FIXED POINT THEORY FOR SIMULATION FUNCTIONS IN G-METRIC SPACES: A NOVEL APPROACH

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Abstract: In this paper, with the aid of simulation mapping $\eta : [0, \infty) \times [0, \infty) \to \mathbb{R}$, we prove some Lemmas and fixed point result for generalized $\mathcal{Z}$ – contraction of the mapping $g : X \to X$ satisfying the following conditions:

$$
\eta(g(x, gy, gz), M(x, y, z)) \geq 0,
$$

for all $x, y, z \in X$, where

$$
M(x, y, z) = \max\{g(x, gy, gz), g(y, gx, gz), g(y, gz, gx), g(z, gy, gz), g(z, gz, gx), g(x, gz, gz)\}.
$$

and $(X, \mathcal{G})$ is a $\mathcal{G}$ – metric space. An example is also given to support our results.

Keywords: fixed point; generalized $\mathcal{Z}$ – contraction; simulation function; $\mathcal{G}$ – metric spaces.

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1. INTRODUCTION

A metric space is a nonempty set $X$ with a two-variable map $d$ that allows us to calculate the distance between two points. We must find the distance not just between integers and vectors,
but also between sequences and functions in higher mathematics. Numerous approaches exist in this sector in order to discover a suitable concept of a metric space. Many renowned mathematicians have considered various generalizations of a metric space. Mustafa and Sims [1] presented $G$-metric space in 2006 and provided an essential generalization of a metric space as follows:

**Definition 1.1.** [1] Let $X$ be a non empty set and $G: X^3 \to [0, \infty)$ be a map which satisfies the following properties:

1. $G(x,y,z) = 0$ if $x = y = z,$
2. $0 < G(x,x,y)$ whenever $x \neq y,$
3. $G(x,x,y) \leq G(x,y,z), y \neq z,$
4. $G(x,y,z) = G(x,z,y) = G(y,x,z) = G(z,x,y) = G(y,z,x) = G(z,y,x),$
5. $G(x,y,z) \leq G(x,a,a) + G(a,y,z), \ \forall x,y,z,a \in X.$

Then, the function $G$ is said to be $G$-metric on $X$ and the pair $(X,G)$ is known as $G$-metric space.

Banach [2] established the Banach contraction principle, a useful conclusion in fixed point theory involving a contraction mapping, in 1922.

**Definition 1.2.** [2] Let $(X,d)$ be a complete metric space and let $f: X \to X$ be a self-mapping. Let $d(fx, fy) < d(x, y)$ holds for all $x,y \in X$ with $x \neq y.$ Then, $f$ is called a contraction known as Banach contraction.

Following this approach, a number of scholars expanded on it by offering various contractions on metric spaces [3, 4-9]. We introduce a mapping, namely the simulation function, and the concept of generalized $\Gamma$-contraction in this paper. Khojasteh et al. [10] have proposed a new class of mappings known as simulation functions. Later, Argoubi et al. [11] made a minor change to the definition of simulation functions by removing a constraint.

**Definition 1.3.** [11] A simulation function is a mapping $\zeta: [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:

$(\zeta_1) \ \zeta(t,s) < s - t \text{ for all } t, s > 0$
(ζ₂) if \( \{t_n\} \) and \( \{s_n\} \) are sequences in \((0, \infty)\) such that
\[
\lim_{n \to \infty} \{t_n\} = \lim_{n \to \infty} \{s_n\} = l \in (0, \infty),
\]
then
\[
\limsup_{n \to \infty} \zeta(t_n, s_n) < 0.
\]

2. MAIN RESULTS

In this section, we prove certain Lemmas and some fixed point results for generalized \( \mathcal{Z} \) - contraction in \( \mathcal{G} \) - metric space.

**Definition 2.1.** Let \( (X, \mathcal{G}) \) be a \( \mathcal{G} \) - metric space, \( g : X \to X \) a mapping and \( \eta \in \mathbb{Z} \). Then \( g \) is called a generalized \( \mathcal{Z} \) - contraction with respect to \( \eta \) if the following condition is satisfied
\[
\eta(\mathcal{G}(gx, gy, gz), \mathcal{M}(x, y, z)) \geq 0, \tag{1}
\]
for all \( x, y, z \in X \), where \( \mathcal{M}(x, y, z) = \max \{\mathcal{G}(x, gy, gy), \mathcal{G}(y, gx, gx), \mathcal{G}(y, gz, gz), \mathcal{G}(z, gy, gy), \mathcal{G}(z, gz, gz), \mathcal{G}(x, gx, gx), \mathcal{G}(x, gz, gz)\} \).

**Lemma 2.2.** Let \( (X, \mathcal{G}) \) denote a \( \mathcal{G} \) - metric space and \( g : X \to X \) denote a generalized \( \mathcal{Z} \) - contraction with regard to \( \mathcal{G} \). Then, for all \( x \in X \), \( g \) is asymptotically regular.

**Proof:** Let \( x \in X \) be arbitrary. If for some \( k \in \mathbb{N} \), \( g^k x = g^{k-1}x \), then \( g^{k-1}x \) is a fixed point of \( g \). Therefore, we have
\[
\mathcal{G}(g^{n+1}x, g^n x) = \mathcal{G}(g^{n+k+1} x, g^{n-k+2} g^{k-1} x, g^{n-k+2} g^{k-1} x) \\
= \mathcal{G}(g^{n-k+1} y, g^{n-k+2} y, g^{n-k+2} y) \\
= \mathcal{G}(y, y, y) = 0.
\]
Therefore, \( \lim_{n \to \infty} \mathcal{G}(g^nx, g^{n+1}x, g^{n+1}x) = 0 \).

So, let us suppose that \( g^nx \neq g^{n-1}x \) for all \( n \in \mathbb{N} \), then it follows from (1) that
\[
\eta(\mathcal{G}(g^{n+1}x, g^nx, g^nx), \mathcal{M}(g^nx, g^{n-1}x, g^{n-1}x)) \geq 0,
\]
since \( g \) is a generalized contraction, where
\[
\mathcal{M}(g^nx, g^{n-1}x, g^{n-1}x) = \max \{\mathcal{G}(g^nx, g^{n}x, g^{n}x), \mathcal{G}(g^{n-1}x, g^{n+1}x, g^{n+1}x), \mathcal{G}(g^{n-1}x, g^{n+1}x, g^{n+1}x), \mathcal{G}(g^{n-1}x, g^{n}x, g^{n}x), \mathcal{G}(g^{n-1}x, g^{n}x, g^{n}x), \mathcal{G}(g^{n-1}x, g^{n}x, g^{n}x), \mathcal{G}(g^{n-1}x, g^{n}x, g^{n}x)\}.
\]
\[ \mathcal{G}(g^{n-1}x, g^n x, g^n x), \mathcal{G}(g^{n+1}x, g^{n+1} x, g^{n+1} x), \]
\[ \mathcal{G}(g^n x, g^n x, g^n x) \]
\[ = \max\{\mathcal{G}(g^n x, g^{n-1} x, g^n x), \mathcal{G}(g^{n+1} x, g^n x, g^n x)\}, \]
\[ \text{since} \]
\[ \mathcal{G}(g^{n+1} x, g^n x, g^n x) \leq \mathcal{G}(g^{n+1} x, g^n x, g^n x) + \mathcal{G}(g^n x, g^{n-1} x, g^n x). \]
\[ \text{If } \max\{\mathcal{G}(g^n x, g^{n-1} x, g^n x), \mathcal{G}(g^{n+1} x, g^n x, g^n x)\} = \mathcal{G}(g^{n+1} x, g^n x, g^n x), \text{ then} \]
\[ \eta(\mathcal{G}(g^{n+1} x, g^n x, g^n x), M(g^n x, g^{n-1} x, g^{n-1} x)) \]
\[ = \eta(\mathcal{G}(g^{n+1} x, g^n x, g^n x), \mathcal{G}(g^{n+1} x, g^n x, g^n x)) \geq 0, \]
which is a contradiction. So, \( \mathcal{G}(g^{n+1} x, g^n x, g^n x) < \mathcal{G}(g^n x, g^{n-1} x, g^n x) \) holds. This shows that \( \mathcal{G}(g^n x, g^{n-1} x, g^n x) \) is monotonically decreasing sequence of non-negative reals and so it must be convergent.

Let \( \lim_{n \to \infty} \mathcal{G}(g^n x, g^{n-1} x, g^n x) = s. \)

If \( s > 0, \) then by contraction condition
\[ 0 \leq \lim_{n \to \infty} \sup \eta(\mathcal{G}(g^{n+1} x, g^n x, g^n x), M(g^n x, g^{n-1} x, g^{n-1} x)) \]
\[ = \lim_{n \to \infty} \sup \eta(\mathcal{G}(g^{n+1} x, g^n x, g^n x), M(g^n x, g^{n-1} x, g^{n-1} x)) < 0, \]
a contradiction and thus \( s > 0 \) and \( g \) is asymptotically regular.

**Lemma 2.3.** Every Picard sequence converges to its unique fixed point, which is found in every generalized \( Z - \) contraction mapping on a complete \( \mathcal{G} - \) metric space where \( x_n = gx_{n-1} \) for all \( n \in \mathbb{N}. \)

**Proof:** Let \( (X, \mathcal{G}) \) denote a \( \mathcal{G} - \) metric space and \( g : X \to X \) a mapping and \( \zeta \in \mathbb{Z}. \)

Let us first demonstrate that if \( g \) has a fixed point, it is unique.

If the mapping \( g \) has two fixed points \( p, r \in X, \) then \( d(p, r) > 0. \)

By (1), we get
\[ \eta(\mathcal{G}(gp, gr, gr), M(p, r, r)) > 0, \]
where
\[ M(p, r, r) \]
\[ = \max\{\mathcal{G}(p, gr, gr), \mathcal{G}(r, gp, gp), \mathcal{G}(r, gr, gr), \mathcal{G}(r, gr, gr), \mathcal{G}(p, gr, gr), \mathcal{G}(p, gr, gr)\}. \]
\[ = \mathcal{G}(p, r, r), \text{which contradicts}(\zeta_2). \]
As a result, there is just one fixed point.

Now, we shall show that if \( \{x_n\} \) is a Picard sequence created by \( g \) then \( \lim_{n \to \infty} x_n = z \) is only fixed point.

Let \( x_0 \in X \) be any number, and \( \{x_n\} \) be the Picard sequence, with \( x_n = g \) for all \( n \in \mathbb{N} \). Assume, on the other hand, that \( \{x_n\} \) is not bounded. We can assume that \( x_{n+k} \neq x_n \) for any \( n, k \in \mathbb{N} \) without losing generality. Because \( \{x_n\} \) is unbounded, there is a subsequence \( \{x_{n_k}\} \) such that \( n_1 = 1 \) and for each \( k \in \mathbb{N}, n_{k+1} \) is the smallest integer.

\[
G(x_{n_{k+1}}, x_{n_k}, x_{n_k}) > 1 \text{ and }
\]
\[
G(x_m, x_{n_k}, x_{n_k}) \leq 1 \text{ for } n_k \leq m \leq n_{k+1} - 1.
\]

Therefore, by triangle inequality, we have
\[
1 < G(x_{n_{k+1}}, x_{n_k}, x_{n_k})
\]
\[
\leq G(x_{n_{k+1}}, x_{n_k-1} - 1, x_{n_{k+1}} - 1) + G(x_{n_{k+1}} - 1, x_{n_k}, x_{n_k})
\]
\[
\leq G(x_{n_{k+1}}, x_{n_{k+1}} - 1, x_{n_k} - 1) + 1.
\]

Letting \( k \to \infty \) and using Lemma 2.2, we get
\[
\lim_{k \to \infty} G(x_{n_{k+1}}, x_{n_k}, x_{n_k}) = 1,
\]
\[
M(x_{n_{k+1}} - 1, x_{n_k - 1} - 1, x_{n_k - 1} - 1)
\]
\[
= \max \{G(x_{n_{k+1}} - 1, g x_{n_k - 1} - 1, g x_{n_{k+1}} - 1), G(x_{n_k - 1} - 1, x_{n_{k+1}} - 1, x_{n_{k+1}} - 1),
\]
\[
G(x_{n_{k+1}} - 1, x_{n_k - 1} - 1, x_{n_k - 1} - 1), G(x_{n_k - 1} - 1, x_{n_{k+1}} - 1, x_{n_{k+1}} - 1),
\]
\[
G(x_{n_{k+1}} - 1, g x_{n_k - 1} - 1, g x_{n_{k+1}} - 1), G(x_{n_k - 1} - 1, g x_{n_{k+1}} - 1, g x_{n_{k+1}} - 1)\}.
\]

Now, since \( g \) is a generalized \( Z \) contraction, so that
\[
0 \leq \lim_{k \to \infty} \sup \eta (G(g x_{n_{k+1}} - 1, g x_{n_k - 1} - 1, g x_{n_{k+1}} - 1)) = \lim_{k \to \infty} \sup \eta (G(x_{n_{k+1}} - 1, x_{n_{k-1}}, x_{n_{k-1}})) < 0,
\]
a contradiction. This contradiction proves the result.
**Theorem 2.4.** Let \((X, \mathcal{G})\) be a complete \(\mathcal{G} -\) metric space and \(g : X \to X\) a mapping and \(\eta \in \mathbb{Z}\) and this is a generalized \(\mathcal{G} -\) contraction. Then, \(g\) has a unique fixed point \(u\) in \(X\) and for every \(x_0 \in X\) the Picard sequence \(\{x_n\}\), where \(x_n = gx_{n-1}\) for all \(n \in \mathbb{N}\) converges to the fixed point of \(g\).

**Proof:** Let \(x_0 \in X\) be arbitrary and \(\{x_n\}\) be the Picard sequence, i.e. \(x_n = gx_{n-1}\ \forall\ n \in \mathbb{N}\). We shall show that this sequence is a Cauchy sequence.

For this, let \(C_n = \sup\{x_p, x_r, x_r: p, r \geq n\}\).

Note that the sequence \(\{x_n\}\) is monotonically decreasing sequence of the reals and by Lemma 2.3, the sequence \(\{x_n\}\) is bounded, therefore \(C_n < \infty\) for all \(n \in \mathbb{N}\). Thus, \(\{C_n\}\) is monotonic bounded sequence, therefore converges, that is \(\exists \ C \geq 0\) such that \(C_n = C\). We shall show that \(C = 0\). If \(C > 0\), then by the definition of \(C_n\), for every \(k \in \mathbb{N}\), \(\exists m_k > n_k \geq k\) and \(C_k - \frac{1}{k} \mathcal{G}(x_{m_k}, x_{n_k}, x_{n_k}) \leq C_k\).

Hence,

\[
\lim_{k \to \infty} \mathcal{G}(x_{m_k}, x_{n_k}, x_{n_k}) \leq C_k
\]

Using (1) and the triangular inequality, we obtain

\[
\mathcal{G}(x_{m_k}, x_{n_k}, x_{n_k}) \leq \mathcal{G}(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}})
\]

\[
\leq \mathcal{G}(x_{m_{k-1}}, x_{m_k}, x_{m_k}) + \mathcal{G}(x_{m_k}, x_{n_k}, x_{n_k}) + \mathcal{G}(x_{n_k}, x_{n_{k-1}}, x_{n_{k-1}}).
\]

\[
\mathcal{G}(x_{m_{k-1}}, x_{m_k}, x_{m_k}) \to 0, \mathcal{G}(x_{n_k}, x_{n_{k-1}}, x_{n_{k-1}}) \to 0 \ \text{as} \ k \to \infty.
\]

Then, by Squeeze Theorem, we have

\[
\lim_{k \to \infty} \mathcal{G}(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}}) = C \ \text{as well}
\]

\[
\mathcal{M}(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}})
\]

\[
= \max \{\mathcal{G}(x_{m_{k-1}}, gx_{n_{k-1}}, gx_{n_{k-1}}), \mathcal{G}(x_{n_{k-1}}, gx_{m_{k-1}}, gx_{m_{k-1}}), \mathcal{G}(x_{n_{k-1}}, gx_{n_{k-1}}, gx_{n_{k-1}}), \mathcal{G}(x_{n_{k-1}}, gx_{n_{k-1}}, gx_{n_{k-1}}), \mathcal{G}(x_{n_{k-1}}, gx_{n_{k-1}}, gx_{n_{k-1}}), \mathcal{G}(x_{m_{k-1}}, gx_{n_{k-1}}, gx_{n_{k-1}}), \mathcal{G}(x_{m_{k-1}}, gx_{n_{k-1}}, gx_{n_{k-1}}), \mathcal{G}(x_{m_{k-1}}, gx_{n_{k-1}}, gx_{n_{k-1}}), \mathcal{G}(x_{m_{k-1}}, gx_{n_{k-1}}, gx_{n_{k-1}})\}.
\]
We know that \( G(x_{m_{k-1}}, g_{x_{n_{k-1}}}, g_{x_{n_{k-1}}}) = C \) as well
\[
M(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}}) \to 0, G(x_{n_{k}}, x_{n_{k-1}}, x_{n_{k-1}}) \to 0 \text{ as } k \to \infty.
\]
0 ≤ \( \lim_{k \to \infty} \sup \eta(G(x_{m_{k}}, x_{n_{k}}, x_{n_{k}}), M(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}})) < 0. \)

This contradiction proves that \( C = 0 \) and so \( \{x_{n}\} \) is a Cauchy sequence. Since \( X \) is a complete \( G \) - metric space, \( \exists u \in X \) such that \( \lim_{n \to \infty} x_{n} = u. \) We shall show that the point \( u \) is a fixed point of \( g. \) Suppose \( gu \neq u, \) then \( G(u, gu, gu) > 0. \)

Again, using (1), we have
\[
0 \leq \lim_{k \to \infty} \sup \eta(G(x_{m_{k}}, x_{n_{k}}, x_{n_{k}}), M(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}})) < 0.
\]
\[
\lim_{n \to \infty} G(gz, gx_{n}, gx_{n}) = \lim_{n \to \infty} G(gz, gx_{n+1}, gx_{n+1}) = G(gz, z, z) > 0,
\]
and
\[
M(z, x_{n}, x_{n}) = \max\{G(gz, gx_{n}, gx_{n}), G(x_{n}, gz, gz), G(x_{n}, gx_{n}, gx_{n}), G(x_{n}, gz, gz), G(z, gx_{n}, gx_{n})\}.
\]

Therefore, \( M(z, x_{n}, x_{n}) \to G(gz, z, z) \) as \( n \to \infty. \)

By contractive condition,
\[
0 \leq \eta(G(gz, gx_{n}, gx_{n}), M(z, x_{n}, x_{n})) \to \eta(G(gz, z, z), M(gz, z, z)) \text{ as } n \to \infty.
\]

By (\( \xi_{2} \)), we have \( \eta(G(gz, z, z), M(gz, z, z)) < 0 \) which contradicts the contraction condition.

That means \( gz = z \) and \( z \) is the unique fixed point of \( g. \)

**Example 2.5.** Let \( X = [0, 1] \) and \( g : X \to X \to \mathbb{R} \) be defined by
\[
G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}.
\]

Then \( (X, G) \) is a complete \( G \)-Metric space.

Define a mapping \( g : X \to X \) as \( gx = \frac{x}{x + 1} \) for all \( x \in X. \) \( g \) is a continuous function but it is not a Banach contraction. But it is a generalized \( Z \) - contraction with respect to \( \eta \in Z, \)

where
\[
\eta(t, s) = \frac{s}{s + 1} - t \text{ for all } t, s \in [0, \infty).
\]
Indeed, if \( x, y \in X \), then by a simple calculation it can be shown that
\[
\eta(G(gx, gy, gz), M(x, y, z)) \geq 0 \quad \text{for all } x, y \in X,
\]
where
\[
M(x, y, z) = \max \{ G(x, gy, gy), G(y, gx, gx), G(y, gz, gz), G(z, gy, gy), G(z, gx, gx),
G(x, gz, gz) \}.
\]
Clearly, 0 is the fixed point of \( g \).

**CONFLICT OF INTERESTS**

The authors declare that there is no conflict of interests.

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