ON ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN 
\(q\)-HYPERCONVEX \(T_0\)-QUASI-METRIC SPACES

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Abstract. In this note a well known result of Khamsi [Proc. Amer. Math. Soc. 132 (2004), 365-373] on approximate fixed points for asymptotically nonexpansive mappings on bounded hyperconvex spaces is generalized to the setting of \(q\)-hyperconvex \(T_0\)-quasi-metric spaces.

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1. Introduction

Let \((X, d)\) be a metric space. A mapping \(T : X \rightarrow X\) is called nonexpansive if

\[
d(T(x), T(y)) \leq d(x, y)
\]

for all \(x, y \in X\). \(T : X \rightarrow X\) is called asymptotically nonexpansive (see Goebel and Kirk [3]) if there exists a sequence of positive numbers \((k_n)_{n \in \mathbb{N}},\) with \(\lim_{n \to \infty} k_n = 1\), such that

\[
d(T(x), T(y)) \leq k_n d(x, y)
\]

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for all \( x, y \in X \). It is known (see [3]) that the class of \emph{asymptotically nonexpansive mappings} is wider than the class of \emph{nonexpansive mappings}.

A well known result which was proved independently by Sine [9] and Soardi [10] in hyperconvex spaces (see [1], [2]) states that the fixed point property for noexpansive mappings holds in a bounded hyperconvex space. Further, it has been proved by Khamsi [5] that: if \( T : H \to H \), where \((H, \rho)\) is a bounded hyperconvex metric space and \( T \) is an asymptotically nonexpansive mapping, then \( T \) has approximate fixed points, that is, \( \inf \{ \rho(x, Tx) : x \in H \} = 0 \). Recently, Künzi and Otafudu [6] have introduced and studied the concept of \( q \)-hyperconvexity in \( T_0 \)-quasi-metric spaces and obtained certain fixed point theorems there in. In this note we continue our studies of this concept by generalizing the above result of Khamsi [5] and show that an asymptotically nonexpansive mapping on a bounded \( q \)-hyperconvex \( T_0 \)-quasi-metric space has approximate fixed points.

2. Preliminaries

For the convenience of the reader and in order to fix our terminology we recall the following concepts.

\textbf{Definition 2.1.} Let \( X \) be a set and let \( d : X \times X \to [0, \infty) \) be a function mapping into the set \([0, \infty)\) of the nonnegative reals. Then \( d \) is called a \emph{quasi-pseudometric} on \( X \) if

(a) \( d(x, x) = 0 \) for all \( x \in X \),

(b) \( d(x, z) \leq d(x, y) + d(y, z) \) for all \( x, y, z \in X \).

We shall say that \( d \) is a \emph{\( T_0 \)-quasi-metric} provided that \( d \) also satisfies the following condition: For each \( x, y \in X \),

\( d(x, y) = 0 = d(y, x) \) implies that \( x = y \).

\textbf{Remark 2.2.} In some cases we need to replace \([0, \infty)\) by \([0, \infty]\) (where for a \( d \) attaining the value \( \infty \) the triangle inequality is interpreted in the obvious way). In such a case we shall speak of an \emph{extended quasi-pseudometric}. In the following we sometimes apply concepts from the theory of quasi-pseudometrics to extended quasi-pseudometrics (without changing the usual definitions of these concepts).
Remark 2.3. Let \( d \) be a quasi-pseudometric on a set \( X \), then \( d^{-1} : X \times X \to [0, \infty) \) defined by \( d^{-1}(x, y) = d(y, x) \) whenever \( x, y \in X \) is also a quasi-pseudometric, called the \textit{conjugate quasi-pseudometric of} \( d \). As usual, a quasi-pseudometric \( d \) on \( X \) such that \( d = d^{-1} \) is called a \textit{pseudometric}. Note that for any \( T_0 \)-quasi-pseudometric \( d \), \( d^* = \max\{d, d^{-1}\} = d \vee d^{-1} \) is a pseudometric (metric).

Let \((X, d)\) be a quasi-pseudometric space. For each \( x \in X \) and \( \epsilon > 0 \), \( B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\} \) denotes the \textit{open} \( \epsilon \)-ball at \( x \). The collection of all “open” balls yields a base for a topology \( \tau(d) \). It is called the \textit{topology induced by} \( d \) on \( X \). Similarly we set for each \( x \in X \) and \( \epsilon \geq 0 \), \( C_d(x, \epsilon) = \{y \in X : d(x, y) \leq \epsilon\} \). Note that this latter set is \( \tau(d^{-1}) \)-closed, but not \( \tau(d) \)-closed in general.

3. \textit{q-}Hyper convexity

In this section we recall some results on \( q \)-hyperconvexity. Some recent further work about \( q \)-hyperconvexity can be found in [4], [6] and [7].

Definition 3.1. [4, Definition 2]. A quasi-pseudometric space \((X, d)\) is called \textit{\( q \)-hyperconvex} provided that for each family \((x_i)_{i \in I}\) of points in \( X \) and families of nonnegative real numbers \((r_i)_{i \in I}\) and \((s_i)_{i \in I}\) the following condition holds: If \( d(x_i, x_j) \leq r_i + s_j \) whenever \( i, j \in I \), then

\[
\bigcap_{i \in I} (C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \neq \emptyset.
\]

Remark 3.2. If \( d \) and \( d^{-1} \) are identical and \( r_i = s_i \) for \( i \in I \) in Definition 3.1, then \((C_d(x_i, r_i))\) and \((C_{d^{-1}}(x_i, s_i))\) coincide and then we recover the well known definition of hyperconvexity due to Aronszajn and Panitchpakdi [1].

The following examples are basic, but important.

Example 3.3. ([4, Example 1], compare [8, Example 2]). Let the set \( \mathbb{R} \) of the reals be equipped with the \( T_0 \)-quasi-metric \( u(x, y) = \max\{x - y, 0\} \) whenever \( x, y \in \mathbb{R} \). Then \((\mathbb{R}, u)\) is \( q \)-hyperconvex.
Corollary 3.4. ([4, Corollary 1]). The quasi-pseudometric subspace \([0, \infty)\) of \((\mathbb{R}, u)\) is \(q\)-hyperconvex.

Example 3.5. ([4, Example 2]). Let \(\mathbb{R}\) be equipped with its standard metric \(u^*(x, y) = |x - y|\) whenever \(x, y \in \mathbb{R}\). Then \((\mathbb{R}, u^*)\) is not \(q\)-hyperconvex.

Proposition 3.6. ([4, Proposition 2]) (a) If \((X, d)\) is a(n extended) \(q\)-hyperconvex (resp. \(q\)-hypercomplete, metrically convex) quasi-pseudometric space, then \((X, d^{-1})\) is \(q\)-hyperconvex (resp. \(q\)-hypercomplete, metrically convex).

(b) If \((X, d)\) is a \(q\)-hyperconvex (resp. \(q\)-hypercomplete) quasi-pseudometric space, then the metric space \((X, d^*)\) is hyperconvex (resp. hypercomplete). However, the corresponding statement for “metrically convex” does not hold.

The following definition can be found in [6] (compare [5] and [9]).

Definition 3.7. ([6, Definition 8]). Let \((X, d)\) be a \(T_0\)-quasi-metric space. We say that a mapping \(T : (X, d) \to (X, d)\) has approximate fixed points if \(\inf_{x \in X} d^*(x, T(x)) = 0\).

4. Main Result

We first recall the following interesting result due to Khamsi [5].

Theorem 4.1. Let \((H, \rho)\) be a bounded hyperconvex metric space and \(T : H \to H\) be asymptotically nonexpansive mapping. Then \(T\) has approximate fixed points, i.e. \(\inf\{\rho(x, T(x)) : x \in H\} = 0\).

The following result generalizes the above theorem to the setting of \(q\)-hyperconvex \(T_0\)-quasi-metric spaces.

Theorem 4.2. Let \((X, d)\) be a bounded \(q\)-hyperconvex \(T_0\)-quasi-metric space and \(T : X \to X\) be asymptotically nonexpansive mapping. Then \(T\) has approximate fixed points, i.e. \(\inf_{x \in X} d^*(x, T(x)) = 0\).
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**Proof.** Since $T : X \to X$ is asymptotically nonexpansive, there exists a sequence of nonnegative real numbers $(k_n)_{n \in \mathbb{N}}$, with $\lim_{n \to \infty} k_n = 1$, such that

$$d(T^n(x), T^n(y)) \leq k_n d(x, y)$$

for all $x, y \in X$.

We shall first show that $T : (X, d^s) \to (X, d^s)$ is asymptotically nonexpansive. Since for any $x, y \in X$, we have

$$d^{-1}(T^n(x), T^n(y)) = d(T^n(y), T^n(x)) \leq k_n d(y, x) = k_n d^{-1}(x, y)$$

with $\lim_{n \to \infty} k_n = 1$, we see that $T : (X, d^{-1}) \to (X, d^{-1})$ is asymptotically nonexpansive. Therefore

$$d(T^n(x), T^n(y)) \leq k_n d(x, y) \leq k_n d^s(x, y)$$

and

$$d^{-1}(T^n(x), T^n(y)) \leq k_n d^{-1}(x, y) \leq k_n d^s(x, y)$$

for all $x, y \in X$. Hence

$$d^s(T^n(x), T^n(y)) \leq k_n d^s(x, y)$$

for all $x, y \in X$ with $\lim_{n \to \infty} k_n = 1$ and so, $T : (X, d^s) \to (X, d^s)$ is asymptotically nonexpansive.

By assumption $(X, d^s)$ is bounded and by Proposition 3.1 (b) it is hyperconvex. Therefore by Theorem 4.1 $T$ has approximative fixed points, i.e. $\inf_{x \in X} d^s(x, T(x)) = 0$ and the conclusion holds. □

**References**


