|\nu - TI\nu\|]^2, [\|\nu_n - \nu\| + \|\nu_n - TI\nu_n\|]^2\}\}^{\frac{1}{2}}.\end{aligned}$$

Hence taking the limit as $n \to \infty$, we get $\nu = TI\nu$, which from (h_2) implies that $T\nu = I\nu$. Since T and I are compatible mapping of type (T), we have

$$\lim_{n\to\infty} \|ITx_n - Ix_n\| + \lim_{n\to\infty} \|ITx_n - TIx_n\| \leq \lim_{n\to\infty} \|TIx_n - Tx_n\|$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t, \text{ for some } t \in X.$$

Using (h_2) and (h_3), we obtain that

$$\begin{aligned}\|v - Tv\|^2 &= \|T^2v - Tv\|^2 \leq q\max\{\|ITv - Iv\|^2, \|ITv - T^2v\|^2, \\ &\quad \|Iv - Tv\|^2, \|ITv - Tv\|^2, \|Iv - TIv\|^2\} \\ &= q\max\{\|v - Tv\|^2, 0, 0, \|v - Tv\|^2, \|v - Tv\|^2\},\end{aligned}$$

which implies that $\nu = T\nu$. Then $I\nu = IT\nu = TI\nu = \nu$, and ν is also a fixed point of I. Now, for the class of weakly G-biased mappings, we obtain the following theorem:

Theorem 3.2. *Let $\{f_n\}$ and $\{g_n\}$ be sequences of elements in a Banach space X. Let $\{\nu_n\}$ be the unique solution of the system of equations $u - FGu = f_n$ and $u - HGu = g_n$, where F, G and $H : X \to X$ satisfying the following conditions:*

(I) $\{F, G\}$ and $\{H, G\}$ are weakly G-biased pairs ,

(II) $F^2 = G^2 = H^2 = \mathbf{I}$, where $\mathbf{I}$ denotes the identity mapping,

(III) $\|Fx - Hy\|^2 \leq q\max\{\|Gx - Gy\|^2, \|Gx - Fx\|^2, \|Gy - Hy\|^2, \|Gx - Fy\|^2, \|Gy - Hx\|^2\}$

for all $x, y \in X$, where $q \in (0, 1)$. If $\|f_n\|, \|g_n\| \to 0$ as $n \to \infty$, then the sequence $\{\nu_n\}$ converges to the solution of the equation

$$u = Fu = Gu = Hu.$$

Proof. We will show that $\{\nu_n\}$ is a Cauchy sequence

$$\begin{aligned}\|\nu_n-\nu_m\|^2 &\le [\|\nu_n-FG\nu_n\|+\|FG\nu_n-HG\nu_m\|+\|HG\nu_m-\nu_m\|]^2\\ &\le \{\|\nu_n-FG\nu_n\|+\|HG\nu_m-\nu_m\|\}^2+2[\|\nu_n-FG\nu_n\|+\|HG\nu_n-\nu_m\|]\\ &\times[\|FG\nu_n-\nu_n\|+\|\nu_n-\nu_m\|+\|\nu_m-HG\nu_m\|]+\|FG\nu_n-HG\nu_n\|^2\\ &\le \{\|\nu_n-FG\nu_n\|+\|HG\nu_m-\nu_m\|\}^2+2[\|\nu_n-FG\nu_n\|+\|HG\nu_n-\nu_m\|]\\ &\times[\|FG\nu_n-\nu_n\|+\|\nu_n-\nu_m\|+\|\nu_m-HG\nu_m\|]+q\max\{\|G^2\nu_n-G^2\nu_m\|\\ &\quad \|G^2\nu_n-FG\nu_n\|,\|G^2\nu_n-HG\nu_m\|^2,\|G^2\nu_n-FG\nu_m\|^2,\|G^2\nu_m-HG\nu_m\|^2\}\\ &\le [\|\nu_n-FG\nu_n\|+\|HG\nu_m-\nu_m\|]^2+2[\|\nu_n-FG\nu_n\|+\|FG\nu_m-\nu_m\|]\\ &\quad\times[\|FG\nu_n-\nu_n\|+\|\nu_n-\nu_m\|+\|\nu_m-HG\nu_m\|]\\ &\quad+q\max\{\|\nu_n-\nu_m\|^2,\|\nu_n-FG\nu_n\|^2,\|\nu_m-HG\nu_m\|^2,[\|\nu_n-\nu_m\|\\ &\quad+\|\nu_m-FG\nu_m\|]^2,[\|\nu_m-\nu_n\|+\|\nu_n-HG\nu_n\|]^2\}.\end{aligned}$$

Letting $n\to\infty$ with $m>n$, we have

$$\lim_{m,n\to\infty}\|\nu_n-\nu_m\|^2\le q\lim_{m,n\to\infty}\|\nu_n-\nu_m\|^2,$$

which implies that

$$\lim_{m,n\to\infty}\|\nu_n-\nu_m\|^2=0.$$

Thus $\{\nu_n\}$ is a Cauchy sequence in X. And converges to a point ν in X. Further;

$$\begin{aligned}\|\nu-HG\nu| &\le \|\nu-\nu_n\|+\|\nu_n-FG\nu_n\|+\|FG\nu_n-HG\nu\|\\ &\le \|\nu-\nu_n\|+\|\nu_n-FG\nu_n\|+\{q\max\{\|\nu_n-\nu\|^2,\|\nu_n-FG\nu_n\|^2,\|\nu-HG\nu\|^2,\\ &[\|\nu_n-\nu\|+\|\nu-FG\nu\|]^2,[\|\nu_n-\nu\|+\|\nu_n-HG\nu_n\|]^2\}\}^{\frac{1}{2}}\end{aligned}$$

Hence taking the limit as $n\to\infty$, we get $\nu=HG\nu$, which from (I) implies that $H\nu=G\nu$. Similarly, $G\nu=F\nu$. From (I), we have

$$d(GFz,Gz)\le d(FGz,Fz)\ \text{ and }\ d(GHz,Gz)\le d(HGz,Hz),$$

which implies

$$d(G^2z, Gz) \leq d(F^2z, Fz) \text{ and } d(G^2z, Gz) \leq d(H^2z, Hz),$$

so we obtain

$$H^2z = Hz = F^2z = Fz \text{ and } FGv = GFv = v = HGv = GHv.$$

Using (II) and (III),

$$\begin{aligned} \|v - Hv\|^2 &= \|F^2v - Hv\|^2 \\ &\leq q \max\{\|GFv - Gv\|^2, \|GFv - F^2v\|^2, \\ &\quad \|Gv - Hv\|^2, \|GFv - Fv\|^2, \|Gv - HFv\|^2\} \\ &= q \max\{\|v - Hv\|^2, 0, 0, \|v - Hv\|^2, \|v - Hv\|^2\}, \end{aligned}$$

which implies that $v = Hv$. Then

$$Gv = GFv = FGv = v, Gv = GHv = HGv = v,$$

this completing the proof of the theorem.

Corollary 3.1. *Let $\{f_n\}$ be sequence of elements in a Banach space X. Let ν_n be the unique solution of the equation $u - FGu = f_n$, where $F, G : X \to X$ satisfying the following conditions:*

(d_1) *F and G are weakly G-biased mappings*

(d_2) *$F^2 = G^2 = \mathbf{I}$, where $\mathbf{I}$ denotes the identity mapping,*

(d_3) $\|Fx - Fy\|^2 \leq q \max\{\|Gx - Gy\|^2, \|Gx - Fx\|^2, \|Gy - Fy\|^2, \|Gx - Fy\|^2, \|Gy - Fx\|^2\}$ *for all $x, y \in X$, where $q \in (0, 1)$. If $\|f_n\| \to 0$ as $n \to \infty$, then the sequence $\{\nu_n\}$ converges to the solution of the equation*

$$u = Fu = Gu.$$

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