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COUPLED COINCIDENCE POINTS FOR HYBRID PAIR OF MAPPINGS VIA MIXED MONOTONE PROPERTY IN BIPOLAR METRIC SPACES

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Abstract. In this paper, we establish a coupled coincidence fixed point results for a hybrid pair of single valued and multivalued mappings satisfying generalized contractive conditions, defined on a partially ordered bipolar metric space. Our results unify, generalize and complement several known comparable results from the current literature. An example is given.

Keywords: bipolar metric space; partial ordered set; coupled coincidence point; coupled common fixed point; multivalued mapping.

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1. INTRODUCTION

Fixed point theory plays a vital role in applications of many branches of mathematics. Finding fixed points of generalized contraction mappings has become the focus of well-built research activity in fixed point theory. Recently, many investigators have published various papers on fixed point theory and applications in different ways. One of the recently popular topics in fixed point theory is to cast the existence of fixed points of contraction mappings in bipolar metric

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spaces which can be considered as generalizations of Banach contraction principle. In 2016, Mutlu and Gürdal [1] has been introduced the concepts of bipolar metric space and they investigated certain basic fixed point and coupled fixed point theorems for covariant and contravariant map with contractive conditions see ([2]-[6]).

The study of mixed monotone mappings is an active area of research due to its wide scope of applications. The theory of mixed monotone multivalued mappings in ordered Banach spaces was extensively investigated in [7]. Existence of fixed points in ordered metric spaces was initiated by Ran and Reurings [8], and further studied by Nieto and López [9]. Several authors studied the problem of existence and uniqueness of a fixed point for mappings satisfying different contractive conditions in the framework of partially ordered metric spaces and references therein ([10]-[13]).

In [14], Bhaskar and Lakshmikantham introduced the concept of coupled fixed point and proved some coupled fixed point theorems in partially ordered sets. As an application, they studied the existence and uniqueness of solution for a periodic boundary value problem associated with a first order ordinary differential equation. Subsequently, Lakshmikantham and Ćirić [15] proved coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces. The study of fixed points for multivalued contractions mappings using the Hausdorff metric was initiated by Markin [16] and [17]. Later, Dhage [18] established hybrid fixed point theorems and gave applications of their results (see also [19]). Hong in his recent work [20] proved hybrid fixed point theorems involving multi-valued operators which satisfy weakly generalized contractive conditions in ordered complete metric spaces.

This paper aims to introduce some coupled coincidence point theorems for a hybrid pair $\{g,F\}$ of single valued and multivalued mappings satisfying generalized contractive conditions defined on partially ordered bipolar metric spaces. We have illustrated the validity of the hypotheses of our results.

First we recall some basic definitions and results.

2. Preliminaries

Definition 2.1:([1]) Let A and B be a two non-empty sets. Suppose that $d : A \times B \to [0, \infty)$ be a mapping satisfying the following properties :

(B_1) $d(a,b) = 0$ if and only if $a = b$ for all $(a,b) \in A \times B$,

(B_2) $d(a,b) = d(b,a)$, for all $a,b \in A \cap B$,

(B_3) $d(a_1,b_2) \leq d(a_1,b_1) + d(a_2,b_1) + d(a_2,b_2)$, for all $a_1,a_2 \in A$, $b_1,b_2 \in B$.

Then the mapping d is called a Bipolar-metric on the pair (A,B) and the triple (A,B,d) is called a Bipolar-metric space.

Definition 2.2:([1]) Assume (A_1,B_1) and (A_2,B_2) as two pairs of sets.The function $F : A_1 \cup B_1 \to A_2 \cup B_2$ is said to be a covariant map if $F(A_1) \subseteq A_2$ and $F(B_1) \subseteq B_2$ and denote this as $F : (A_1,B_1) \rightrightarrows (A_2,B_2)$.

The mapping $F : A_1 \cup B_1 \to A_2 \cup B_2$ is said to be a contravariant map, if $F(A_1) \subseteq B_2$ and $F(B_1) \subseteq A_2$ and this as $F : (A_1,B_1) \leftrightarrows (A_2,B_2)$.

In particular, if d_1 and d_2 are bipolar metrics in (A_1,B_1) and (A_2,B_2) respectively. Then in some times we use the notations $F : (A_1,B_1,d_1) \rightrightarrows (A_2,B_2,d_2)$ and $F : (A_1,B_1,d_1) \leftrightarrows (A_2,B_2,d_2)$.

Definition 2.3:([1]) Let (A,B,d) be a bipolar metric space. A point $v \in A \cup B$ is said to be left point if $v \in A$, a right point if $v \in B$ and a central point if both.

Similarly, a sequence $\{a_n\}$ on the set A and a sequence $\{b_n\}$ on the set B are called a left and right sequence respectively.

In a bipolar metric space, sequence is the simple term for a left or right sequence.

A sequence $\{v_n\}$ is convergent to a point v if and only if $\{v_n\}$ is a left sequence, v is a right point and $\lim_{n\to\infty} d(v_n,v) = 0$; or $\{v_n\}$ is a right sequence, v is a left point and $\lim_{n\to\infty} d(v,v_n) = 0$.

A bisequence $(\{a_n\},\{b_n\})$ on (A,B,d) is sequence on the set $A \times B$. If the sequence $\{a_n\}$ and $\{b_n\}$ are convergent, then the bisequence $(\{a_n\},\{b_n\})$ is said to be convergent. $(\{a_n\},\{b_n\})$ is Cauchy sequence , if $\lim_{n,m\to\infty} d(a_n,b_m) = 0$.

A bipolar metric space is called complete, if every Cauchy bisequence is convergent, hence biconvergent.

3. Main Results

Following definitions and results will be need in the sequel.

Let (A,B,d) be a bipolar metric space. For a points $a \in A, b \in B$ and a subsets $X \subseteq A, Y \subseteq B$, Consider the bipolar metric $d(a,Y) = \{d(a,y)/y \in Y\}$ and $d(X,b) = \{d(x,b)/x \in X\}$. We denote by $CB(A)$ and $CB(B)$ be a class of all nonempty closed and bounded subsets of A and B respectively. Let H be the Hausdorff bipolar metric induced by the bipolar metric d on (A,B), that is $H(X,Y) = \max\left\{\sup_{x\in X} d(x,B), \sup_{y\in Y} d(A,y)\right\}$ for every $X \in CB(A)$ and $Y \in CB(B)$.

Definition 3.1: Let be given the mapping $F : A^2 \cup B^2 \to CB(A \cup B)$ and $g : A \cup B \to A \cup B$. An element $(a,b) \in A^2 \cup B^2$ is called

(i) a coupled fixed point of a set valued mapping F if $a \in F(a,b)$ and $b \in F(b,a)$

(ii) a coupled coincidence point of a pair $\{F,g\}$ if $ga \in F(a,b)$ and $gb \in F(b,a)$

(iii) a coupled common fixed point of a pair $\{F,g\}$ if $a = ga \in F(a,b)$ and $b = gb \in F(b,a)$.

Lemma 3.1:([17]) Let $\kappa \geq 0$. If $X \in CB(A), Y \in CB(B)$ with $H(X,Y) \leq \kappa$, then for each $x \in X$ there exist an element $y \in Y$ such that $d(x,y) \leq \kappa$

Lemma 3.2:([17]) Let (X_n,Y_n) be a bisequence in $(CB(A),CB(B))$ with $\lim_{n\to\infty} H(X_n,Y) = \lim_{n\to\infty} H(X,Y_n) = 0$ for $X \in CB(A), Y \in CB(B)$. If $a_n \in X_n$, $b_n \in Y_n$ and $\lim_{n\to\infty} d(a_n,b) = \lim_{n\to\infty} d(a,b_n) = 0$, then $a \in X$ and $b \in Y$.

Definition 3.2: Let $(A,B,\leq)$ be a partially ordered set and $F : (A^2,B^2) \rightrightarrows CB(A,B)$ be a covariant map. The map F is called a mixed monotone mapping if for $(a_i,b_i) \in A^2 \cup B^2$ $(i = 1,2)$, with $a_1 \leq a_2$ and $b_1 \geq b_2$ then for all $u_1 \in F(a_1,b_1)$ there exist $u_2 \in F(a_2,b_2)$ such that $u_1 \leq u_2$ and for all $v_1 \in F(b_1,a_1)$ there exist $v_2 \in F(b_2,a_2)$ such that $v_1 \geq v_2$.

Definition 3.3: Let $(A,B,\leq)$ be a partially ordered set and $F : (A^2,B^2) \rightrightarrows CB(A,B)$ and $g : (A,B) \rightrightarrows (A,B)$ be two covariant maps. We say F has the mixed g-monotone property if F is monotone g-non-decreasing in its first argument a and is monotone g-non-increasing in its second argument b, that is, for any $(a,b) \in A^2 \cup B^2$.

$$(a_1,a_2) \in A^2,\ g(a_1) \leq g(a_2) \Rightarrow F(a_1,b) \leq F(a_2,b)$$

$$(b_1,b_2) \in B^2,\ g(b_1) \geq g(b_2) \Rightarrow F(a,b_1) \geq F(a,b_2)$$

Note that if $g(a_1) \leq g(a_2)$, $g(b_1) \geq g(b_2)$ and F has mixed g-monotone property, by Definition-(3.3), we obtain $F(a_1,b_1) \leq F(a_2,b_2)$ and $F(b_1,a_1) \geq F(b_2,a_2)$.

Let $\Psi = \{\psi : \psi : R \to R\}$ be a family of non-decreasing continuous function satisfying the following three properties: such that

(a) $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t > 0$,

(b) $0 < \psi(t) < t$ for all $t > 0$

(c) let $\psi \in \Psi$, $\lim_{n\to+\infty} \psi^n(t) = 0$ for all $t > 0$

Theorem 3.1: Let $(A,B,\leq)$ be a partially ordered set such that there exist a bipolar metric d on (A,B) .Consider the covariant mappings $F : (A^2,B^2) \rightrightarrows CB(A,B)$ and $g : (A,B) \rightrightarrows (A,B)$ such that $F(A^2 \cup B^2) \subseteq g(A\cup B)$ and satisfies the following conditions.

$$H\left(F(a,b),F(p,q)\right) \leq \psi\left(\frac{d(g(a),g(p))+d(g(b),g(q))}{2}\right) \tag{1}$$

for all $a,b \in A$ and $p,q \in B$ with $g(a) \geq g(p)$ and $g(b) \leq g(q)$. Suppose also that

(3.1.1) $g(A\cup B)$ is complete subspaces of (A,B,d)

(3.1.2) F has a mixed g-monotone property

(3.1.3) There exist $(a_0,b_0) \in A^2 \cup B^2$ and for some $x_1 \in F(a_0,b_0)$, $y_1 \in F(b_0,a_0)$ we have $ga_0 \leq x_1$ and $gb_0 \geq y_1$

(3.1.4) If a non-decreasing sequence $(\{a_n\},\{p_n\})$ is convergent to (p,a) for $a \in A, p \in B$, then $a_n \leq p$, $p_n \leq a$ for all n and if a non-increasing sequence $(\{b_n\},\{q_n\})$ is convergent to (q,b) for $b \in A, q \in B$, then $b_n \geq q$, $q_n \geq b$ for all n.

Then F and g have a coupled coincidence point, that is there exist $(a,b) \in A^2 \cup B^2$ such that $ga \in F(a,b)$ and $gb \in F(b,a)$.

Proof: Let $a_0,b_0 \in A$ and $p_0,q_0 \in B$. By (3.1.3) there exist $x_1 \in F(a_0,b_0)$, $y_1 \in F(b_0,a_0)$ and $w_1 \in F(p_0,q_0)$, $z_1 \in F(q_0,p_0)$ we have $ga_0 \leq w_1$ and $gb_0 \geq z_1$ and $gp_0 \leq x_1$ and $gq_0 \geq y_1$. By assumption $F(A^2 \cup B^2) \subseteq g(A\cup B)$, there exist $a_1,b_1 \in A$ and $p_1,q_1 \in B$ such that

$$ga_1 = x_1 \in F(a_0,b_0), gb_1 = y_1 \in F(b_0,a_0) \text{ and } gp_1 = w_1 \in F(p_0,q_0), gq_1 = z_1 \in F(q_0,p_0)$$

and $ga_0 \leq gp_1$ and $gb_0 \geq gq_1$ and $gp_0 \leq ga_1$ and $gq_0 \geq gb_1$.

Applying this in inequality (1), we have

$$H(F(a_0,b_0),F(p_1,q_1)) \leq \psi\left(\frac{d(ga_0,gp_1)+d(gb_0,gq_1)}{2}\right) \tag{2}$$

and

$$H(F(b_0,a_0),F(q_1,p_1)) \leq \psi\left(\frac{d(gb_0,gq_1)+d(ga_0,gp_1)}{2}\right) \tag{3}$$

On adding (2) and (3), we get

$$\frac{H(F(a_0,b_0),F(p_1,q_1))+H(F(b_0,a_0),F(q_1,p_1))}{2} \leq \psi\left(\frac{d(ga_0,gp_1)+d(gb_0,gq_1)}{2}\right) \tag{4}$$

On the other hand

$$H(F(a_1,b_1),F(p_0,q_0)) \leq \psi\left(\frac{d(ga_1,gp_0)+d(gb_1,gq_0)}{2}\right) \tag{5}$$

and

$$H(F(b_1,a_1),F(q_0,p_0)) \leq \psi\left(\frac{d(gb_1,gq_0)+d(ga_1,gp_0)}{2}\right) \tag{6}$$

On adding (5) and (6), we get

$$\frac{H(F(a_1,b_1),F(p_0,q_0))+H(F(b_1,a_1),F(q_0,p_0))}{2} \leq \psi\left(\frac{d(ga_1,gp_0)+d(gb_1,gq_0)}{2}\right) \tag{7}$$

Moreover,

$$H(F(a_0,b_0),F(p_0,q_0)) \leq \psi\left(\frac{d(ga_0,gp_0)+d(gb_0,gq_0)}{2}\right) \tag{8}$$

and

$$H(F(b_0,a_0),F(q_0,p_0)) \leq \psi\left(\frac{d(gb_0,gq_0)+d(ga_0,gp_0)}{2}\right) \tag{9}$$

On adding (8) and (9), we get

$$\frac{H(F(a_0,b_0),F(p_0,q_0))+H(F(b_0,a_0),F(q_0,p_0))}{2} \leq \psi\left(\frac{d(ga_0,gp_0)+d(gb_0,gq_0)}{2}\right) \tag{10}$$

From (4), (7) and (10),

If $d(ga_0,gp_1) = d(gb_0,gq_1) = 0$, then $ga_0 = gp_1 \in F(p_0,q_0), gb_0 = gq_1 \in F(q_0,p_0)$

and

If $d(ga_1,gp_0) = d(gb_1,gq_0) = 0$, then $gp_0 = ga_1 \in F(a_0,b_0), gq_0 = gb_1 \in F(b_0,a_0)$

If $d(ga_0, gp_0) = d(gb_0, gq_0) = 0$ then $ga_0 = gp_0, gb_0 = gq_0$. Consequently (ga_0, gb_0) is a coupled coincidence point of F and g.

Assume that either $d(ga_0, gp_1) \neq 0$ or $d(gb_0, gq_1) \neq 0$ and $d(ga_1, gp_0) \neq 0$ or $d(gb_1, gq_0) \neq 0$ also $d(ga_0, gp_0) \neq 0$ or $d(gb_0, gq_0) \neq 0$. Take

$$\frac{\varepsilon}{2} = \frac{d(ga_0, gp_1) + d(gb_0, gq_1)}{2} > 0, \frac{\varepsilon}{2} = \frac{d(ga_1, gp_0) + d(gb_1, gq_0)}{2} > 0$$

$$\text{and } \frac{\varepsilon}{2} = \frac{d(ga_0, gp_0) + d(gb_0, gq_0)}{2} > 0$$

Using a property of ψ, we have $\psi(\frac{\varepsilon}{2}) > 0$, so from (4), (7) and (10)

$$\frac{H(F(a_0, b_0), F(p_1, q_1)) + H(F(b_0, a_0), F(q_1, p_1))}{2} \leq \psi(\frac{\varepsilon}{2}) \tag{11}$$

and

$$\frac{H(F(a_1, b_1), F(p_0, q_0)) + H(F(b_1, a_1), F(q_0, p_0))}{2} \leq \psi(\frac{\varepsilon}{2}) \tag{12}$$

also

$$\frac{H(F(a_0, b_0), F(p_0, q_0)) + H(F(b_0, a_0), F(q_0, p_0))}{2} \leq \psi(\frac{\varepsilon}{2}) \tag{13}$$

Since $ga_1 \in F(a_0, b_0)$, $gb_1 \in F(b_0, a_0)$ then (11) and Lemma-(3.1) there exists $w_2 \in F(p_1, q_1)$, $z_2 \in F(q_1, p_1)$ such that

$$\frac{d\left(ga_1, w_2\right) + d\left(gb_1, z_2\right)}{2} \leq \psi(\frac{\varepsilon}{2})$$

Since $F(A^2 \cup B^2) \subseteq g(A \cup B)$ there exist $p_2, q_2 \in B$ such that $w_2 = gp_2, z_2 = gq_2$. Thus

$$\frac{d\left(ga_1, gp_2\right) + d\left(gb_1, gq_2\right)}{2} \leq \psi(\frac{\varepsilon}{2}) \tag{14}$$

and Since $gp_1 \in F(p_0, q_0)$, $gq_1 \in F(q_0, p_0)$ then (12) and Lemma-(3.1) there exists $x_2 \in F(a_1, b_1)$, $y_2 \in F(b_1, a_1)$ such that

$$\frac{d\left(x_2, gp_1\right) + d\left(y_2, gq_1\right)}{2} \leq \psi(\frac{\varepsilon}{2})$$

Since $F(A^2 \cup B^2) \subseteq g(A \cup B)$ there exist $a_2, b_2 \in A$ such that $x_2 = ga_2, y_2 = gb_2$. Thus

$$\frac{d\left(ga_2, gp_1\right) + d\left(gb_2, gq_1\right)}{2} \leq \psi(\frac{\varepsilon}{2}) \tag{15}$$

also Since $ga_1 \in F(a_0,b_0)$, $gb_1 \in F(b_0,a_0)$ and $gp_1 \in F(p_0,q_0)$, $gq_1 \in F(q_0,p_0)$ then (13) and Lemma-(3.1) there exists $x_1 \in F(a_0,b_0)$, $y_1 \in F(b_0,a_0)$ and $w_1 \in F(p_0,q_0)$, $z_1 \in F(q_0,p_0)$ such that

$$\frac{d(x_1,w_1)+d(y_1,z_1)}{2} \leq \psi(\frac{\varepsilon}{2})$$

Since $F(A^2 \cup B^2) \subseteq g(A \cup B)$ there exist $a_1,b_1 \in A, p_1,q_1 \in B$ such that $x_1 = ga_1, y_1 = gb_1, w_1 = gp_1, z_1 = gq_1$. Thus

$$\frac{d(ga_1,gp_1)+d(gb_1,gq_1)}{2} \leq \psi(\frac{\varepsilon}{2}) \tag{16}$$

Since, we have $ga_0 \leq gp_1$, $gb_0 \geq gq_1$ and $gp_0 \leq ga_1$, $gq_0 \geq gb_1$, $ga_1 \in F(a_0,b_0)$, $gp_1 \in F(p_0,q_0), gb_1 \in F(b_0,a_0), gq_1 \in F(q_0,p_0)$ and $ga_2 \in F(a_1,b_1), gp_2 \in F(p_1,q_1)$, $gb_2 \in F(b_1,a_1), gq_2 \in F(q_1,p_1)$, by assumption (3.1.2), we get

$$ga_1 \leq gp_2, gb_1 \geq gq_2 \text{ and } gp_1 \leq ga_2, gq_1 \geq gb_2$$

Applying this in (1) and using (14),(15), (16) we obtain

$$\begin{aligned}\frac{H(F(a_1,b_1),F(p_2,q_2))+H(F(b_1,a_1),F(q_2,p_2))}{2} &\leq \psi\left(\frac{d(ga_1,gp_2)+d(gb_1,gq_2)}{2}\right)\\ &\leq \psi^2(\frac{\varepsilon}{2}) \end{aligned} \tag{17}$$

and

$$\begin{aligned}\frac{H(F(a_2,b_2),F(p_1,q_1))+H(F(b_2,a_2),F(q_1,p_1))}{2} &\leq \psi\left(\frac{d(ga_2,gp_1)+d(gb_2,gq_1)}{2}\right)\\ &\leq \psi^2(\frac{\varepsilon}{2}) \end{aligned} \tag{18}$$

also

$$\begin{aligned}\frac{H(F(a_1,b_1),F(p_1,q_1))+H(F(b_1,a_1),F(q_1,p_1))}{2} &\leq \psi\left(\frac{d(ga_1,gp_1)+d(gb_1,gq_1)}{2}\right)\\ &\leq \psi^2(\frac{\varepsilon}{2}) \end{aligned} \tag{19}$$

Since $ga_2 \in F(a_1,b_1)$, $gb_2 \in F(b_1,a_1)$then (17) and Lemma-(3.1) there exists $w_3 \in F(p_2,q_2)$, $z_3 \in F(q_2,p_2)$ such that

$$\frac{d(ga_2,w_3)+d(gb_2,z_3)}{2} \leq \psi^2(\frac{\varepsilon}{2})$$

Since $F(A^2 \cup B^2) \subseteq g(A \cup B)$ there exist $p_3, q_3 \in B$ such that $w_3 = gp_3, z_3 = gq_3$. Thus

$$\frac{d(ga_2,gp_3)+d(gb_2,gq_3)}{2} \leq \psi^2(\frac{\varepsilon}{2}) \tag{20}$$

and Since $gp_2 \in F(p_1,q_1)$, $gq_2 \in F(q_1,p_1)$then (18) and Lemma-(3.1) there exists $x_3 \in F(a_2,b_2)$, $y_3 \in F(b_2,a_2)$ such that

$$\frac{d(x_3,gp_2)+d(y_3,gq_2)}{2} \leq \psi^2(\frac{\varepsilon}{2})$$

Since $F(A^2 \cup B^2) \subseteq g(A \cup B)$ there exist $a_3, b_3 \in A$ such that $x_3 = ga_3, y_3 = gb_3$. Thus

$$\frac{d(ga_3,gp_2)+d(gb_3,gq_2)}{2} \leq \psi^2(\frac{\varepsilon}{2}) \tag{21}$$

also Since $ga_2 \in F(a_1,b_1)$, $gb_2 \in F(b_1,a_1)$ and $gp_2 \in F(p_1,q_1)$, $gq_2 \in F(q_1,p_1)$ then (19) and Lemma-(3.1) there exists $x_2 \in F(a_1,b_1)$, $y_2 \in F(b_1,a_1)$ and
$w_2 \in F(p_1,q_1)$, $z_2 \in F(q_1,p_1)$ such that

$$\frac{d(x_2,w_2)+d(y_2,z_2)}{2} \leq \psi^2(\frac{\varepsilon}{2})$$

Since $F(A^2 \cup B^2) \subseteq g(A \cup B)$ there exist $a_2, b_2 \in A, p_2, q_2 \in B$ such that
$x_2 = ga_2, y_2 = gb_2, w_2 = gp_2, z_2 = gq_2$ Thus

$$\frac{d(ga_2,gp_2)+d(gb_2,gq_2)}{2} \leq \psi^2(\frac{\varepsilon}{2}) \tag{22}$$

Since, we have $ga_1 \leq gp_2$, $gb_1 \geq gq_2$ and $gp_1 \leq ga_2$, $gq_1 \geq gb_2$, $ga_2 \in F(a_1,b_1)$, $gp_2 \in F(p_1,q_1), gb_2 \in F(b_1,a_1), gq_2 \in F(q_1,p_1)$ and $ga_3 \in F(a_2,b_2), gp_3 \in F(p_2,q_2)$, $gb_3 \in F(b_2,a_2), gq_3 \in F(q_2,p_2)$. Again, applying assumption (3.1.2), we get

$$ga_2 \leq gp_3, gb_2 \geq gq_3 \text{ and } gp_2 \leq ga_3, gq_2 \geq gb_3$$

Continuing similarly this process, we have $ga_{n+1} \in F(a_n,b_n)$,
$gp_{n+1} \in F(p_n,q_n), gb_{n+1} \in F(b_n,a_n), gq_{n+1} \in F(q_n,p_n)$ with

$$ga_n \leq gp_{n+1}, gb_n \geq gq_{n+1} \text{ and } gp_n \leq ga_{n+1}, gq_n \geq gb_{n+1}$$

such that

$$\frac{d(ga_n, gp_{n+1}) + d(gb_n, gq_{n+1})}{2} \leq \psi^n(\frac{\varepsilon}{2}) \tag{23}$$

and

$$\frac{d(ga_{n+1}, gp_n) + d(gb_{n+1}, gq_n)}{2} \leq \psi^n(\frac{\varepsilon}{2}) \tag{24}$$

also

$$\frac{d(ga_n, gp_n) + d(gb_n, gq_n)}{2} \leq \psi^n(\frac{\varepsilon}{2}) \tag{25}$$

Put $t_n = d(ga_n, gp_{n+1}) + d(gb_n, gq_{n+1})$ for any $n \in N$ then

$$t_n \leq 2\psi^n(\frac{\varepsilon}{2}) \tag{26}$$

Put $s_n = d(ga_{n+1}, gp_n) + d(gb_{n+1}, gq_n)$ for any $n \in N$ then

$$s_n \leq 2\psi^n(\frac{\varepsilon}{2}) \tag{27}$$

Put $r_n = d(ga_n, gp_n) + d(gb_n, gq_n)$ for any $n \in N$ then

$$r_n \leq 2\psi^n(\frac{\varepsilon}{2}) \tag{28}$$

Since $\lim_{n\to\infty} \psi^n(\frac{\varepsilon}{2}) = 0$. Therefore, (26), (27) and (28) gives

$$\lim_{n\to\infty} t_n = 0, \lim_{n\to\infty} s_n = 0 \text{ and } \lim_{n\to\infty} r_n = 0$$

that is

$$\begin{aligned}
&\lim_{n\to\infty} d(ga_n, gp_{n+1}) = \lim_{n\to\infty} d(gb_n, gq_{n+1}) = 0,\\
&\lim_{n\to\infty} d(ga_{n+1}, gp_n) = \lim_{n\to\infty} d(gb_{n+1}, gq_n) = 0\\
&\lim_{n\to\infty} d(ga_n, gp_n) = \lim_{n\to\infty} d(gb_n, gq_n) = 0
\end{aligned}$$

Using the property (B_3), we have

$$d(ga_n, gp_m) \leq d(ga_n, gp_{n+1}) + d(ga_{n+1}, gp_{n+1}) + \cdots + d(ga_{m-1}, gp_m)$$

$$d(gb_n, gq_m) \leq d(gb_n, gq_{n+1}) + d(gb_{n+1}, gq_{n+1}) + \cdots + d(gb_{m-1}, gq_m)$$

(29)

and

$$d(ga_m, gp_n) \leq d(ga_m, gp_{m-1}) + d(ga_{m-1}, gp_{m-1}) + \cdots + d(ga_{n+1}, gp_n)$$

$$d(gb_m, gq_n) \leq d(gb_m), gq_{m-1}) + d(gb_{m-1}, gq_{m-1}) + \cdots + d(gb_{n+1}, gq_n)$$

(30)

Next, we show that $(\{ga_n\}, \{gp_n\})$ and $(\{gb_n\}, \{gq_n\})$ are Cauchy bisequence in $(g(A), g(B))$ for each $n; m \in N$be such that $n < m$. Then, from (26), (27), (28), (29) and (30), we have

$$\begin{aligned}
& d(ga_n, gp_m) + d(gb_n, gq_m) \\
\leq\ & (d(ga_n, gp_{n+1}) + d(gb_n, gq_{n+1})) + (d(ga_{n+1}, gp_{n+1}) + d(gb_{n+1}, gq_{n+1})) \\
& + \cdots + (d(ga_{m-1}, gp_{m-1}) + d(gb_{m-1}, gq_{m-1})) \\
& \quad + (d(ga_{m-1}, gp_m) + d(gb_{m-1}, gq_m)) \\
\leq\ & t_n + r_{n+1} + \cdots + r_{m-1} + t_{m-1} \\
\leq\ & (t_n + t_{n+1} + \cdots + t_{m-1}) + (r_{n+1} + r_{n+2} + \cdots + r_{m-1}) \\
\leq\ & \sum_{k=n}^{m-1} t_k + \sum_{k=n+1}^{m-1} r_k \\
\leq\ & \sum_{k=n}^{m-1} 2\psi^k(\frac{\varepsilon}{2}) + \sum_{k=n+1}^{m-1} 2\psi^k(\frac{\varepsilon}{2}) \\
\leq\ & 2\sum_{k=n}^{+\infty} \psi^k(\frac{\varepsilon}{2}) + 2\sum_{k=n+1}^{+\infty} \psi^k(\frac{\varepsilon}{2}) \qquad (31)
\end{aligned}$$

and

$$\begin{aligned}
& d(ga_m, gp_n) + d(gb_m, gq_n) \\
\leq\ & (d(ga_m, gp_{m-1}) + d(gb_m, gq_{m-1})) + (d(ga_{m-1}, gp_{m-1}) + d(gb_{m-1}, gq_{m-1})) \\
& + \cdots + (d(ga_{n+1}, gp_{n+1}) + d(gb_{n+1}, gq_{n+1})) \\
& \quad + (d(ga_{n+1}, gp_n) + d(gb_{n+1}, gq_n)) \\
\leq\ & s_{m-1} + r_{m-1} + \cdots + r_{n+1} + s_n
\end{aligned}$$

$$
\begin{aligned}
&\leq (s_n + s_{n+1} + \cdots + s_{m-1}) + (r_{n+1} + r_{n+2} + \cdots + r_{m-1}) \\
&\leq \sum_{k=n}^{m-1} s_k + \sum_{k=n+1}^{m-1} r_k \\
&\leq \sum_{k=n}^{m-1} 2\psi^k(\frac{\varepsilon}{2}) + \sum_{k=n+1}^{m-1} 2\psi^k(\frac{\varepsilon}{2}) \\
&\leq 2\sum_{k=n}^{+\infty} \psi^k(\frac{\varepsilon}{2}) + 2\sum_{k=n+1}^{+\infty} \psi^k(\frac{\varepsilon}{2}) \qquad (32)
\end{aligned}
$$

From (31) and (32), it follows now that $(\{ga_n\},\{gp_n\})$ and $(\{gb_n\},\{gq_n\})$ are Cauchy bisequences in $(g(A),g(B))$. Since, $g(A\cup B)$ is complete, therefore, $(\{ga_n\},\{gp_n\})$ and $(\{gb_n\},\{gq_n\})$ are convergent in $(g(A),g(B),d)$. There exists, $a,b\in A$ and $p,q\in B$ such that

$$\lim_{n\to\infty} ga_{n+1} = gp, \lim_{n\to\infty} gb_{n+1} = gq, \lim_{n\to\infty} gp_{n+1} = ga, \lim_{n\to\infty} gq_{n+1} = gb \qquad (33)$$

Now we will show that $ga\in F(a,b), gb\in F(b,a)$ and $gp\in F(p,q), gq\in F(q,p)$ As $(\{ga_n\},\{gp_n\})$ is a non-decreasing bisequence and $(\{gb_n\},\{gq_n\})$ is a non-increasing bisequence in (A, B), we have $ga_n\to gp, gb_n\to gq$ and $gp_n\to ga, gq_n\to gb$.

By assumption (3.1.4) we get $ga_n\leq gp, gp_n\leq ga$ and $gb_n\geq gq, gq_n\geq gb$ for all n.

If $ga_n = gp, gp_n = ga$ and $gb_n = gq, gq_n = gb$ for some $n\geq 0$, then

$gp = ga_n \leq gp_{n+1} \leq ga = gp_n$ $gq = gb_n \geq gq_{n+1} \geq gb = gq_n$ and

$ga = gp_n \leq ga_{n+1} \leq gp = ga_n$, $gb = gq_n \geq gb_{n+1} \geq gq = gb_n$ implies $ga = gp$ and $gb = gq$ therefore, $ga_n = ga_{n+1}\in F(a_n,b_n)$ and $gb_n = gb_{n+1}\in F(b_n,a_n)$.

So (ga_n, gb_n) is coupled coincidence point of F and g.

Suppose that $(ga_n, gp_n)\neq(gp,ga)$ and $(gb_n,gq_n)\neq(gq,gb)$ for all $n\geq 0$.

From (1), we have

$$H(F(a,b),F(p_n,q_n)) \leq \psi\left(\frac{d(ga,gp_n)+d(gb,gq_n)}{2}\right) < \frac{d(ga,gp_n)+d(gb,gq_n)}{2},$$

because $d(ga,gp_n)+d(gb,gq_n) > 0$ and $\psi(t) < t$ for all $t > 0$. By (33)

$$\lim_{n\to\infty} H(F(a,b),F(p_n,q_n)) = 0$$

Since $gp_{n+1}\in F(p_n,q_n)$ and $\lim_{n\to\infty} d(ga, gp_{n+1}) = 0$, we have $ga\in F(a,b)$.

Similarly, we can prove $gb\in F(b,a)$ and $gp\in F(p,q)$, $gq\in F(q,b)$

On the other hand,

$$d(ga,gp) = d\left(\lim_{n\to\infty} gp_n, \lim_{n\to\infty} ga_n\right) = \lim_{n\to\infty} d(ga_n, gp_n) = 0$$

and

$$d(gb,gq) = d\left(\lim_{n\to\infty} gq_n, \lim_{n\to\infty} gb_n\right) = \lim_{n\to\infty} d(gb_n, gq_n) = 0$$

Therefore, $ga = gp$ and $gb = gq$ and hence F ang g have a coupled coincidence point.

Theorem 3.2: Let $(A,B,\leq)$ be a partially ordered set such that there exist a bipolar metric d on (A,B) .Consider $F:(A^2,B^2)\rightrightarrows CB(A,B)$ be a covariant set valued mapping, such that

$$H(F(a,b),F(p,q)) \leq \psi\left(\frac{d(a,p)+d(b,q)}{2}\right) \tag{34}$$

for all $a,b\in A$ and $p,q\in B$ with $a\geq p$ and $b\leq q$. Suppose also that

(3.2.1) F has a mixed monotone property

(3.2.2) There exist $a_0,b_0\in A\cup B$ and for some $x_1\in F(a_0,b_0)$, $y_1\in F(b_0,a_0)$ we have $a_0\leq x_1$ and $b_0\geq y_1$

(3.2.3) If a non-decreasing sequence $(\{a_n\},\{p_n\})$ is convergent to (p,a) for $a\in A, p\in B$, then $a_n\leq p$, $p_n\leq a$ for all n and if a non-increasing sequence $(\{b_n\},\{q_n\})$ is convergent to (q,b) for $b\in A, q\in B$, then $b_n\geq q$, $q_n\geq b$ for all n.

Then F has a coupled fixed point, that is there exist $a,b\in A\cup B$ such that $a\in F(a,b)$ and $b\in F(b,a)$.

Example 3.1: Let $A=\{U_m(R)/U_m(R)\ \textit{is upper triangular matrices over}\ R\}$ and $B=\{L_m(R)/L_m(R)\ \textit{is lower triangular matrices over}\ R\}$ with the bipolar metric $d(P,Q)=\sum_{i,j=1}^{m}|p_{ij}-q_{ij}|$, for all $P=(p_{ij})_{m\times m}\in U_m(R)$ and $Q=(q_{ij})_{m\times m}\in L_m(R)$. On the set (A,B), consider the following relation :$(P,Q)\in A^2\cup B^2, P\preceq Q \Leftrightarrow p_{ij}\leq q_{ij}$ where $\leq$ is usual ordering. Then clearly, (A,B,d) is a complete bipolar metric space and $(A,B,\preceq)$ is a partially ordered set. Let $F:\left(A^2,B^2\right)\rightrightarrows CB(A,B)$ be defined as

$$F(P,Q)=\left(\frac{p_{ij}+q_{ij}}{3}\right)_{m\times m}+\frac{2}{3}(I_{ij})_{m\times m}\ \textit{for all}\ \left(P=(p_{ij})_{m\times m}, Q=(q_{ij})_{m\times m}\right)\in A^2\cup B^2$$

and $g:A\cup B\to A\cup B$ by $g(P)=(p_{ij})_{m\times m}$ and let $\psi:(0,1)\to(0,1)$ by $\psi(t)=\frac{2t}{3}$ for $t\in(0,1)$ Let the bisequence (P_n,Q_n) in (A, B) such that $\lim_{n\to\infty} g(P_n)=X$, $\lim_{n\to\infty} g(Q_n)=Y$. Then obviously,

$X=Y=\frac{2}{3}(I_{ij})_{m\times m}$. Now for all $n\geq 0$, $g(P_n)=(p_{nij})_{m\times m}$ and $g(Q_n)=(q_{nij})_{m\times m}$ and $F(P_n,Q_n)=\left(\frac{p_{nij}+q_{nij}}{3}\right)_{m\times m}+\frac{2}{3}(I_{ij})_{m\times m}$, $F(Q_n,P_n)=\left(\frac{q_{nij}+p_{nij}}{3}\right)_{m\times m}+\frac{2}{3}(I_{ij})_{m\times m}$. Then obviously, F has the g-mixed monotone property, also there exist $P=(O_{ij})_{m\times m}$ and $Q=(I_{ij})_{m\times m}$ such that $F\left((O_{ij})_{m\times m},(I_{ij})_{m\times m}\right)=(\frac{O_{ij}+I_{ij}}{3})_{m\times m}+\frac{2}{3}(I_{ij})_{m\times m}\succeq(O_{ij})_{m\times m}$
and $F\left((I_{ij})_{m\times m},(O_{ij})_{m\times m}\right)=(\frac{O_{ij}+I_{ij}}{3})_{m\times m}+\frac{2}{3}(I_{ij})_{m\times m}\preceq(I_{ij})_{m\times m}$
Taking $\left(P=(p_{ij})_{m\times m},Q=(q_{ij})_{m\times m}\right),\left(R=(r_{ij})_{m\times m},S=(s_{ij})_{m\times m}\right)\in A^2\cup B^2$ with $P\succeq R$, $Q\preceq S$ that is $p_{ij}\geq r_{ij}$, $q_{ij}\leq s_{ij}$, we have

$$\begin{aligned}
d(F(P,Q),F(R,S)) &= d(\tfrac{p_{ij}+q_{ij}}{3}+\tfrac{2}{3}(I_{ij}),\tfrac{r_{ij}+s_{ij}}{3}+\tfrac{2}{3}(I_{ij}))\\
&= \tfrac{1}{3}\sum_{i,j=1}^{m}|(p_{ij}+q_{ij})-(r_{ij}+s_{ij})|\\
&\leq \tfrac{1}{3}\left(\sum_{i,j=1}^{m}|p_{ij}-r_{ij}|+\sum_{i,j=1}^{m}|q_{ij}-s_{ij}|\right)\\
&\leq \tfrac{1}{3}\left(d(g(P),g(R))+d(g(Q),g(S))\right)\\
&\leq \tfrac{2}{3}\left(\tfrac{d(g(P),g(R))+d(g(Q),g(S))}{2}\right)
\end{aligned}$$

Therefore, all the conditions of Theorem (3.1) holds for $\psi(t)=\frac{2t}{3}$ for all $t>0$ and we see that $F(A^2\cup B^2)\subseteq g(A\cup B)$ and also verified that F has g-monotone property and $((\frac{2}{3}I_{ij})_{m\times m},(\frac{2}{3}I_{ij})_{m\times m})$ is a coupled coincidence point of F and g.

Corollary 3.1: Let $(A,B,\leq)$ be a partially ordered set such that there exist a bipolar metric d on (A,B). Consider the covariant mappings $F:(A^2,B^2)\rightrightarrows CB(A,B)$ and $g:(A,B)\rightrightarrows(A,B)$ such that $F(A^2\cup B^2)\subseteq g(A\cup B)$ and satisfies the following conditions. There exists $\kappa\in(0,1)$ such that

$$H\left(F(a,b),F(p,q)\right)\leq\frac{\kappa}{2}\left(d(g(a),g(p))+d(g(b),g(q))\right) \tag{35}$$

for all $a,b\in A$ and $p,q\in B$ with $g(a)\geq g(p)$ and $g(b)\leq g(q)$. Suppose also that

(i) $g(A\cup B)$ is complete subspaces of (A,B,d)

(ii) F has a mixed g-monotone property

(iii) There exist $a_0,b_0\in A\cup B$ and for some $x_1\in F(a_0,b_0)$, $y_1\in F(b_0,a_0)$ we have $ga_0\leq x_1$ and $gb_0\geq y_1$

(iv) If a non-decreasing sequence $(\{a_n\},\{p_n\})$ is convergent to (p,a) for $a \in A, p \in B$, then $a_n \leq p$, $p_n \leq a$ for all n and if a non-increasing sequence $(\{b_n\},\{q_n\})$ is convergent to (q,b) for $b \in A, q \in B$, then $b_n \geq q$, $q_n \geq b$ for all n.

Then F and g have a coupled coincidence point, that is there exist $a,b \in A \cup B$ such that $ga \in F(a,b)$ and $gb \in F(b,a)$.

proof: It follows by taking $\psi(t) = \kappa t$ in Theorem (3.1).

4. CONCLUSIONS

In the present research, we introduced and proved a coupled coincidence fixed point results for a hybrid pair of single valued and multivalued mappings, satisfying generalized contractive conditions, defined on a partially ordered bipolar metric space and gave suitable example that support our main result.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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