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A NEW FIXED POINT RESULT IN GENERALIZED METRIC SPACE WITH A GRAPH

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Abstract. In this paper, we introduce the concepts of G-Kannan contraction in a generalized metric space and G-Chatterjea contraction in dislocated metric space by combining fixed point theory with a graph theory.

Keywords: G-Kannan S-mapping; G-Chatterjea S-mapping; Directed graph; generalized metric space; dislocated metric space; fixed point.

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1. Introduction and Preliminaries

It is well known that the Banach contraction principle was published in 1922 by S. Banach as follows:

Theorem 1. [1] *Let (X,d) be a complete metric space and a self mapping $T : X \longrightarrow X$. T is said to be contraction if there exists $k \in [0,1)$ such that for all $x,y \in X$, $d(Tx,Ty) \leq kd(x,y)$ then T has a unique fixed point in X.*

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Currently, fixed point is a very active area of research because of the importance of its applications in multiple fields. The Banach contraction principle has been extensively studied in various spaces, and different generalizations have been proposed (see [2, 3, 4, 5]).

Kannan established his known extension of this contraction in 1968 [6]. He proved in a complete metric space (X,d), the self-mapping $T : X \longrightarrow X$ has a fixed point in X if the following condition is satisfied: $d(Tx,Ty) \leq k\,[d(x,Tx)+d(y,Ty)]$ for all $x,y \in X$ where $0 \leq k < \frac{1}{2}$.

Chattergea introduced a similar contractive condition in 1972 [3] as follows:

Let $T : X \longrightarrow X$, where (X,d) is a complete metric space if there exists $0 \leq k < \frac{1}{2}$ such that $d(Tx,Ty) \leq k\,[d(y,Tx)+d(Ty,x)]$ for all $x,y \in X$. Then T has a fixed point in X.

Sabiri and al [4] investigated convergence and existence results of the proximity points for p-cyclic contraction in (S) convex metric space in 2020 and developed a novel notion of the measure between p-points where $p \geq 2$.

In 2021, Sabiri and al [7] proved the existence and uniqueness of a fixed point for various types of tricyclic contractions.

In 2008, Jachymski [8] extented the Banach contraction principle in metric space endowed with a graph.

In this work, inspired by the idea given in [9, 10, 11, 12, 13, 14], we investigate Kannan and Chatterjea's fixed point theorem in generalized metric space with graphs, introduced by Jleli and Samet. In addition, several interesting results about the existence and uniqueness of fixed points in generalized metric space were demonstrated (see [15, 16]).

We start by recalling some basic concepts of graphs which are used in this paper.

A directed graph or digraph G is determined by a nonempty set $V(G)$ of its vertices and the set $E(G) \subset V(G) \times V(G)$ of its arcs. Let Δ denote the diagonal of the Cartisian product $V(G) \times V(G)$. A digraph is said to be reflexive if the set $E(G)$ of its edges contains all loops, i.e., $\Delta \subset E(G)$. G is said to be transitive if and only if for any $x,y,z \in V(G)$

$$[(x,y) \in E(G) \text{ and } (y,z) \in E(G)\ ] \Longrightarrow (x,z) \in E(G).$$

We say that a vertex x in $V(G)$ is isolated if for any vertex y in $V(G)$ such that $x \neq y$ we have $(x,y) \notin E(G)$ and $(y,x) \notin E(G)$.

We denote by G^{-1} the converse of a digraph G, that is, the digraph obtained from G by reversing the direction of arcs. Then we have $E\left(G^{-1}\right) = \{(x,y) \in X \times X : (y,x) \in E\left(G\right)\}$.

Also, $\widetilde{G}$ denotes the indirected graph obtained from G by ignoring the direction of the edges. So it is clear that $E\left(\widetilde{G}\right) = E\left(G\right) \cup E\left(G^{-1}\right)$.

The notion of a G-monotone sequence was introduced in [11]. It is claimed that a sequence $\{x_n\} \in V\left(G\right)$ is G-increasing if $(x_n, x_{n+1}) \in E\left(G\right)$ for all $n \in \mathbb{N}$, G-decreasing if $(x_{n+1}, x_n) \in E\left(G\right)$ for all $n \in \mathbb{N}$, and G-monotone if it is either G-increasing or G-decreasing.

Definition 1. *Let X be a nonempty set and $D : X \times X \longrightarrow [0,+\infty[$ be a function satisfies the following conditions:*

(D_1) For every $(x,y) \in X \times X$, $D(x,y) = 0 \Longrightarrow x = y$.

(D_2) For every $(x,y) \in X \times X$, $D(x,y) = D(y,x)$.

(D_3) For every $(x,y,z) \in X \times X \times X$ $D(x,y) \leq D(x,z) + D(z,y)$

(D_4) There exists $C > 0$ such that if $(x,y) \in X \times X$ and $\{x_n\} \in C\left(D,X,x\right)$, then

$$D(x,y) \leq C \limsup_{n \longrightarrow +\infty} D(x_n, y).$$

where $C\left(D,X,x\right) = \{\{x_n\} \subset X : \lim_{n \longrightarrow +\infty} D(x_n, x) = 0\}$.

If D satisfies the conditions (D_1), (D_2), and (D_4), then D is called a generalized metric on X. In this case, we say that the pair (X,D) is the generalized metric space.

If D satisfies the conditons (D_1), (D_2), and (D_3), then (X,D) is called a dislocated metric space.

Definition 2. *Let $\{x_n\}$ a sequence in generalized metric space (X,D) is to be D-convergente to x if $\{x_n\} \in C\left(D,X,x\right)$ and a D-Cauchy sequence if*

$\lim_{n,m \longrightarrow +\infty} D(x_n, x_m) = 0$. Note that in generalized metric spaces, a sequence has at most one limit, and a D-convergent sequence may not be a D-Cauchy sequence. Furthermore, (X,D) is to be D-complete if every D-Cauchy sequence in X is a D-convergent to some element in X.

Definition 3. *A generalized metric space (X,D) endowed with a graph G is said to be G-complete if every D-Cauchy G-monotone sequence $\{x_n\} \subset V\left(G\right)$ D-convergent to a point in $V\left(G\right)$.*

The digraph G is said to satisfy the property (P) that is for any G-monotone increasing (resp. decreasing) sequence $\{x_n\}$ which D-convergent to some element $x \in V(G)$, we have $(x_n, x) \in E(G)$,(resp $(x, x_n) \in E(G)$) for all $n \in \mathbb{N}$.

Definition 4. *A self mapping T on X is called*

(i) weak continuous if the following condition holds: if $\{x_n\} \subset X$ is a D-convergent to $x \in X$, then there exists a subsequence $\{x_{n_q}\}$ of $\{x_n\}$ such that $\{Tx_{n_q}\}$ D-converges to Tx $(as\ q \longrightarrow +\infty)$.

(ii) Orbitally G-continuous if for all $x, y \in V(G)$ and any sequence $\{k_n\}$ of positive integres, $\{T^{k_n}x\}$ D-converges to y and $\left(T^{k_n}x, T^{k_n+1}x\right) \in E(G)$ implies $\{T(T^{k_n}x)\}$ D-converges to Ty.

In 2019, Chaira et al. introduced G-Kannan mapping in generalized metric space (X,D) with the digraph G. A mapping $T : X \longrightarrow X$ is said to be a G-Kannan mapping if the following conditions are satisfied:

(i) T is G-monotone, that is, for all $x, y \in X$, $(x, y) \in E(G) \Longrightarrow (Tx, Ty) \in E(G)$.

(ii) There exists $k \in [0, \frac{1}{2})$ such for every $x, y \in V(G)$

$$(x, y) \in E(G) \Longrightarrow D(Tx, Ty) \leq k(D(Tx, x) + D(y, Ty)).$$

Lemma 2. [9] *Let $T : X \longrightarrow X$ a G-monotone mapping and suppose there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$ (respectively, $(Tx_0, x_0) \in E(G)$) and the complete subgraph $G[\theta_T(x_0)]$ is transitive, then $\{T^n x_0\}$ is a G-increasing (respectively, G-decreasing) sequence and $(T^m x_0, T^n x_0) \in E(G)$ (respectively $(T^n x_0, T^m x_0) \in E(G)$) for any $m, n \in \mathbb{N}$ such $m \leq n$, with $G[\theta_T(x_0)]$ induced by the orbit $\theta_T(x_0)$:=$\{T^n x_0, n \in \mathbb{N}\}$.*

In 2021 (see[7]) we defined Kannan S-type tricyclic contraction and we proved the existence and uniqueness of fixed point in metric space (X, d).

Definition 5. *Let A, B and C be nonempty subsets of a metric space (X, d)*

$$T : A \cup B \cup C \longrightarrow A \cup B \cup C$$

be a Kannan-S-type tricyclic contraction, if there exists $k \in [0, \frac{1}{3})$ such that

(1) $T(A) \subseteq B, T(B) \subseteq C, T(C) \subseteq A$.

(2) $D(Tx,Ty,Tz) \leq k[d(x,Tx)+d(y,Ty)+d(z,Tz)]$ *for all* $(x,y,z) \in A \times B \times C$.

Definition 6. *The metric space* (X,d) *be a Chatterjea-S-type tricyclic contraction, if there exist* $k \in [0,\frac{1}{3})$ *such that*

(1) $T(A) \subseteq B, T(B) \subseteq C, T(C) \subseteq A$.

(2) $D(Tx,Ty,Tz) \leq k[d(y,Tx)+d(z,Ty)+d(x,Tz)]$ *for all* $(x,y,z) \in A \times B \times C$.

In this paper, we prove the existence and uniqueness of the fixed point based on the technique of graph theory in generalized metric space.

2. Main Results

Definition 7. *Let* (X,D) *a generalized metric space we define the map:*

$$D_3 : X \times X \times X \longrightarrow [0,+\infty[$$

such that

$$D_3(x,y,z) = D(x,y)+D(y,z)+D(x,z) \text{ for all } x,y,z \in X.$$

Remark 1. *For every* $x,y,z \in X$, $D_3(x,y,z) = 0 \Longrightarrow x = y = z$.

For every $x,y,z \in X$, $D_3(x,y,z) = D_3(x,z,y) = D_3(y,z,x) = ...D_3(z,y,x)$.

For every $x,y,z \in X$, $D(x,y) \leq D_3(x,y,z)$.

For every $x,y,z \in X$, $D(x,x) \leq D_3(x,y,z)$.

Definition 8. *Let a metric space* (X,D). *A mapping* $T : X \longrightarrow X$ *is said to be a Kannan-S-type contraction if there exists* $k \in [0,\frac{1}{3})$ *such that for every* $x,y,z \in X$: $D_3(Tx,Ty,Tz) \leq k[D(Tx,x)+D(y,Ty)+D(Tz,z)]$.

Definition 9. *Let a generalized metric space* (X,D) *with digraph* G. *A mapping* $T : X \longrightarrow X$ *is said to be a G-Kannan S-mapping if the following conditions are satisfied:*

(i) T *is G-monotone, that is for all* $x,y \in X$, $(x,y) \in E(G) \Longrightarrow (Tx,Ty) \in E(G)$.

(ii) *there exists* $k \in [0,\frac{1}{3})$ *such for every* $x,y,z \in V(G)$.

$$\begin{cases} (x,y) \in E(G) \\ (y,z) \in E(G) \end{cases} \Longrightarrow D_3(Tx,Ty,Tz) \leq k[D(Tx,x)+D(y,Ty)+D(Tz,z)].$$

In this work we shall adopt the following notations:

$$\begin{aligned}\delta_0 &= \delta(D_3,T,x_0)=\sup\{D_3\left(T^{i+1}x_0,T^ix_0,x_0\right): i\in\mathbb{N}\}\\ \beta &= \frac{k}{1-2k} \text{ and with } k\in[0,\frac{1}{3}).\end{aligned}$$

Example 1. *Let $X=\{0,1,2,3,4\}$. Consider the function D defined on X by $D(x,y)=(x-y)^2$. We have D is a generalized metric with constant $C\geq 3$.*

Define the mapping $T: X\longrightarrow X$ by $T(0)=T(4)=1$ and $T(1)=T(2)=T(3)=0$.

We have $D_3(T(0),T(1),T(2))=2$ and $D(0,T(0))+D(1,T(1))+D(2,T(2))=6$, then T is not Kannan-S-type contraction.

But by considering the digraph $G=(X,E)$ represented in Figure 1

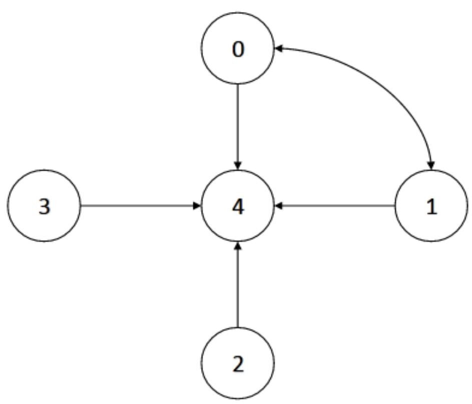

FIGURE 1. Digraph $G=(X,E)$

We have T is a G-Kannan-S-mapping with constant $k\in[\frac{2}{11},\frac{1}{3})$.

Proposition 3. *Let $T: X\longrightarrow X$ a G-Kannan S-mapping and suppose there exists $x_0\in X$ such that $(x_0,Tx_0)\in E(G)$ (respectively, $(Tx_0,x_0)\in E(G)$) and the complete subgraph $G[\theta_T(x_0)]$ is transitive then:*

(1) For every $n\geq 2$, we have:

$$D_3\left(T^nx_0,T^{n-1}x_0,T^{n-2}x_0\right)\leq\delta_0\beta^{n-2},$$

(2) For every $n,m,s\geq 2$, we have:

$$D_3\left(T^nx_0,T^mx_0,T^sx_0\right)\leq k\delta_0(\beta^{n-2}+\beta^{m-2}+\beta^{s-2}).$$

Proof. 1. For $n=2$ is trivial.

Assume that $n>2$. Since T is a G-Kannan S-mapping and $(T^{n-1}x_0, T^{n-2}x_0) \in E(G)$ and $(T^{n-2}x_0, T^{n-3}x_0) \in E(G)$ then:

$$\begin{aligned} & D_3\left(T^n x_0, T^{n-1}x_0, T^{n-2}x_0\right) \\ \leq\ & k[D(T^{n-1}x_0, T^n x_0) + D(T^{n-2}x_0, T^{n-1}x_0) + D(T^{n-2}x_0, T^{n-3}x_0)] \\ \leq\ & k[2D_3\left(T^n x_0, T^{n-1}x_0, T^{n-2}x_0\right) + D_3\left(T^{n-1}x_0, T^{n-2}x_0, T^{n-3}x_0\right)] \\ \leq\ & \frac{k}{1-2k}[D_3\left(T^{n-1}x_0, T^{n-2}x_0, T^{n-3}x_0\right)]. \end{aligned}$$

By induction on n, we prove that:

$$\begin{aligned} D_3\left(T^n x_0, T^{n-1}x_0, T^{n-2}x_0\right) &\leq (\frac{k}{1-2k})^{n-2} D_3\left(T^2x_0, Tx_0, x_0\right) \\ &\leq \delta_0 \beta^{n-2} \end{aligned}$$

2. For $n,m,s \geq 2$, Since T is a G-Kannan S-mapping, we have:

$$D_3\left(T^n x_0, T^m x_0, T^s x_0\right) \leq k[D(T^{n-1}x_0, T^n x_0) + D(T^{m-1}x_0, T^m x_0) + D(T^{s-1}x_0, T^s x_0)].$$

By using 1. We have:

$$D_3\left(T^n x_0, T^m x_0, T^s x_0\right) \leq \delta_0(\beta^{n-2} + \beta^{m-2} + \beta^{s-2}).$$

□

Theorem 4. *Let (X,D) be a generalized G-complete metric space endowed with a reflixive digraph G such that $V(G)=X$ and $T : X \longrightarrow X$ is a G-Kannan S-mapping with $k \in [0, \inf\{\frac{1}{3}, \frac{1}{C}\})$. Suppose that there exist $x_0 \in X$ such that $\delta\left(D_3, T, x_0\right) < +\infty$, $(x_0, Tx_0) \in E(\widetilde{G})$ and the subgraph $G\left[\theta_T(x_0)\right]$ is transitive, then the sequence $\{T^n x_0\}$ converge to some point $w \in X$.*

Moreover, if one of the following conditions holds:

1. T is weak continuous.

2. T is orbitally G-continuous.

3. G satisfies the property (P) and $D(x_0, Tw) < +\infty$.

Then w is the fixed point of T.

Proof. We assume that $(x_0, Tx_0) \in E(G)$. Let $(m,n,s) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ such that $2 < n < m < s$, we have $(T^m x_0, T^n x_0) \in E(G)$ and $(T^n x_0, T^s x_0) \in E(G)$.

If T is G-Kannan S-mapping, we obtain:

$$D(T^n x_0, T^m x_0) \leq D_3\left(T^n x_0, T^m x_0, T^s x_0\right) \leq 3k\delta_0(\beta^{n-2}).$$

where $\beta = \frac{k}{1-2k} \in [0,1[$. Thus $\{T^n x_0\}$ is a D- Cauchy sequence.

From the G-completeness of (X,D) the sequence $\{T^n x_0\}$ is D-converges to some $w \in X$.

1. Assume that T is weak continuous, then there exists a subsequente $\{T^{n_q} x_0\}$ such that $\{T^{n_q+1} x_0\}$ D- converges to Tw when $n_q \to +\infty$. Using the uniqueness of the limit, we get $Tw = w$.

2. Assume that T is orbitally G-continuous, since $\{T^n x_0\}$ is D-converges to w and $(T^n x_0, T^{n+1} x_0) \in E(G)$, then $T(T^n x_0) \to Tw$, therefore $Tw = w$.

3. Assume that G satisfaies the property (P) and $D(x_0, Tw) < +\infty$. Since $\{T^n x_0\}$ is a G-monotone increasing sequence which D-converges to $w \in X$, we have $(T^n x_0, w) \in E(G)$ for any $n \in \mathbb{N}$.

Let $n,m \in \mathbb{N}$ such that. If T is a G-Kannan S-mapping such that $2 < n < m$, then:

$$\begin{aligned} D_3\left(T^n x_0, T^m x_0, Tw\right) &\leq k[D(T^n x_0, T^{n-1} x_0) + D(T^m x_0, T^{m-1} x_0) + D(Tw, w)] \\ &\leq k\delta_0(\beta^{n-2} + \beta^{m-2}) + kD(Tw, w) \\ &\leq 2k\delta_0\beta^{n-2} + kC\limsup_{p\longrightarrow+\infty} D(T^p x_0, Tw). \end{aligned}$$

Taking limit superior as $n \to +\infty$ we get:

$$\begin{aligned} \limsup_{n\longrightarrow+\infty} D(T^n x_0, Tw) &\leq \limsup_{n\longrightarrow+\infty}(D_3\left(T^n x_0, T^m x_0, Tw\right)) \\ &\leq \limsup_{n\longrightarrow+\infty} 2k\delta_0\beta^{n-2} + kC\limsup_{p\longrightarrow+\infty} D(T^p x_0, Tw). \end{aligned}$$

Thus:

$$(1-kC)\limsup_{n\longrightarrow+\infty} D(T^n x_0, Tw) \leq \limsup_{n\longrightarrow+\infty} 2k\delta_0\beta^{n-2}.$$

Since $k < \inf\{\frac{1}{3}, \frac{1}{C}\}$ and $\beta \in [0,1[$ then $D(T^n x_0, Tw) \to 0$. So $\{T^n x_0\}$ D-converges to Tw. By the uniqueness of the limit we get $Tw = w$. $\square$

Proposition 5. *Let (X,D) a generalized metric space, and suppose that T is a G-kannan S mapping. If $w \in X$ is a fixed point of T satisfying $D_3(w,w,w) < +\infty$ then $D_3(w,w,w) = 0$.*

Proof. Let $w \in X$ be a fixed point of T such that $D_3(w,w,w) < +\infty$

since $(w,w) \in E(G)$ we have:

$$\begin{aligned} D_3(w,w,w) &= D_3(Tw,Tw,Tw) \\ &\leq k(D(Tw,w)+D(Tw,w)+D(Tw,w)), \\ &\leq 3kD_3(w,w,w), \end{aligned}$$

which implies $(1-3k)\,D_3(w,w,w) \leq 0$.Thus, $D_3(w,w,w) = 0$. □

Proposition 6. *Let (X,D) be a generalized metric space, and suppose that T is a G-kannan S mapping. If T has three fixed points w_1,w_2,w_3 in X such that $D(w_1,w_2,w_3) < +\infty$, then $w_1 = w_2 = w_3$.*

Proof. Suppose that w_1,w_2,w_3 in X are three fixed points of T then:

$$\begin{aligned} D_3(w_1,w_2,w_3) &= D_3(Tw_1,Tw_2,Tw_3) \\ &\leq k\,[D(Tw_1,w_1)+D(Tw_2,w_2)+D(Tw_3,w_3)] \\ &= k\,[D(w_1,w_1)+D(w_2,w_2)+D(w_3,w_3)] \\ &\leq k[D_3(w_1,w_1,w_1)+D_3(w_2,w_2,w_2)+D_3(w_3,w_3,w_3)] = 0. \end{aligned}$$

Thus $D_3(w_1,w_2,w_3) = 0$, then $w_1 = w_2 = w_3$. □

Example 2. *Let $X = [0,1]$.Consider the generalized distance function D defined on X by $D(x,y) = (x-y)^2$ and the self mapping T on X defined by:*

$$Tx = \begin{cases} \frac{x}{6} \; si \; x \in \{0\} \cup \{\frac{1}{6^n}, n \in \mathbb{N}\}, \\ \frac{1}{2} \qquad\qquad otherwise. \end{cases}$$

Consider the graph G on X consisting of the transitive closure of the graph represented in Figure 2

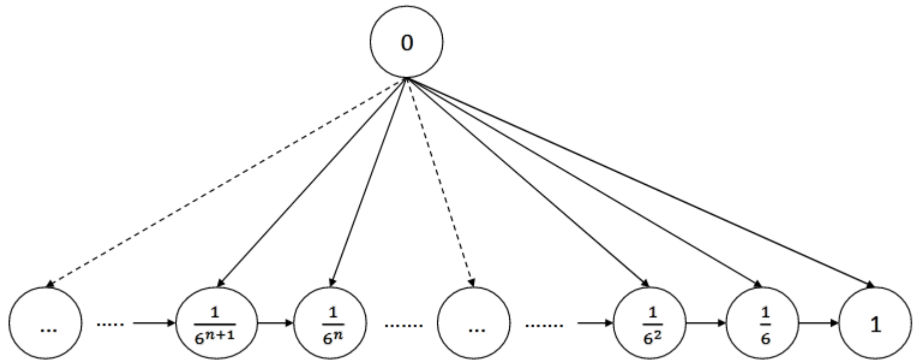

FIGURE 2. Digraph $G=(X,E)$

We have:

$$E(G)=\Delta\cup\{(0,\frac{1}{6^n}):n\in\mathbb{N}\}\cup\{(\frac{1}{6^n},\frac{1}{6^m}):n,m\in\mathbb{N}\text{ and }n\geq m\}$$

and

$$(T0,T\tfrac{1}{6^n})=(0,\tfrac{1}{6^{n+1}})\in E(G)\text{ for any }n\in\mathbb{N},$$
$$(T\tfrac{1}{6^n},T\tfrac{1}{6^m})=(\tfrac{1}{6^{n+1}},\tfrac{1}{6^{m+1}})\in E(G)\text{ for any }n,m\in\mathbb{N},$$

then T is G-monotone.

For $x_0=1$,we have $(Tx_0,x_0)\in E(G)$, $G[\theta_T(x_0)]$ is transitive and

$\delta(D_3,T,x_0)=\sup\{D_3\left(T^{i+1}x_0,T^ix_0,x_0\right):i\in\mathbb{N}\}=3<\infty$.

Let $x,y,z\in X$ such that $(x,y)\in E(G)$, and $(y,z)\in E(G)$.

If $x=y=z$ then:

$$D_3(Tx,Tx,Tx)=0\leq k(D(Tx,x)+D(Ty,y)+D(Tz,z)).$$

If $(x,y,z)=(0,0,\frac{1}{6^n})$ then:

$$\begin{aligned} D_3\left(T0,T0,T\frac{1}{6^n}\right) &= \frac{2}{6^{2(n+1)}}\\ &\leq \frac{25k}{6^{2(n+1)}}\\ &= k(D(T0,0)+D(T0,0)+D(T\frac{1}{6^n},\frac{1}{6^n})). \end{aligned}$$

If $(x,y,z) = (0, \frac{1}{6^n}, \frac{1}{6^m})$ *then:*

$$\begin{aligned}
D_3\left(T0, T\frac{1}{6^n}, T\frac{1}{6^m}\right) &= \frac{2}{6^{2(n+1)}} + \frac{2}{6^{2(m+1)}} + \frac{(6^n-6^m)^2}{6^{2(n+m+1)}} \\
&\leq \frac{2}{6^{2(n+1)}} + \frac{2}{6^{2(m+1)}} + \frac{6^{2n}}{6^{2(n+m+1)}} \\
&= \frac{2}{6^{2(n+1)}} + \frac{3}{6^{2(m+1)}} \\
&\leq 25k(\frac{1}{6^{2(n+1)}} + \frac{1}{6^{2(m+1)}}) \text{ with } k \leq \frac{2}{25} \\
&= k(D(T0,0) + D(T\frac{1}{6^n}, \frac{1}{6^n}) + D(T\frac{1}{6^m}, \frac{1}{6^m})).
\end{aligned}$$

If $(x,y,z) = (\frac{1}{6^n}, \frac{1}{6^m}, \frac{1}{6^s})$ *with* $s \geq m \geq n$ *then:*

$$\begin{aligned}
D_3(Tx,Ty,Tz) &= D_3\left(T\frac{1}{6^n}, T\frac{1}{6^m}, T\frac{1}{6^s}\right) \\
&= \frac{(6^n-6^m)^2}{6^{2(n+m+1)}} + \frac{(6^n-6^s)^2}{6^{2(n+s+1)}} + \frac{(6^m-6^s)^2}{6^{2(m+s+1)}} \\
&= \frac{6^{2n}-2(6^{n+m})+6^{2m}}{6^{2(n+m+1)}} + \frac{6^{2n}-2(6^{n+s})+6^{2s}}{6^{2(n+s+1)}} + \frac{6^{2m}-2(6^{m+s})+6^{2s}}{6^{2(m+s+1)}} \\
&\leq \frac{6^{2n}+6^{2m}}{6^{2(n+m+1)}} + \frac{6^{2n}+6^{2s}}{6^{2(n+s+1)}} + \frac{6^{2m}+6^{2s}}{6^{2(m+s+1)}} \\
&= \frac{6^{2n}}{6^{2(n+m+1)}} + \frac{6^{2m}}{6^{2(n+m+1)}} + \frac{6^{2n}}{6^{2(n+s+1)}} + \frac{6^{2s}}{6^{2(n+s+1)}} + \frac{6^{2m}}{6^{2(m+s+1)}} + \frac{6^{2s}}{6^{2(m+s+1)}} \\
&= \frac{2}{6^{2(m+1)}} + \frac{2}{6^{2(n+1)}} + \frac{2}{6^{2(s+1)}} \\
&\leq 25k(\frac{1}{6^{2(n+1)}} + \frac{1}{6^{2(m+1)}} + \frac{1}{6^{2(s+1)}}) \text{ with } k \leq \frac{2}{25} \\
&= k(D(T\frac{1}{6^n}, \frac{1}{6^n}) + D(T\frac{1}{6^m}, \frac{1}{6^m}) + D(T\frac{1}{6^s}, \frac{1}{6^s})).
\end{aligned}$$

Then, for all $x,y,z \in X$ *such that* $(x,y) \in E(G)$*,and* $(y,z) \in E(G)$*.We have:*

$$D_3(Tx,Ty,Tz) \leq k(D(Tx,x) + D(Ty,y) + D(Tz,z)) \text{ with } k \in [\frac{2}{25}, \frac{1}{3}[.$$

Then T *is a G-kannan S mapping, The sequence* $\{T^n x_0\} = \{\frac{1}{6^n}\}$ *is G-decreasing, D-convergent to* 0 *and* $(0, \frac{1}{6^n}) \in E(G)$*, then G has the* (P) *Property. Thus implies that* T *has a fixed point* 0*.*

Now we define a G-Chatterjea S-mapping in dislocated metric space (X,D) as follows:

Definition 10. *Let a dislocated metric space (X,D) with the digraph G. A mapping $T:X\longrightarrow X$ is said to be a G-Chatterjea S-mapping if the following conditions are satisfied:*

(i) *T is G-monotone, that is for all $x,y\in X$, $(x,y)\in E(G)\Longrightarrow (Tx,Ty)\in E(G)$,*

(ii) *There exists $k\in[0,\frac{1}{3})$ such for every $x,y,z\in V(G)$:*

$$\begin{cases} (x,y)\in E(G) \\ (y,z)\in E(G) \end{cases} \Longrightarrow D_3(Tx,Ty,Tz)\leq k[D(Tx,y)+D(Ty,z)+D(Tz,x)].$$

Lemma 7. *Let (X,D) be a dislocated metric space with digraph G, $T:X\longrightarrow X$ is a G-Chatterjea S-mapping and suppose there exists $x_0\in X$ such that $(x_0,Tx_0)\in E(G)$ (respectively, $(Tx_0,x_0)\in E(G)$) and the complete subgraph $G[\theta_T(x_0)]$ is transitive then:*

(1) For every $n\geq 2$, we have:

$$D_3\left(T^n x_0,T^{n-1}x_0,T^{n-2}x_0\right)\leq \delta_0\beta^{n-2},$$

(2) For every for $s>m>n\geq 2$, we have:

$$D_3\left(T^n x_0,T^m x_0,T^s x_0\right)\leq \delta_0 3k(\frac{\beta^{n-1}}{1-\beta})$$

Proof. 1. For $n=2$ is trivial.

Assume that $n>2$. Since T is a G-Chatterjea S-mapping and $\left(T^{n-1}x_0,T^{n-2}x_0\right)$, $\left(T^n x_0,T^{n-2}x_0\right)$, $\left(T^{n-1}x_0,T^{n-3}x_0\right)\in E(G)$, then:

$$\begin{aligned}
& D_3\left(T^n x_0,T^{n-1}x_0,T^{n-2}x_0\right) \\
\leq\ & k[D(T^n x_0,T^{n-2}x_0)+D(T^{n-1}x_0,T^{n-3}x_0)+D(T^{n-2}x_0,T^{n-1}x_0)] \\
\leq\ & k[2D_3\left(T^n x_0,T^{n-1}x_0,T^{n-2}x_0\right)+D_3\left(T^{n-1}x_0,T^{n-2}x_0,T^{n-3}x_0\right)].
\end{aligned}$$

Thus is

$$D_3\left(T^n x_0,T^{n-1}x_0,T^{n-2}x_0\right)\leq \frac{k}{1-2k}(D(T^{n-1}x_0,T^{n-2}x_0,T^{n-3}x_0).$$

By induction on n,we prove that

$$\begin{aligned}
D_3\left(T^n x_0,T^{n-1}x_0,T^{n-2}x_0\right) &\leq (\frac{k}{1-2k})^{n-2}D_3\left(T^2x_0,Tx_0,x_0\right) \\
&\leq \delta_0\beta^{n-2}
\end{aligned}$$

2. for $s \geq m \geq n > 2$, we have

$$\begin{aligned}
& D_3\left(T^n x_0, T^m x_0, T^s x_0\right) \\
\leq\ & k[D(T^n x_0, T^{m-1} x_0) + D(T^m x_0, T^{s-1} x_0) + D(T^s x_0, T^{n-1} x_0)]. \\
\leq\ & k\left[\begin{array}{c} \sum_{i=n-1}^{i=m-2} D_3\left(T^i x_0, T^{i+1} x_0, T^{i+2} x_0\right) + \sum_{i=m-1}^{i=s-2} D_3\left(T^i x_0, T^{i+1} x_0, T^{i+2} x_0\right) + \\ \sum_{i=n}^{i=s-3} D_3\left(T^i x_0, T^{i+1} x_0, T^{i+2} x_0\right) \end{array}\right]
\end{aligned}$$

By using 1. We have:

$$\begin{aligned}
D(T^n x_0, T^m x_0) &\leq D_3\left(T^n x_0, T^m x_0, T^s x_0\right) \\
&\leq \delta_0 k\left(\frac{\beta^{n-1} - \beta^m}{1-\beta} + \frac{\beta^{m-1} - \beta^s}{1-\beta} + \frac{\beta^n - \beta^{s-2}}{1-\beta}\right) \\
&\leq \delta_0 k\left(\frac{\beta^{n-1}}{1-\beta} + \frac{\beta^{m-1}}{1-\beta} + \frac{\beta^n}{1-\beta}\right). \\
&\leq \delta_0 k\left(\frac{\beta^{m-1}}{1-\beta} + \frac{2\beta^{n-1}}{1-\beta}\right) \\
&\leq \delta_0 3k\left(\frac{\beta^{n-1}}{1-\beta}\right)
\end{aligned}$$

□

Theorem 8. *Let a dislocated (X,D) G-complete metric space endowed with a reflixive digraph G such that $V(G) = X$ and $T : X \longrightarrow X$ is a G-Chatterjea S-mapping. Suppose that there exist $x_0 \in X$ such that $\delta\left(D_3, T, x_0\right) < +\infty$, $(x_0, T_{X_0}) \in E(\widetilde{G})$ and the subgraph $G\left[\theta_T(x_0)\right]$ is transitive, then the sequence $\{T^n x_0\}$ converge to some point $w \in X$.*

Moreover, if one of the following conditions holds

1. T is weak continuous.

2. T is orbitally G-continuous.

3. G satisfies the Property (P) and $D(x, Tw) < \infty$.

Then w is a fixed point of T.

Proof. We assume that $(x_0, T_{X_0}) \in E(\widetilde{G})$. Let $(m,n,s) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that, $s \geq m \geq n > 2$. We have $(T^m x_0, T^n x_0) \in E(G)$ and $(T^n x_0, T^s x_0) \in E(G)$.

If T is G-Chatterjea S-mapping, by lemma, we get:

$$D_3\left(T^n x_0, T^m x_0, T^s x_0\right) \leq \delta_0 3k\left(\frac{\beta^{n-1}}{1-\beta}\right).$$

For every $(m,n,s) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ such that $s \geq m \geq n > 2$, where $\beta = \frac{k}{1-2k} \in [0,1[$. Thus $\{T^n x_0\}$ is D-Cauchy sequence.

From the completeness of (X,D) the sequence $\{T^n x_0\}$ D-converge to some point $w \in X$.

1. Assume that T is weak continuous, then there exists a subsequente $\{T^{n_q} x_0\}$ such that $\{T^{n_q+1} x_0\}$ D-converges to Tw when $n_q \to +\infty$. using the uniqueness of the limit, we get $Tw = w$.

2. Assume that T is orbitally G-continuous, since $\{T^n x_0\}$ is D- converges to w and $(T^n x_0, T^{n+1} x_0) \in E(G)$, then $T(T^n x_0) \to Tw$.

3. Assume that G satisfaies the property (P) and $D(x_0, Tw) < +\infty$. Since $\{T^n x_0\}$ is a G-monotone increasing sequence which D-converges to $w \in X$, we have $(T^n x_0, w) \in E(G)$ for any $n \in \mathbb{N}$.

If T is a G-Chatterjea S-mapping such that $m \geq n > 2$, then:

$$\begin{aligned} D_3\left(T^n x_0, T^m x_0, Tw\right) &\leq k[D(T^{n-1} x_0, T^m x_0) + D(T^{m-1} x_0, Tw) + D(T^n x_0, w)] \\ &\leq k[\delta_0\left(\frac{\beta^{n-1}}{1-\beta}\right) + D(T^{m-1} x_0, Tw) + D(T^n x_0, w)] \end{aligned}$$

Since $\{T^n x_0\}$ D-converge to w we have $D_3\left(T^n x_0, T^m x_0, Tw\right) \leq k\delta_0\left(\frac{\beta^{n-1}}{1-\beta}\right)$.

Taking limit superior as $m \to +\infty$, then $\{T^n x_0\}$ D-converges to Tw by the uniqueness of the limit we get $Tw = w$. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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