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INEXACT GENERALIZED PROXIMAL POINT ALGORITHM WITH ALTERNATING INERTIAL STEPS FOR MONOTONE INCLUSION PROBLEM

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Abstract. In this paper, we propose and study an inexact generalized proximal point algorithm with alternated inertial steps for solving monotone inclusion problem and obtain weak convergence results under some mild conditions. In the case when the operator T is such that T^{-1} is Lipschitz continuous at 0, we prove that the sequence of the iterates is linearly convergent. Fejér monotonicity of even subsequences of the iterates is also obtained. Finally, we give some priori and posteriori error estimates of our generated sequences.

Keywords: proximal-point algorithm; alternated inertial step; weak convergence; linear convergence; Hilbert spaces.

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1. Introduction

Let H be a real Hilbert space with inner product $\langle , \rangle$ and induced norm $\|\cdot\|$. Given a maximal monotone set-valued operator, $T : H \to 2^H$, we consider the following inclusion problem

$$\text{find } x \in H \ \text{ such that } \ 0 \in T(x). \tag{1}$$

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We denote by $zer(T)$ the set of solutions of (1), and assume throughout this paper that $zer(T) \neq \emptyset$. It is well known that (1) serves as a unifying model for many problems of fundamental importance, including fixed point problem, variational inequality problem, minimization of closed proper convex functions, and their variants and extensions. Therefore, its efficient solution is of practical interest in many situations.

The proximal point algorithm (PPA), which was first studied by Martinet and further developed by Rockafella and others (see e.g., [1, 2, 3]) , has been used for many years for studying problem(1). Let $c > 0$ be a constant, the resolvent operator of T is defined by $J_{cT} := (I + cT)^{-1}$, i.e., for any $x \in H$, $J_{cT}(x)$ is the unique solution of $0 \in x + cT(x)$. It is a single valued operator (see [4]). Starting from an arbitrary point $z^0 \in H$, the exact form of the PPA iteratively generates its sequence $\{z^k\}$ by the scheme

$$z^{k+1} = J_{cT}(z^k) \tag{2}$$

which is equivalent to

$$0 \in cT(z^{k+1}) + z^{k+1} - z^k \tag{3}$$

where c, called proximal parameter, is a positive real number. The inexact version of the PPA is defined as:

$$z^{k+1} \approx J_{cT}(z^k) \tag{4}$$

In (64), the tolerance of accuracy is zero, and so (4) contains (64). However, (64) is of interest in its own right, since it requires estimating the resolvent accurately. Under different settings, both the exact and inexact versions of the PPA have been investigated in the literature. In [5], the convergence of both the exact and inexact versions of PPA was comprehensively studied. It turns out that the PPA is a very powerful algorithmic tool and contains many well known algorithms as special cases, including the classical augmented Lagrangian method [6, 7], the Douglas-Rachford splitting method [8] and the alternating direction method of multipliers [9, 10]. For more facts about the PPA and generalizations of the PPA, one can consult the references [4, 11, 12]. The equivalent representation of the PPA (3), can be written as

$$0 \in \frac{z^{k+1} - z^k}{c} + T(z^{k+1}). \tag{5}$$

This can be viewed as an implicit discretization of the evolution differential inclusion problem

$$0 \in \frac{dx}{dt} + T(x(t)) \tag{6}$$

It has been shown that the solution trajectory of (6) converges to a solution of (1) provided that T satisfies certain conditions (see e.g., [13]). To speed up convergence, the following second order evolution differential inclusion problem was introduced in the literature:

$$0 \in \frac{d^2x}{dt^2} + c\frac{dx}{dt} + T(x(t)), \tag{7}$$

where $c > 0$ is a friction parameter. If $T = \nabla f$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable convex function with attainable minimum, the system (7) characterizes roughly the motion of a heavy ball which rolls under its own inertia over the graph of f until friction stops it at a stationary point of f. In this case, the three terms in (7) denote, respectively, inertial force, friction force and gravity force. Consequently, the system (7) is usually referred to as the heavy-ball with friction (HBF) system. In theory, the convergence of the solution trajectories of the HBF system to a solution of (1) can be faster than those of the first-order system (6), while in practice the second order inertial term $\frac{d^2x}{dt^2}$ can be exploited to design faster algorithms (see e.g., [14, 15]) . As a result of the properties of (7), an implicit discretization method was proposed in [16, 17] as follows, given z^{k-1} and z^k, the next point z^{k+1} is determined via

$$0 \in \frac{z^{k+1} - 2z^k + z^{k-1}}{h^2} + \gamma\frac{z^{k+1} - z^k}{h} + T(z^{k+1}), \tag{8}$$

which results to an iterative algorithm of the form

$$z^{k+1} = J_{cT}(z^k + \alpha(z^k - z^{k-1})) \tag{9}$$

where $c = \frac{h^2}{1+ch}$ and $\alpha = \frac{1}{1+ch}$. Observe that (9) is the proximal point step applied to the extrapolated point $z^k + \alpha(z^k - z^{k-1})$, rather than x^k itself as in the classical PPA. Hence the iterative algorithm (9) is a two-step method generally called inertial PPA (iPPA). Convergence properties of (9) were studied in [16, 17] under some assumptions on the parameters α and c.

Since the introduction of the iPPA, inexact and other forms of iPPAs have been studied by many authors (see e.g., [18, 19, 3] and the references contained therein). Recently, there are increasing interests in studying inertial type algorithms, for example, inertial forward-backward splitting methods (see for instance [20, 21, 8, 22]), inertial Douglas-Rachford splitting method [23, 24], inertial alternating method of multipliers, and inertial forward-backward-forward method (see the refernces [25, 26, 24, 10, 11]).
A major drawback of the iPPA is that Fejér monotonicity of $\|z^k - z^*\|,\ z^* \in zer(T)$ is lost in many cases and hence, makes the sequence $\{z^k\}$ generated by the methods to swing back and forth around $zer(T)$. This situation makes these methods sometimes not converge faster than their counterpart non-inertial methods (see, e.g., [27, 28, 32]). Furthermore, no linear convergence rate of iPPA has been obtained in the literature (see, for example, [31]).

A search to overcome this drawback led to introduction of the so-called *alternated iPPA* in the literature. It has been shown that with the alternated iPPA, some sort of Fejér monotonicity of $\|z^k - z^*\|,\ z^* \in zer(T)$ is recovered and that the method out-performs their non-inertial counterparts. Please, refer to [32, 34] for more details. In [12], the authors studied the generalized PPA for maximal monotone set-valued operator T such that T^{-1} is Lipschitz continuous at $\mathbf{0}$ in real Hilbert space and obtained both weak and linear convergence results.

Motivated by the results mentioned above, Shehu and Ezeora [29] proposed and studied an alternated inertial exact generalized PPA for solving monotone inclusion problem in real Hilbert space. They obtained the following results:

- Fejér monotonicity of $\|z^k - z^*\|, z^* \in zer(T)$ to some extent, which is not obtained for iPPA in [16, 17] and other related works;
- weak convergence of the generated sequence $\{z^k\}$ to a point in $zer(T)$ and thereby generalizes the results obtained in [34];

- linear convergence of the generated sequence $\{z^k\}$ to a unique point z^* under the condition that T is maximal monotone and T^{-1} is Lipschitz continuous at $\mathbf{0}$, which has not been obtained for iPPA in [16, 17, 34] and other related works
- priori and posteriori error estimates of the generated sequence $\{z^k\}$.

As we mentioned earlier, the inexact PPA is more general than the exact PPA.

Question of Interest: Can an inexact generalized PPA with alternated inertial terms be developed for solving problem (1) such that some or all the results of [29] are recovered?

It is our aim in this article to answer the question in the positive.

2. Preliminaries

In this section, we present some definitions and known results needed for further discussions.

2.1. Some Definitions.

Let $T : H \to 2^H$ be a set-valued map. T is said to be :

(i) nonexpansive if

$$||u-v|| \leq ||x-y|| \ \forall\, x,y \in H,\ u \in T(x),\ v \in T(y). \tag{10}$$

(ii) firmly nonexpansive if

$$||u-v|| \leq \langle u-v, x-y\rangle \forall\, x,y \in H,\ u \in T(x),\ v \in T(y). \tag{11}$$

(iii) $\theta-$ strongly monotone if there exists $\theta > 0$ such that

$$\langle u-v, x-y\rangle \geq \theta ||x-y||^2 \ \forall\, x,y \in H,\ u \in T(x),\ v \in T(y). \tag{12}$$

Definition 2.1. *Let $T : H \to 2^H$ be a set-valued map. T^{-1} is called Lipschitz continuous at* $\mathbf{O}$ *with modulus $\alpha \geq 0$ if there is a unique solution z^* to $0 \in T(z)$ $(i.e.T^{-1}(0) = z^*)$, and for some $\tau > 0$, we have $||x-x^*|| \leq \alpha ||w||$ whenever $x \in T^{-1}(w)$ and $||w|| \leq \tau$.*

Definition 2.2. *A sequence $\{z^k\}$ in H is said to converge weakly to $z^* \in H$ if*

$$\textit{for all } \ q \in H, \lim_{k\to\infty} \langle z^k, q\rangle = \langle z^*, q\rangle. \tag{13}$$

Definition 2.3. *Suppose a sequence $\{x_n\}$ in H converges in norm to $x^* \in H$. We say that $\{x_n\}$ converges to x^* $R-$ linearly if $\limsup_{n\to\infty} ||x_n - x^*|| < 1$. $\{x_n\}$ is said to converge to x^* $Q-$ linearly if there exists $\sigma \in (0,1)$ such that $||x_{n+1} - x^*|| \leq \sigma ||x_n - x^*||$ for n sufficiently large.*

It is well known that $Q-$ linear convergence implies $R-$ linear convergence, but the reverse implication is not true.

Remark 2.1.
- *Definition* (2.1) *is as given in* [5].
- *From the definitions, the following conclusions hold. If T is nonexpansive, then it is Lipschitz continuous. Furthermore, the problem* (1) *has a unique solution point when T^{-1} is Lipschitz continuous at* $\mathbf{0}$.
- *Example of a set-valued map T such that T^{-1} is Lipschitz continuous at* $\mathbf{O}$ *is given in* [12]. *It is also shown in* [12] *that Lipschitz continuity at* $\mathbf{O}$ *is weaker than strong monotonicity as assumed in the results of many authors (see e.g.,* [33]*).*

2.2. Some Known Results.

Lemma 2.1. *We have the following facts.*
(i) All firmly nonexpansive operators are nonexpansive.
(ii) An operator T is firmly nonexpansive if and only if $2T-I$ is nonexpansive.
(iii) An operator is firmly nonexpansive if and only if it is of the form $\frac{1}{2}(K+I)$, where K is nonexpansive.
iv) An operator T is firmly nonexpansive if and only if $I-T$ is firmly nonexpansive.

Lemma 2.2. *(see* [12]*) Let $T: H \to 2^H$ be set-valued and maximal monotone, define $J_{\lambda T} = (I+\lambda T)^{-1}$ with $\lambda > 0$. Then, we have*
(i) $\langle J_{\lambda T}(z) - J_{\lambda T}(z'), (I-J_{\lambda T})(z) - (I-J_{\lambda T})(z')\rangle \geq 0 \; \forall z, z' \in H$
(ii) $||z-z'||^2 \geq ||J_{\lambda T}(z) - J_{\lambda T}(z')||^2 + ||(I-J_{\lambda T})(z) - (I-J_{\lambda T})(z')||^2 \; \forall z, z' \in H.$

Lemma 2.3. *The following statements hold in H. (i)* $||x+y||^2 = ||x||^2 + 2\langle x, y\rangle + ||y||^2 \;\; \forall\, x, y \in H$.
(ii) $||x+y||^2 \leq ||x||^2 + \langle y, x+y\rangle \; \forall x, y \in H$.
(iii) $||\alpha x + \beta y||^2 = \alpha(\alpha+\beta)||x||^2 + \beta(\alpha+\beta)||y||^2 - \alpha\beta||x-y||^2 \;\; \forall x, y \in H, \; \forall \alpha, \beta \in \mathbb{R}.$

Lemma 2.4. *Let C be a nonempty set of H and $\{z^k\}$ be a sequence in H such that the following two conditions hold:*

(i). for any $x \in C$, $\lim_{n\to\infty} ||z^k - x||$ *exists;*

(ii) every sequential weak cluster point of $\{z^k\}$ *is in* C. *Then* $\{z^k\}$ *converges weakly to a point in* C.

3. Convergence of Inexact GPPA with Alternated Inertia

In this section, we introduce and study the following inexact GPPA with alternated inertal term.

$$\begin{aligned} z^{k+1} &= (1-\gamma)w^k + \gamma\bar{w}^k \ \forall\ k \geq 1, \\ ||\bar{w}^k - J_{cT}(w^k)|| &\leq \delta_k ||w^k - z^{k+1}|| \text{ where} \end{aligned} \tag{14}$$

$$(i)\ w^k = \begin{cases} z^k \ \text{ if } k = even \\ z^k + \alpha(z^k - z^{k-1}) \ \text{ if } k = odd, \end{cases}$$

(ii). $\delta_k > 0 : \sum_{k=1}^{\infty} \delta_k < \infty$.

(iii). $0 \leq \alpha < \frac{(2-\gamma)}{\gamma}$, $\gamma \in (0,2)$.

To establish the results of this section, we need the following Lemmas.

Lemma 3.1. *(*[12]*) Let* $\{\alpha_k\}$ *be a sequence of positive real numbers satisfying* $\sum_{k=1}^{\infty} \alpha_k < \infty$ *Then,*

$$\prod_{k=1}^{\infty} (1+\alpha_k) < \infty.$$

Lemma 3.2. *(*[12]*) Let* $\{\delta_k\}$ *be a sequence of positive real numbers satisfying* $\sum_{k=1}^{\infty} \delta_k < \infty$ *and* $\gamma > 0$. *Then,*

$$\prod_{k=1}^{\infty} \frac{(1+\gamma\delta_k)}{(1-\gamma\delta_k)} < \infty.$$

Lemma 3.3. *(see* [12]*) Let* $\{a_k\}$ *and* $\{b_k\}$ *be sequences of positive real numbers with* $\sum_{k=1}^{\infty} b_k < \infty$ *and*

$$a_{k+1} \leq a_k + b_k,\ \forall\ k. \tag{15}$$

Then, $\{a_k\}$ *is convergent.*

Theorem 3.1. *Let $\{z^k\}$ be the sequence defined by* (14). *Then,*

(1) The sequence $\{z^{2k}\}$ is bounded.

(2) The following limit holds

$$\lim_{k\to\infty} ||z^{2k} - \bar{z}^{2k}|| = 0.$$

Proof. Observe that: (a) the operator $(I - J_{cT})$, is firmly nonexpansive for any $c > 0$.

(b) Any zero of T is a zero point of the operator $(I - J_{cT})$.

Let $z^* \in zer(T)$, we estimate as follows:

$$\begin{aligned} ||z^{k+1} - z^*||^2 &= ||w^k - \gamma(I - J_{cT})(w^k) - z^*||^2 \\ &= ||w^k - z^*||^2 - 2\gamma\langle w^k - z^*, (I - J_{cT})(w^k)\rangle + \gamma^2||(I - J_{cT})(w^k)||^2 \\ &\leq ||w^k - z^*||^2 - \gamma(2-\gamma)||(I - J_{cT})(w^k)||^2 \\ &\leq ||w^k - z^*||^2 - \gamma(2-\gamma)||w^k - \bar{w}^k||^2 \end{aligned} \tag{16}$$

Set $\hat{z}^{k+1} = (1-\gamma)w^k + \gamma\bar{w}^k$. Then $\hat{z}^{k+1} - z^{k+1} = \gamma(\bar{w}^k - \bar{w}^k)$, where $\bar{w}^k = J_{cT}(w^k)$.

Hence

$$\begin{aligned} ||\hat{z}^{k+1} - z^{k+1}|| &= \gamma||\bar{w}^k - \bar{w}^k|| \\ &\leq \gamma\delta_k||w^k - z^{k+1}|| \text{ by (14)} \end{aligned} \tag{17}$$

Notice that

$$\begin{aligned} ||z^{k+1} - z^*|| &\leq ||z^{k+1} - \hat{z}^{k+1}|| + ||\hat{z}^{k+1} - z^*|| \\ &\leq \gamma\delta_k||w^k - z^{k+1}|| + ||\hat{z}^{k+1} - z^*|| \text{ by (17)} \\ &\leq \gamma\delta_k\big(||w^k - z^*|| + ||z^{k+1} - z^*||\big) + ||\hat{z}^{k+1} - z^*||. \end{aligned} \tag{18}$$

That is,

$$\begin{aligned} ||z^{k+1} - z^*|| &\leq \gamma\delta_k||w^k - z^*|| + \gamma\delta_k||z^{k+1} - z^*|| + ||w^k - z^*|| \\ &\leq \frac{1+\gamma\delta_k}{1-\gamma\delta_k}||w^k - z^*|| \end{aligned} \tag{19}$$

From (19) we have that

$$||z^{2k+2}-z^*|| \leq \frac{1+\gamma\delta_{2k+1}}{1-\gamma\delta_{2k+1}}||w^{2k+1}-z^*|| \tag{20}$$

Observe that

$$\begin{aligned}\|w^{2k+1}-z^*\|^2 &= \|z^{2k+1}+\alpha(z^{2k+1}-z^{2k})-z^*\|^2\\ &= (1+\alpha)\|z^{2k+1}-z^*\|^2-\alpha\|z^{2k}-z^*\|^2\\ &+ \alpha(1+\alpha)\|z^{2k+1}-z^{2k}\|^2.\end{aligned} \tag{21}$$

Also,

$$\begin{aligned}\|z^{2k+1}-z^*\|^2 &\leq \|w^{2k}-z^*\|^2-\gamma(2-\gamma)\|w^{2k}-J_{cT}(w^{2k})\|^2\\ &= \|z^{2k}-z^*\|^2-\gamma(2-\gamma)\|w^{2k}-J_{cT}(w^{2k})\|^2.\end{aligned} \tag{22}$$

And

$$\begin{aligned}\|z^{2k+1}-z^{2k}\| &= \|w^{2k}-\gamma(w^{2k}-J_{cT}(w^{2k}))-w^{2k}\|\\ &= \gamma\|w^{2k}-J_{cT}(w^{2k}))\|\\ &= \gamma\|z^{2k}-J_{cT}(z^{2k}))\|.\end{aligned} \tag{23}$$

So,

$$\begin{aligned}\|w^{2k+1}-z^*\|^2 &\leq (1+\alpha)\left[\|z^{2k}-z^*\|^2-\gamma(2-\gamma)\|z^{2k}-J_{cT}(z^{2k})\|^2\right]\\ &- \alpha\|z^{2k}-z^*\|^2+\alpha(1+\alpha)\|z^{2k+1}-z^{2k}\|^2 \text{ by (22)}\\ &= \|z^{2k}-z^*\|^2-(1+\alpha)\gamma(2-\gamma)\|z^{2k}-J_{cT}(z^{2k})\|^2\\ &\quad +\alpha(1+\alpha)\gamma\|z^{2k}-J_{cT}(z^{2k}))\| \text{ by (23)}\\ &\leq \|z^{2k}-z^*\|^2-(1+\alpha)\gamma\big((2-\gamma)-\alpha\gamma\big)\|z^{2k}-J_{cT}(z^{2k}))\|\\ &\leq \|z^{2k}-z^*\|^2\end{aligned} \tag{24}$$

That is,

$$\|w^{2k+1}-z^*\| \leq \|z^{2k}-z^*\| \tag{25}$$

Using (25) in (20), gives

$$
\begin{aligned}
||z^{2k+2}-z^*|| &\leq \frac{1+\gamma\delta_{2k+1}}{1-\gamma\delta_{2k+1}}||w^{2k+1}-z^*|| \\
&\leq \frac{1+\gamma\delta_{2k+1}}{1-\gamma\delta_{2k+1}}\|z^{2k}-z^*\| \\
&\vdots \\
&\leq \prod_{j=1}^{k+1}\frac{1+\gamma\delta_{2j-1}}{1-\gamma\delta_{2j-1}}||z^0-z^*|| \\
&\leq \prod_{j=1}^{\infty}\frac{1+\gamma\delta_j}{1-\gamma\delta_j}||z^0-z^*|| \qquad (26)
\end{aligned}
$$

By Lemma 3.2, we conclude that $\{z^{2k}\}$ is bounded establishing (1). That is, there exists $M>0$ such that $||z^{2k}-z^*||\leq M\ \forall k\geq 1$.

Observe that

$$
\gamma^2\delta_k^2||w^k-z^{k+1}||^2 \leq \gamma^2\delta_k^2\big(||w^k-z^*||+||z^{k+1}-z^*||\big)^2 \qquad (27)
$$

Hence,

$$
\begin{aligned}
\gamma^2\delta_{2k+1}^2||w^{2k+1}-z^{2k+2}||^2 &\leq \gamma^2\delta_{2k+1}^2\big(||w^{2k+1}-z^*||+||z^{2k+2}-z^*||\big)^2 \\
&\leq 4M^2\gamma^2\delta_{2k+1}^2 \qquad (28)
\end{aligned}
$$

On the other hand, from (16),

$$
\begin{aligned}
||z^{2k+2}-z^*||^2 &= ||\hat{z}^{2k+2}-z^*+(z^{2k+2}-\hat{z}^{2k+2})||^2 \\
&\leq ||\hat{z}^{2k+2}-z^*||^2+||\hat{z}^{2k+2}-z^{2k+2}||^2+2\langle\hat{z}^{2k+2}-z^*,\hat{z}^{2k+2}-z^{2k+2}\rangle \\
&\leq ||\hat{z}^{2k+2}-z^*||^2+2||\hat{z}^{2k+2}-z^*||^2||\hat{z}^{2k+2}-z^{2k+2}||+||\hat{z}^{2k+2}-z^{2k+2}||^2 \\
&\leq ||\hat{z}^{2k+2}-z^*||^2+2M\gamma\delta_{2k+1}||w^{2k+1}-z^{2k+2}||+\gamma^2\delta_{2k+1}^2||w^{2k+1}-z^{2k+2}||^2 \\
&\leq ||w^{2k+1}-z^*||^2-\gamma(2-\gamma)||w^{2k+1}-\bar{w}^{2k+1}||^2 \\
&+ 2M\gamma\delta_{2k+1}||w^{2k+1}-z^{2k+2}||+\gamma^2\delta_{2k+1}^2||w^{2k+1}-z^{2k+2}||^2 \qquad (29)
\end{aligned}
$$

Therefore, we have using (24) that

$$\begin{aligned} ||z^{2k+2}-z^*||^2 &\le ||z^{2k}-z^*||^2-\gamma(2-\gamma)||w^{2k+1}-\bar{w}^{2k+1}||^2 \\ &+ 2M\gamma\delta_{2k+1}||w^{2k+1}-z^{2k+2}||+\gamma^2\delta_{2k+1}^2||w^{2k+1}-z^{2k+2}||^2 \end{aligned} \tag{30}$$

and so by (28),

$$||z^{2k+2}-z^*||^2 \le ||z^{2k}-z^*||^2+4M^2\gamma\delta_{2k+1}+4M^2\gamma^2\delta_{2k+1}^2. \tag{31}$$

Applying Lemma 3.3 and condition (ii) to (31), we conclude that $\lim_{k\to\infty}||z^{2k}-z^*||$ exists. From (14) we get $z^{2k+2}=w^{2k+1}-\gamma(w^{2k+1}-J_{cT}(w^{2k+1}))$. So,

$$\begin{aligned} \|z^{2k+2}-z^*\|^2 &= \|w^{2k+1}-\gamma(w^{2k+1}-J_{cT}(w^{2k+1}))-z^*\|^2 \\ &= \|w^{2k+1}-z^*\|^2-2\gamma\langle w^{2k+1}-z^*,w^{2k+1}-J_{cT}(w^{2k+1})\rangle \\ &+ \gamma^2\|w^{2k+1}-J_{cT}(w^{2k+1})\|^2 \\ &\le \|w^{2k+1}-z^*\|^2-\gamma(2-\gamma)\|w^{2k+1}-J_{cT}(w^{2k+1})\|^2. \\ &\le \|z^{2k}-z^*\|^2-(1+\alpha)\gamma(2-\gamma)\|z^{2k}-J_{cT}(z^{2k})\|^2 \\ &\quad +\alpha(1+\alpha)\|z^{2k+1}-z^{2k}\|^2-\gamma(2-\gamma)\|w^{2k+1}-J_{cT}(w^{2k+1})\|^2. \end{aligned} \tag{32}$$

$$\begin{aligned} \|z^{2k+1}-z^{2k}\| &= \|w^{2k}-\gamma(w^{2k}-J_{cT}(w^{2k}))-w^{2k}\| \\ &= \gamma\|w^{2k}-J_{cT}(w^{2k}))\| \\ &= \gamma\|z^{2k}-J_{cT}(z^{2k}))\|. \end{aligned} \tag{33}$$

Hence,

$$\begin{aligned} \|z^{2k+2}-z^*\|^2 &\le \|z^{2k}-z^*\|^2-(1+\alpha)\gamma(2-\gamma)\|z^{2k}-J_{cT}(z^{2k})\|^2 \\ &+ \alpha(1+\alpha)\gamma^2\|z^{2k}-J_{cT}(z^{2k})\|^2 \\ &- \gamma(2-\gamma)\|w^{2k+1}-J_{cT}(w^{2k+1})\|^2 \\ &\le \|z^{2k}-z^*\|^2-(1+\alpha)\gamma\big((2-\gamma)-\alpha\gamma\big)\|z^{2k}-J_{cT}(z^{2k})\|^2 \\ &\quad -\gamma(2-\gamma)\|w^{2k+1}-J_{cT}(w^{2k+1})\|^2. \end{aligned} \tag{34}$$

From (33) and (34), we have that

$$\lim_{k\to\infty}||z^{2k+1}-z^{2k}|| = \lim_{k\to\infty}\gamma||z^{2k}-\bar{z}^{2k}|| = 0. \tag{35}$$

That is

$$||z^{2k}-\bar{z}^{2k}|| \to 0 \; k\to\infty. \tag{36}$$

This completes the proof. □

3.1. Weak Convergence Results.

Theorem 3.2. *Let $T : H \to 2^H$ be a maximal monotone operator and suppose the following assumptions hold:*

(i) $\gamma \in (0,2)$*;*

(ii) $0 \leq \alpha < \frac{2-\gamma}{\gamma}$*;*

(iii) $T^{-1}(0) \neq \emptyset$.

(iv). $\delta_k > 0 : \sum_{k=1}^{\infty}\delta_k < \infty$.

For given $z^0, z^1 \in H$, let the sequence $\{z^k\}$ be generated by

$$\begin{aligned} z^{k+1} &= (1-\gamma)w^k + \gamma\bar{w}^k \;\forall\; k\geq 1, \\ ||\bar{w}^k - J_{c_kT}(w^k)|| &\leq \delta_k||w^k - z^{k+1}|| \; where \end{aligned} \tag{37}$$

$$(i)\; w^k = \begin{cases} z^k \;\; if\, k = even \\ z^k + \alpha(z^k - z^{k-1}) \;\; if\, k = odd, \end{cases}$$

Then $\{z^k\}$ converges weakly to $z^* \in T^{-1}(0)$.

Proof. Since the sequence $\{z^{2k}\}$ is bounded,there exists a subsequence $\{z^{2k_j}\}$ of $\{z^{2k}\}$ which converges weakly to some point z^*.

Recall that

$$c_{2k}^{-1}\left(z^{2k} - J_{c_{2k}T}(z^{2k})\right) \in T(J_{c_{2k}T}(z^{2k})).$$

Since T is monotone, we have

$$\langle z - J_{c_{2k}T}(z^{2k}), v - c_{2k}^{-1}(z^{2k} - J_{c_{2k}T}(z^{2k}))\rangle \geq 0, \forall\, z, v \text{ such that } v \in T(z). \tag{38}$$

Replacing $2k$ with $2k_j$ in (38), gives

$$\langle z - J_{c_{2k_j}T}(z^{2k_j}), v - c_{2k_j}^{-1}(z^{2k_j} - J_{c_{2k_j}T}(z^{2k_j}))\rangle \geq 0, \forall\, z, v \text{ such that } v \in T(z). \tag{39}$$

From Theorem 3.1 (ii), we have

$$\lim_{k\to\infty} \|z^{2k} - \bar{z}^{2k}\| = 0. \tag{40}$$

Since $\{z^{2k_j}\}$ converges weakly to z^* and (40) holds, then $\{\bar{z}^{2k_j}\}$ converges weakly to z^*. Hence, we obtain from (39) that $\langle u - z^*, v\rangle \geq 0$. Since T is maximal monotone, (see, e.g., [5]), we conclude that $z^* \in T^{-1}(0)$. Therefore, by Lemma 2.4 we have that $\{z^{2k}\}$ converges weakly to an element of $T^{-1}(0)$.

Claim: z^* is unique.

Assume for contradiction that there exists $\xi \in H$ such that z^{2k_j} converges weakly to ξ. Then

$$\begin{aligned}
\|z^* - \xi\|^2 &= \langle z^* - \xi, z^* - \xi\rangle = \langle z^*, z^* - \xi\rangle - \langle \xi, z^* - \xi\rangle \\
&= \lim_{k\to\infty}\langle z^{2k}, z^* - \xi\rangle - \lim_{k\to\infty}\langle z^{2k}, z^* - \xi\rangle \\
&= \lim_{k\to\infty}\langle z^{2k} - z^{2k}, z^* - \xi\rangle = 0. \qquad (41)
\end{aligned}$$

Hence, z^* is unique. Definition 2.2 of weak convergence gives

$$\lim_{k\to\infty}\langle z^{2k} - z^*, \mu\rangle = 0\ \forall \mu \in H. \tag{42}$$

From (35) and (42), we have that for all $\mu \in H$,

$$\begin{aligned}
|\langle z^{2k+1} - z^*, \mu\rangle| &= |\langle z^{2k+1} - z^* + z^{2k} - z^{2k}, \mu\rangle| \\
&\leq |\langle z^{2k} - z^*, \mu\rangle| + \|z^{2k+1} - z^{2k}\|\|\mu\| \to 0,\ k \to \infty \qquad (43)
\end{aligned}$$

Hence $\{z^k\}$ converges weakly to $z^* \in T^{-1}(0)$. □

3.2. linear Convergence.

Lemma 3.4. *Let $T : H \to 2^H$ be maximal monotone and z^* be a solution point of* (1)*, let $c > 0$. If T^{-1} is Lipschitz continuous at $\mathbf{0}$ with modulus $a > 0$, then there exists a positive number τ such that*

$$\|J_{cT}(z) - z^*\| \leq \frac{a}{\sqrt{a^2+c^2}}\|z - z^*\| \textit{when } \|c^{-1}(z - J_{cT}(z))\| \leq \tau \; \forall z \in H. \tag{44}$$

Proof. Applying Property (ii) in Lemma 2.2 with $\bar{z} = z^*$ and noticing that $c > 0$, we get

$$\|z - z^*\|^2 \geq \|J_{cT}(z) - z^*\|^2 + \|(I - J_{cT})(z)\|^2 \tag{45}$$

Now, using definition of J_{cT} we get $c^{-1}(I - J_{cT})(z) \in T(J_{cT}(z))$. Since T^{-1} is Lipschitz continuous at 0 with modulus $a > 0$, it follows from definition2.1 that there exists a positive parameter τ such that

$$\|J_{cT}(z) - z^*\| \leq a\|c^{-1}(z - J_{cT}(z))\| \text{when } \|c^{-1}(z - J_{cT}(z))\| \leq \tau \; \forall z \in H. \tag{46}$$

That is,

$$\|(z - J_{cT}(z))\| \geq \frac{c}{a}\|J_{cT}(z) - z^*\| \text{when } \|c^{-1}(z - J_{cT}(z))\| \leq \tau \; \forall z \in H. \tag{47}$$

Substituting this inequality (47) into (45), we obtain

$$\begin{aligned} \|z - z^*\|^2 &\geq \|J_{cT}(z) - z^*\|^2 + \frac{c}{a}\|J_{cT}(z) - z^*\|^2 \\ &= \frac{a^2+c^2}{a^2}\|J_{cT}(z) - z^*\|^2 \text{when } \|c^{-1}(z - J_{cT}(z))\| \leq \tau \; \forall z \in H. \end{aligned} \tag{48}$$

From (48), we get (44). This completes the proof. □

Lemma 3.5. *Let $\{z^k\}$ be the sequence generated by the algorithm* (37)*, with $\gamma \in (0,2)$, and z^* be a solution point of* (1)*. If T^{-1} is Lipschitz continuous at $\mathbf{0}$ with modulus $a > 0$, and the proximal parameter c is positive, then there exists an integer $\hat{k}$ such that*

$$\|\bar{w}^k - z^*\| \leq \frac{a}{\sqrt{a^2+c^2}}\|w^k - z^*\| \; \forall \; k > \hat{k}. \tag{49}$$

Proof. Observe that Lemma 3.4 above holds for all $z \in H$. Set $\bar{w}^k = J_{cT}(w^k)$, then $\bar{w}^k \in H \; \forall \; k$. Also, w^k is in H for all k. Therefore, the proof follows from Lemma 3.4 above . □

Lemma 3.6. *Let $T : H \to 2^H$ be a maximal monotone operator and suppose the following assumptions hold:*

(i) $\gamma \in (0,2)$*;*

(ii) $0 \leq \alpha < \frac{2-\gamma}{\gamma}$*;*

(iii) $T^{-1}(0) \neq \emptyset$.

(iv) T^{-1} *is Lipschitz continuous at* $\mathbf{0}$ *with modulus* $a > 0$, *and the proximal parameter* c *is positive (* $c > 0$ *) Given* $z^0, z^1 \in H$, *let the sequence* $\{z^k\}$ *be generated by*

$$w^k = \begin{cases} z^k, & k = \text{even} \\ z^k + \alpha(z^k - z^{k-1}), & k = \text{odd} \end{cases}$$

and

$$z^{k+1} = w^k - \gamma(w^k - J_{cT}(w^k)) \; \forall \; k \geq 1. \tag{50}$$

then there exists a positive integer k_1 *such that*

$$\|z^{k+1} - z^*\|^2 \leq \tau \|w^k - z^*\|^2 \; \forall \; k > k_1 with \tag{51}$$

$$\tau := 1 - \min\left\{\gamma, 2\gamma - \gamma^2\right\} \frac{c^2}{a^2 + c^2} \in (0,1). \tag{52}$$

and z^* *is the unique solution of* (1)

Proof.

$$\begin{aligned} \|z^{k+1} - z^*\|^2 &= \|w^k - \gamma(w^k - J_c T(w^k)) - z^*\|^2 \\ &= \|(1-\gamma)(w^k - z^*) + \gamma(J_c T(w^k) - z^*)\|^2 \\ &= (1-\gamma)^2 \|w^k - z^*\|^2 + 2\gamma(1-\gamma)\langle w^k - z^*, \bar{w}^k - z^*\rangle + \gamma^2 \|\bar{w}^k - z^*\|^2 \\ &= (1-\gamma)^2 \|w^k - z^*\|^2 + 2\gamma(1-\gamma)\|\bar{w}^k - z^*\|^2 + \gamma^2 \|\bar{w}^k - z^*\|^2 \\ &+ 2\gamma(1-\gamma)\langle w^k - \bar{w}^k, \bar{w}^k - z^*\rangle \text{ with } \bar{w}^k = J_{cT}(w^k) \\ &= (1-\gamma)^2 \|w^k - z^*\|^2 + (2\gamma - \gamma^2)\|\bar{w}^k - z^*\|^2 \\ &+ 2\gamma(1-\gamma)\langle w^k - \bar{w}^k, \bar{w}^k - z^*\rangle. \end{aligned} \tag{53}$$

For $\gamma = 1$, assertions (51) and (52) follow immediately from Lemma 3.5. For $0 < \gamma \leq 1$, we have from (53) the following estimates :

$$\begin{aligned}
\|z^{k+1} - z^*\|^2 &= (1-\gamma)^2\|w^k - z^*\|^2 + (2\gamma - \gamma^2)\|\bar{w}^k - z^*\|^2 \\
&+ 2\gamma(1-\gamma)\langle w^k - \bar{w}^k, \bar{w}^k - z^*\rangle \\
&= (1-\gamma)^2\|w^k - z^*\|^2 + (2\gamma - \gamma^2)\|\bar{w}^k - z^*\|^2 \\
&+ 2\gamma(1-\gamma)\langle w^k - z^* + z^* - \bar{w}^k, \bar{w}^k - z^*\rangle \\
&= (1-\gamma)^2\|w^k - z^*\|^2 + (2\gamma - \gamma^2)\|\bar{w}^k - z^*\|^2 \\
&+ 2\gamma(1-\gamma)\langle w^k - z^*, \bar{w}^k - z^*\rangle - 2\gamma(1-\gamma)\langle \bar{w}^k - z^*, \bar{w}^k - z^*\rangle \\
&= (1-\gamma)^2\|w^k - z^*\|^2 + (2\gamma - \gamma^2)\|\bar{w}^k - z^*\|^2 \\
&+ 2\gamma(1-\gamma)\langle w^k - z^*, \bar{w}^k - z^*\rangle - 2\gamma(1-\gamma)||\bar{w}^k - z^*||^2 \\
&\leq (1-\gamma)^2\|w^k - z^*\|^2 + (2\gamma - \gamma^2)\|\bar{w}^k - z^*\|^2 \\
&+ \gamma(1-\gamma)||w^k - z^*||^2 + \gamma(1-\gamma)||\bar{w}^k - z^*||^2 \\
&- 2\gamma(1-\gamma)||\bar{w}^k - z^*||^2 \text{ since } 2ab \leq a^2 + b^2\ \forall\, a, b \in \mathbb{R} \\
&= (1-\gamma)\|w^k - z^*\|^2 + \gamma\|\bar{w}^k - z^*\|^2 \\
&\leq \big((1-\gamma) + \gamma\frac{a^2}{a^2+c^2}\big)\|w^k - z^*\|^2 \text{ by Lemma 3.5} \\
&= \big(1 - \gamma\frac{c^2}{a^2+c^2}\big)\|w^k - z^*\|^2.
\end{aligned}$$

That is,

$$\|z^{k+1} - z^*\|^2 \leq \Big(1 - \gamma\frac{c^2}{a^2+c^2}\Big)\|w^k - z^*\|^2 \tag{54}$$

From Lemma 2.2 (ii) with $z = w^k, \bar{w}^k = J_{cT}(w^k), z' = z^*$, we have that

$$\|w^k - z^*\|^2 \geq \|\bar{w}^k - z^*\|^2 + \|w^k - \bar{w}^k\|^2.$$

With this, we have that $\langle w^k - \bar{w}^k, \bar{w}^k - z^*\rangle \geq 0$. If $1 < \gamma < 2$, we get

$$2\gamma(1-\gamma)\langle w^k - \bar{w}^k, \bar{w}^k - z^*\rangle \leq 0. \tag{55}$$

Hence, from (53) and Lemma 3.5 we obtain

$$\|z^{k+1}-z^*\|^2 \leq \Big(1-(2\gamma-\gamma^2)\frac{c^2}{a^2+c^2}\Big)\|w^k-z^*\|^2 \ \forall k>\hat{k}. \tag{56}$$

To verify (52), observe that for $\gamma \in (0,2)$ and $c \geq a > 0$, we get

$$\begin{aligned} 0 < 1-\min\Big\{\gamma, 2\gamma-\gamma^2\Big\} &\leq \tau := 1-\min\Big\{\gamma, 2\gamma-\gamma^2\Big\}\frac{c^2}{a^2+c^2} \\ &< 1-\min\Big\{\gamma, 2\gamma-\gamma^2\Big\}\frac{a^2}{a^2+a^2} < 1. \end{aligned}$$

Therefore,

$$\|z^{k+1}-z^*\|^2 \leq \Big(1-\min\Big\{\gamma, 2\gamma-\gamma^2\Big\}\frac{c^2}{a^2+c^2}\Big)\|w^k-z^*\|^2. \tag{57}$$

This establishes (51).

□

Theorem 3.3. *Let $T: H \to 2^H$ be a maximal monotone operator and suppose the following assumptions hold:*

(i) $\gamma \in (0,2)$;

(ii) $0 \leq \alpha < \frac{2-\gamma}{\gamma}$;

(iii) $T^{-1}(0) \neq \emptyset$.

(iv) T^{-1} is Lipschitz continuous at $\mathbf{0}$ with modulus $a>0$, and the proximal parameter c is positive ($c>0$). Let $\{z^k\}$ be the sequence generated by the scheme (37). Then, $\{z^k\}$ converges strongly to the unique solution z^* of (1). Furthermore, there exists an integer $\bar{k}$ such that

$$\begin{aligned} ||z^{k+1}-z^*|| &\leq \theta||w^k-z^*|| \ \text{ when } k>\bar{k}, \text{ and} \\ 0 &< \theta := \frac{\sqrt{1-\min(\gamma,(2\gamma-\gamma^2)\frac{c^2}{a^2+c^2})}+\gamma\delta_k}{1-\delta_k} < 1. \end{aligned} \tag{58}$$

Moreover, for $0 \leq \alpha \leq \frac{1-\theta}{1+\theta}$, the sequence $\{z^k\}$ satisfies

$$||z^k-z^*|| \leq \begin{cases} \frac{||z^2-z^*||}{\theta}\theta^{\frac{k}{2}} & \text{if } k = even \\ \\ \frac{||z^2-z^*||}{\theta}\theta^{\frac{(k-1)}{2}} & \text{if } k = odd, \end{cases}$$

That is, $\{z^k\}$ converges $R-$linearly to the unique element $z^* \in zer(T)$.

Proof. Set $\hat{z}^{k+1} = (1-\gamma)w^k + \gamma\bar{w}^k$. Then, $\hat{z}^{k+1} - z^{k+1} = \gamma(\bar{w}^k - \bar{w}^k)$, where $\bar{w}^k = J_{cT}(w^k)$. From Lemma 3.6,

$$||\hat{z}^{k+1} - z^*||^2 \leq 1 - \min(\gamma, (2\gamma - \gamma^2))\frac{c^2}{a^2+c^2}||w^k - z^*||^2. \tag{59}$$

Further, from (37)

$$\begin{aligned}
||z^{k+1} - z^*|| &\leq \gamma\delta_k\big(||w^k - z^*|| + ||z^{k+1} - z^*||\big) + ||\hat{z}^{k+1} - z^*|| \\
&\leq \gamma\delta_k\big(||w^k - z^*|| + ||z^{k+1} - z^*||\big) + \sqrt{1 - \min(\gamma, (2\gamma - \gamma^2))\frac{c^2}{a^2+c^2}}||w^k - z^*|| \\
&\leq \frac{\sqrt{1 - \min(\gamma, (2\gamma - \gamma^2))\frac{c^2}{a^2+c^2}} + \gamma\delta_k}{1 - \gamma\delta_k}||w^k - z^*|| \qquad (60)
\end{aligned}$$

Since $\delta_k \to 0,\ c > 0$, there exists $\bar{k} \geq k_1$ such that

$$0 < \theta := \frac{\sqrt{1 - \min(\gamma, (2\gamma - \gamma^2))\frac{c^2}{a^2+c^2}} + \gamma\delta_k}{1 - \gamma\delta_k} < 1 \ w\bar{h}en\ k > \bar{k}. \tag{61}$$

Consequently,

$$||z^{k+1} - z^*|| \leq \theta||w^k - z^*|| \ \forall\, k \geq \bar{k}. \tag{62}$$

Establishing (58). From (62),

$$||z^{k+1} - z^*||^2 \leq \theta^2||w^k - z^*||^2 \ \forall\, k \geq \bar{k}. \tag{63}$$

Now,

$$\begin{aligned}
||w^{2k+1} - z^*||^2 &= ||z^{2k+1} + \alpha(z^{2k+1} - z^{2k}) - z^*||^2 \\
&= ||(1+\alpha)(z^{2k+1} - z^*) - \alpha(z^{2k} - z^*)||^2 \\
&= (1+\alpha)||z^{2k+1} - z^*||^2 - \alpha||(z^{2k} - z^*)||^2 \\
&+ \alpha(1+\alpha)||z^{2k+1} - z^{2k}||^2 \text{ by Lemma 2.3(iii)} \qquad (64)
\end{aligned}$$

Using $k = 2k$ in (63) gives

$$||z^{2k+1} - z^*||^2 \leq \theta^2||w^{2k} - z^*||^2 = \theta^2||z^{2k} - z^*||^2 \ \forall\, k \geq \bar{k} \tag{65}$$

Also, putting $k = 2k+1$ in (63) and noticing (64) and (65), we have

$$
\begin{aligned}
||z^{2k+2} - z^*||^2 &\leq \theta^2 ||w^{2k+1} - z^*||^2 \ \forall\, k \geq \bar{k} \\
&= \theta^2 \big[(1+\alpha)||z^{2k+1} - z^*||^2 - \alpha ||z^{2k} - z^*||^2 + \alpha(1+\alpha)||z^{2k+1} - z^{2k}||^2\big] \\
&= \theta^2 \big[\theta^2(1+\alpha)||z^{2k} - z^*||^2 - \alpha ||z^{2k} - z^*||^2 + \alpha(1+\alpha)||z^{2k+1} - z^{2k}||^2\big] \\
&\leq \theta^2 \big[\theta^2(1+\alpha)||z^{2k} - z^*||^2 - \alpha ||z^{2k} - z^*||^2 \\
&+ \alpha(1+\alpha)\big(||z^{2k+1} - z^*|| + ||z^{2k} - z^*||\big)^2\big] \\
&\leq \theta^2 \big[\theta^2(1+\alpha)||z^{2k} - z^*||^2 - \alpha ||z^{2k} - z^*||^2 + \alpha(1+\alpha)(1+\theta)^2||z^{2k} - z^*||^2\big] \\
&\leq \theta^2 \big[\theta^2(1+\alpha) - \alpha + \alpha(1+\alpha)(1+\theta)^2\big]||z^{2k} - z^*||^2 \\
&\leq \theta^2 ||z^{2k} - z^*||^2 \qquad (66)
\end{aligned}
$$

From (66), we have

$$
\begin{aligned}
||z^{2k+2} - z^*|| &\leq \theta ||z^{2k} - z^*|| \\
&\leq \theta^2 ||z^{2k-2} - z^*|| \\
&\vdots \\
&\leq \theta^k ||z^2 - z^*|| \ \forall\, k \geq \bar{k}. \qquad (67)
\end{aligned}
$$

Therefore,

$$
||z^{2k} - z^*|| \leq \frac{||z^2 - z^*||}{\theta} \theta^k \ \forall k \geq \bar{k}. \qquad (68)
$$

Putting (68) into (65), gives

$$
\begin{aligned}
||z^{2k+1} - z^*|| &\leq \theta ||z^{2k} - z^*|| \leq ||z^{2k} - z^*|| \\
&\leq \frac{||z^2 - z^*||}{\theta} \theta^k \ \forall\, k \geq \bar{k}. \qquad (69)
\end{aligned}
$$

That is

$$
||z^k - z^*|| \leq \begin{cases} \frac{||z^2 - z^*||}{\theta} \theta^{\frac{k}{2}} & \text{if } k = even \\ \\ \frac{||z^2 - z^*||}{\theta} \theta^{\frac{(k-1)}{2}} & \text{if } k = odd, \end{cases}
$$

Hence, $\{z^k\}$ converges $R-$linearly to $z^* \in zer(T)$ as required.

$\square$

CONCLUSION

In this paper, we introduce an alternated inertia inexact generalized PPA for solving monotone inclusion problem in infinite dimensional real Hilbert spaces. Weak and linear convergence results are obtained. Fejér monotonicity of $\|z^k - z^*\|, z^* \in zer(T)$ to some extent, which is not obtained for iPPA in [16, 17] and other related works is aslo recovered. It has been recently stated in [31] that one cannot obtain linear convergence result with inertial PPA; here in our results, we are able to show that linear convergence result for alternated inertial PPA is possible even for the more general inexact version of alternated inertial PPA.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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