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FIXED POINT THEOREMS FOR GENERALIZED $\tau-\psi-$CONTRACTION MAPPINGS IN RECTANGULAR QUASI b-METRIC SPACES

ADIL BAIZ*, JAMAL MOULINE, ABDELKARIM KARI

Laboratory of Analysis, Modeling and Simulation, Faculty of Sciences Ben M'Sik, Hassan II University,

B.P. 7955 Casablanca, Morocco

Abstract. In the last few decades, a lot of generalizations of The Banach contraction principle had been introduced. Recently, Piri et al. gave an interesting generalization of this principle in the framework of generalized quasi b-metric spaces. In this paper, we present the notion of $\tau-\psi-$contraction and $\tau-\psi-$contraction in generalized quasi b-metric spaces to study the existence and uniqueness of fixed point for them. We will also provide some illustrative examples. Our results improve many existing results.

Keywords: fixed point; quasi b-metric spaces; $\tau-\psi-$contraction.

2020 AMS Subject Classification: 47H10, 47H09.

1. INTRODUCTION

The problem of the existence of the solution of many mathematical models is equivalent to the existence of a fixed point problem for a certain map. The study of fixed points is, therefore, has a central role in many disciplines of applied sciences. The most essential and key part of the theory of fixed points is the existence of the solution of operator equations satisfying certain conditions, for example, Fredholm integral equations, Voltera integral equations, two point boundary value problems in differential equations as well as some eigenvalue problems.

*Corresponding author
E-mail address: baizadil@gmail.com
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A beautiful blend of analysis, topology and geometry has laid down the foundation of the theory of fixed points.

The Banach contraction principle [1] has become a powerful tool in modern analysis and it is an important tool for solving existence problems in mathematics and physics. Many authors have established the theory of fixed points particularly in two directions. One by stating the conditions on the mapping T and second, taking the set X as a more general structure [5, 11].

Many generalizations of the concept of metric spaces are defined and some fixed point theorems were proved in these spaces. In particular, quasi b-metric spaces were introduced by Wilson [14] as a generalization of metric spaces. Many mathematicians worked on this interesting space. For more, the reader can refer to [8, 10].

A. Branciari in [2] initiated the notions of a generalized metric space as a generalization of a metric space, where the triangular inequality of metric spaces replaced by $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$ (quadrilateral inequality). Various fixed point results were established on such spaces, see [1, 3, 6, 7, 12, 13, 16, 17, 18, 19, 20, 21] and references therein.

Combining conditions used for definitions of asymmetric metric and generalized metric spaces, Piri et al [9] announced the notions of generalized quasi b-metric space.

In this paper, we introduce the notion of $\tau - \psi-$contraction and $\tau - \psi-$contraction and establish some new fixed point theorems for mappings in the setting of complete generalized asymmetric metric spaces. Our result generalizes, improve and extend the corresponding results due to Kannan and Reich. Moreover, an illustrative examples is presented to support the obtained results.

2. PRELIMINARIES

Preliminaries. In what follows, we recall basic definition and results on the topics for the sake of completeness.

Notation We need the following symbols and class of functions to prove certain results of this section: $\mathbb{R}$ is the set of all real numbers. N is the set of all natural numbers. $\Psi = \{\psi : \mathbb{R}^+ \to \mathbb{R}^+$, such that, ψ is non-decreasing, continuous,$\sum_{k=1}^{\infty} s^k \psi^k(t) < \infty$, $s\psi(t) < t$ for $t > 0$ and $\psi(0) = 0$ if and only if $t = 0$, where ψ^k is the kth iterate of ψ and $s \geq 1\}$.

Definition 2.1. [2] Let X be a non-empty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric if and only if for all $x,y,z \in X$, the following conditions are satisfied:

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all distinct points $x,y \in X$;

(iii) $d(x,z) \leq s(d(x,y) + d(y,z))$.

Then (X,d) is called a b-metric space.

It should be noted that the class of b-metric spaces is effectively larger than that of metric spaces since a b-metric is a metric when $s = 1$.

Definition 2.2. [2] Let X be a non-empty set and $d : X \times X \to \mathbb{R}^+$ be a mapping such that for all $x,y \in X$ and for all distinct points $u,v \in X$, each of them different from x and y,

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all distinct points $x,y \in X$;

(iii) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$. (quadrilateral inequality)

Then (X,d) is called a rectangular metric space.

Definition 2.3. [2] Let X be a non-empty set,$s \geq 1$ be a given real number,

and $d : X \times X \to \mathbb{R}^+$ be a mapping such that for all $x,y \in X$ and for all distinct points $u,v \in X$, each of them different from x and y,

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all distinct points $x,y \in X$;

(iii) $d(x,y) \leq s(d(x,u) + d(u,v) + d(v,y))$. (quadrilateral inequality)

Then (X,d) is called a rectangular b-metric space.

Note: Every metric space is a rectangular metric space, and every rectangular metric space is a rectangular b-metric space with coefficient $s = 1$. It is evident that any rectangular metric space is a rectangular b-metric space, but the converse is not true in general. We give an example to show that not every rectangular b-metric space is a rectangular metric space.

Example 2.4. [2] Let $X = N, \alpha \geq 0$ and $d : X \times X \to \mathbb{R}^+$ such that:

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all distinct points $x,y \in X$;

(iii) $d(x,y) = 4\alpha$,.if $x,y \in \{1,2\}$ and $x \neq y$

(iv) $d(x,y) = \alpha$,.if $x,y \notin \{1,2\}$ and $x \neq y$

Then (X,d) is called a rectangular b-metric space with coefficient $s = \frac{4}{3} > 1$.but (X,d) is not rectangular metric space, as $d(1,2) = 4\alpha \nleq 3\alpha = d(1,3) + d(3,4) + d(4,2)$.

The following is the definition of the notion of rectangular quasi metric space.

Definition 2.5. [9] Let X be a non-empty set and $d : X \times X \to \mathbb{R}^+$ be a mapping such that for all $x,y \in X$ and for all distinct points $u,v \in X$, each of them different from x and y,

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$. (quadrilateral inequality)

Then (X,d) is called a rectangular quasi metric space.

Note: Any rectangular metric space is a rectangular quasi metric space, but the converse is not true in general.

We give an example to show that not every rectangular quasi metric on a set X is a rectangular metric space on X.

Example 2.6. Let $X = \{t, 2t, 3t, 4t, 5t\}$,with $t > 0$ as a constant,$\alpha > 0$ and define

$d : X \times X \to \mathbb{R}^+$ by:

$d(x,x) = 0$ for all $x \in X$;

$d(t,2t) = d(2t,t) = 3\alpha$;

$d(t,3t) = d(2t,3t) = d(3t,t) = d(3t,2t) = \alpha$,;

$d(t,4t) = d(2t,4t) = d(3t,4t) = d(4t,t) = d(4t,2t) = d(4t,3t) = 2\alpha$,;

$d(t,5t) = d(2t,5t) = d(3t,5t) = d(4t,5t) = \frac{3}{2}\alpha$,;

$d(5t,t) = d(5t,2t) = d(5t,3t) = d(5t,4t) = \frac{5}{4}\alpha$,;

Then (X,d) is a rectangular quasi metric space, but for the fact that

$d(t,5t) = \frac{3}{2}\alpha \nleq \frac{5}{4}\alpha = d(5t,t)$.

but (X,d) is not a rectangular metric space.

Definition 2.7. [9]. Let (X,d) is a rectangular quasi metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X, and $x\in X$. Then

(i) We say that $\{x_n\}_{n\in\mathbb{N}}$ forward (backward) converges to x if and only if

$$\lim_{n\to+\infty} d(x,x_n) = \lim_{n\to+\infty} d(x_n,x) = 0.$$

(ii) We say that $\{x_n\}_{n\in\mathbb{N}}$ forward (backward) Cauchy if

$$\lim_{n,m\to+\infty} d(x_n,x_m) = \lim_{n,m\to+\infty} d(x_m,x_n) = 0.$$

(iii) (X,d) is called complete rectangular quasi metric space if every Cauchy sequence in X converges to some $x\in X$.

Lemma 2.8. [9]. *Let (X,d) be a rectangular quasi metric space and $\{x_n\}_n$ be a forward (or backward) Cauchy sequence with pairwise disjoint elements in X. If $\{x_n\}_n$ forward converges to $x\in X$ and backward converges to $y\in X$, then $x = y$.*

Definition 2.9. [9]. Let (X,d) be a rectangular quasi metric space. X is said to be complete if X is forward and backward complete.

3. Main Results

In this section, we introduce rectangular quasi b-metric spaces, define generalized $\tau-\psi-$contraction mappings, and study fixed point results for the mappings introduced in the setting of rectangular quasi b-metric spaces. We start by introducing the notion of a rectangular quasi b-metric space as follows:

Definition 3.1. Let X be a non-empty set,$s\geq 1$ be a given real number, and $d: X\times X\to\mathbb{R}^+$ be a mapping such that for all $x,y\in X$ and for all distinct points $u,v\in X$, each of them different from x and y,

(i) $d(x,y)=0$ if and only if $x=y$;

(ii) $d(x,y)\leq s(d(x,u)+d(u,v)+d(v,y))$ (quadrilateral inequality).

Then (X,d) is called a rectangular quasi b-metric space.

Now, we give an example of a rectangular quasi b-metric space.

Example 3.2. Let $X = A\bigcup B$ where $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ and $B = [1,2]$.Define the generalized metric d on X as follows
$d(\frac{1}{2},\frac{1}{3}) = d(\frac{1}{4},\frac{1}{5}) = 0.3; d(\frac{1}{3},\frac{1}{2}) = d(\frac{1}{5},\frac{1}{4}) = d(\frac{1}{3},\frac{1}{4}) = 0.1;$
$d(\frac{1}{2},\frac{1}{4}) = d(\frac{1}{3},\frac{1}{5}) = 0.6; d(\frac{1}{4},\frac{1}{2}) = d(\frac{1}{5},\frac{1}{3}) = 0.4;$
$d(\frac{1}{2},\frac{1}{5}) = 1.05; d(\frac{1}{5},\frac{1}{2}) = d(\frac{1}{4},\frac{1}{3}) = 0.5;$
$d(\frac{1}{2},\frac{1}{2}) = d(\frac{1}{3},\frac{1}{3}) = d(\frac{1}{4},\frac{1}{4}) = d(\frac{1}{5},\frac{1}{5}) = 0;$
and $d(x,y) = |x-y|$ if $x,y \in B$ or $x \in B, y \in A$ Then (X,d) is a rectangular quasi b-metric space with coefficient $s = \frac{3}{2} \geq 1$. Indeed Condition (i)in Definition 7 trivially holds. Now, we show condition (ii) in Definition 7 holds: Case (i) If $x,y \in A$, then
$d(x,y) = d(\frac{1}{2},\frac{1}{3}) = 0.3 \leq s[d(\frac{1}{2},u)+d(u,v)+d(v,\frac{1}{3})]$ when $u,v \in \{\frac{1}{4},\frac{1}{5}\}$.
$d(x,y) = d(\frac{1}{3},\frac{1}{2}) = 0.1 \leq s[d(\frac{1}{3},u)+d(u,v)+d(v,\frac{1}{2})]$ when $u,v \in \{\frac{1}{4},\frac{1}{5}\}$.
$d(x,y) = d(\frac{1}{3},\frac{1}{4}) = 0.1 \leq s[d(\frac{1}{3},u)+d(u,v)+d(v,\frac{1}{4})]$ when $u,v \in \{\frac{1}{2},\frac{1}{5}\}$.
$d(x,y) = d(\frac{1}{4},\frac{1}{3}) = 0.5 \leq s[d(\frac{1}{4},u)+d(u,v)+d(v,\frac{1}{3})]$ when $u,v \in \{\frac{1}{2},\frac{1}{5}\}$.
$d(x,y) = d(\frac{1}{4},\frac{1}{5}) = 0.3 \leq s[d(\frac{1}{4},u)+d(u,v)+d(v,\frac{1}{5})]$ when $u,v \in \{\frac{1}{2},\frac{1}{3}\}$.
$d(x,y) = d(\frac{1}{5},\frac{1}{4}) = 0.1 \leq s[d(\frac{1}{5},u)+d(u,v)+d(v,\frac{1}{4})]$ when $u,v \in \{\frac{1}{2},\frac{1}{3}\}$.
$d(x,y) = d(\frac{1}{2},\frac{1}{4}) = 0.6 \leq s[d(\frac{1}{2},u)+d(u,v)+d(v,\frac{1}{4})]$ when $u,v \in \{\frac{1}{3},\frac{1}{5}\}$.
$d(x,y) = d(\frac{1}{4},\frac{1}{2}) = 0.4 \leq s[d(\frac{1}{4},u)+d(u,v)+d(v,\frac{1}{2})]$ when $u,v \in \{\frac{1}{3},\frac{1}{5}\}$.
$d(x,y) = d(\frac{1}{2},\frac{1}{5}) = 1.05 \leq s[d(\frac{1}{2},u)+d(u,v)+d(v,\frac{1}{5})]$ when $u,v \in \{\frac{1}{3},\frac{1}{4}\}$.
$d(x,y) = d(\frac{1}{5},\frac{1}{2}) = 0.5 \leq s[d(\frac{1}{5},u)+d(u,v)+d(v,\frac{1}{2})]$ when $u,v \in \{\frac{1}{3},\frac{1}{4}\}$.
$d(x,y) = d(\frac{1}{3},\frac{1}{5}) = 0.6 \leq s[d(\frac{1}{3},u)+d(u,v)+d(v,\frac{1}{5})]$ when $u,v \in \{\frac{1}{2},\frac{1}{4}\}$.
$d(x,y) = d(\frac{1}{5},\frac{1}{3}) = 0.6 \leq s[d(\frac{1}{5},u)+d(u,v)+d(v,\frac{1}{3})]$ when $u,v \in \{\frac{1}{2},\frac{1}{4}\}$.
Case (ii) If $x,y \in B$ or $x \in A, y \in B$ or $x \in B, y \in A$, then $d(x,y) = |x-y| \leq s|x-u|+|u-v|+|v-y|$ for all distinct points $u,v \in X, \{x,y\}$.

But (X,d)is neither a metric space, a rectangular metric space nor a rectangular quasi metric space because the triangle inequality, symmetry, and rectangular inequality fail respectively as follows:
$d(\frac{1}{2},\frac{1}{4}) = 0.6 \nleq 0.4 = d(\frac{1}{2},\frac{1}{3}) + d(\frac{1}{3},\frac{1}{4}) = 0.3+0.1,$

$d(\frac{1}{2},\frac{1}{4}) = 0.6 \neq 0.4 = d(\frac{1}{4},\frac{1}{2})$, and
$d(\frac{1}{2},\frac{1}{5}) = 1.05 \nleq 0.7 = d(\frac{1}{2},\frac{1}{3}) + d(\frac{1}{3},\frac{1}{4}) + d(\frac{1}{4},\frac{1}{5})$.

We next give the definitions of rectangular quasi b-convergence of a sequence and completeness of rectangular quasi b-metric spaces.

Definition 3.3. Let (X,d) is a rectangular quasi b-metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X, and $x \in X$. Then

(i) We say that $\{x_n\}_{n\in\mathbb{N}}$ forward (backward) converges to x if and only if

$$\lim_{n\to+\infty} d(x,x_n) = \lim_{n\to+\infty} d(x_n,x) = 0.$$

(ii) We say that $\{x_n\}_{n\in\mathbb{N}}$ forward (backward) Cauchy if

$$\lim_{n,m\to+\infty} d(x_n,x_m) = \lim_{n,m\to+\infty} d(x_m,x_n) = 0.$$

(iii) (X,d) is called complete rectangular quasi b-metric space if every Cauchy sequence in X converges to some $x \in X$.

Remark 3.4. Let (X,d) be a rectangular quasi b-metric space. For $x \in X$, we define the open ball with center x and radius $r > 0$ by $B_r(x,y) = \{y \in X : max\{d(x,y),d(y,x)\} < r\}$.

In general, an open ball in a rectangular metric space need not be an open set. A rectangular quasi b-metric space need not be continuous. A convergent sequence in a rectangular quasi b-metric space need not be a Cauchy. A rectangular quasi b-metric space need not be a Hausdorff, and hence the uniqueness of limits cannot be guaranteed. Now, we give an example to support Remark.

Example 3.5. Let $X = A \cup B$, where $A = \{\frac{1}{n}, n \in \mathbb{N}$, and $B = \{0,3\}$. Define the function $d : X \times X \to \mathbb{R}^+$ such that

$$d(x,y) = \begin{cases} 0 & \text{if} \quad x = y; \\ \frac{9}{2} & \text{if} \quad x,y \in A; \\ \frac{1}{n} & \text{if} \quad x \in A, y \in B; \\ 2 & \text{if} \quad x,y \in B; \end{cases}$$

The function d is a rectangular quasi b-metric space with $s = 2$. But d is neither a rectangular quasi metric nor a rectangular b-metric space because

$$d(\frac{1}{2},\frac{1}{3}) = \frac{9}{2} \not\leq \frac{11}{4} = d(\frac{1}{2},0)+d(0,3)+d(3,\frac{1}{3})$$

and

$$d(\frac{1}{3},0) = \frac{1}{3} \neq \frac{1}{4} = d(0,\frac{1}{3})$$

It is also clear that

$$\lim_{n\to\infty} d\left(\frac{1}{2n},0\right) = \lim_{n\to\infty}\frac{1}{n} = 0 = \lim_{n\to\infty} d\left(0,\frac{1}{2n}\right) = \lim_{n\to\infty}\frac{1}{n+1}$$

and

$$\lim_{n\to\infty} d(\frac{1}{2n},3) = \lim_{n\to\infty}\frac{1}{n} = 0 = \lim_{n\to\infty} d(3,\frac{1}{2n}) = \lim_{n\to\infty}\frac{1}{n+1})$$

that is, the sequence $\{\frac{1}{2n}\}$ has two different limits the numbers 0 and 3.
In addition, the sequence $\{\frac{1}{2n}\}$ is rectangular quasi b-convergent, but not a rectangular quasi b-Cauchy sequence, because

$$\lim_{n\to\infty} d(x_n,x_{n+p}) = \lim_{n\to\infty} d(\frac{1}{2n},\frac{1}{2n+p}) = \frac{9}{2} = \lim_{n\to\infty} d(x_{n+p},x_n) = \lim_{n\to\infty} d(\frac{1}{2n+p},\frac{1}{2n}).$$

In the following, we define an (τ,ψ)-contraction mapping in the setting of rectangular quasi b-metric space.

Definition 3.6. Let (X,d) be a rectangular quasi b-metric space and $T : X \to X$ be a given mapping. We say that T is a generalized (τ,ψ)-contraction mapping if there exist a function $\psi \in \Psi$ and $\tau > 1$ such that $\tau d(Tx,Ty) \leq \psi(M(x,y))$ for all $x,y \in X$,
where

$$M(x,y) = max\{d(x,y), \frac{d(x,Tx)d(x,ty)}{1+d(x,Ty)+d(y,Tx)}, d(x,Tx), d(y,Ty)\}$$

Now, we state and prove the following fixed point theorem.

Theorem 3.7. *Let (X,d) be an complete rectangular quasi b-metric space and $T : X \to X$ be generalized τ,ψ-contraction mapping. Suppose that*

(i) *there exist $x_n \in X$ such that $x_{n+1} = Tx_n = T^{n+1}x_0$, for all $n \geq 0$,*

(ii) *T is continuous Then T has a unique fixed point.*

Proof. Step 1: We show that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0 = \lim_{n\to\infty} d(x_{n+1},x_n)$$

and

$$\lim_{n\to\infty} d(x_n,x_{n+2}) = 0 = \lim_{n\to\infty} d(x_{n+2},x_n).$$

Regarding (1), we have

$d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n) \leq \tau d(Tx_{n-1},Tx_n) \leq \psi(M(x_{n-1},x_n))$ for all $n \leq 1$,

where

$$\begin{aligned} M(x_{n-1},x_n) &= max\{d(x_{n-1},x_n), \frac{d(x_{n-1},Tx_{n-1})d(x_{n-1},Tx_n)}{1+d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})}, d(x_{n-1},Tx_{n-1}), d(x_n,Tx_{n+1})\} \\ &= max\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)d(x_{n-1},x_{n+1})}{1+d(x_{n-1},x_{n+1})+d(x_n,x_n)}, d(x_{n-1},x_n), d(x_n,x_{n+1})\} \\ &= max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\} \end{aligned}$$

If $M(x_{n-1},x_n) = d(x_n,x_{n+1})$, we get

$$d(x_n,x_{n+1}) \leq \psi d(x_n,x_{n+1}) \leq \psi d(x_n,x_{n+1}) < d(x_n,x_{n+1}),$$

which is a contradiction. Hence, $M(x_{n-1},x_n) = d(x_{n-1},x_n)$

$$\begin{aligned} d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n) \\ &\leq \psi(d(x_{n-1},x_n)) \\ &= \psi(d(Tx_{n-2},Tx_{n-1})) \\ &\leq \psi^2(d(x_{n-2},x_{n-1})) \\ &\leq .. \\ &.. \leq \psi^n(d(x_0,x_1)) \longrightarrow 0 \quad as \quad n \longrightarrow \infty, \end{aligned}$$

Then

$$d(x_n,x_{n+1}) \longrightarrow 0 \quad as \quad n \longrightarrow \infty, \tag{3.1}$$

Also,

$$d(x_{n+1},x_n) = d(Tx_n,Tx_{n-1}) \leq \tau(d(Tx_n,Tx_{n-1})) \leq \psi(M(x_n,x_{n-1})),\ for\ all\ n \geq 1,$$

where

$$
\begin{aligned}
M(x_n,x_{n-1}) &= max\{d(x_n,x_{n-1}), \frac{d(x_n,Tx_n)d(x_n,Tx_{n-1})}{1+d(x_n,Tx_{n-1})+d(x_{n-1},Tx_n)}, d(x_n,Tx_n), d(x_{n-1},Tx_{n-1})\} \\
&= max\{d(x_n,x_{n-1}), \frac{d(x_n,x_{n+1})d(x_n,x_n)}{1+d(x_n,x_n)+d(x_{n-1},x_{n+1})}, d(x_n,x_{n+1}), d(x_{n-1},x_n)\} \\
&= max\{d(x_n,x_{n-1}), d(x_n,x_{n+1}), d(x_{n-1},x_n)\}
\end{aligned}
$$

We consider three different cases:

Case (i) If $M(x_n,x_{n-1}) = d(x_{n-1},x_n)$, we get

$$d(x_{n+1},x_n) \leq \psi d(x_{n-1},x_n) \leq \psi^n d(x_0,x_1) \longrightarrow 0 \quad as \quad n \longrightarrow \infty,$$

Case (ii) If $M(x_n,x_{n-1}) = d(x_n,x_{n+1})$, we get

$$d(x_{n+1},x_n) \leq \psi d(x_n,x_{n+1}) \leq \psi^n d(x_0,x_1) \longrightarrow 0 \quad as \quad n \longrightarrow \infty,$$

Case (iii) If $M(x_n,x_{n-1}) = d(x_n,x_{n-1})$, we get

$$
\begin{aligned}
d(x_{n+1},x_n) &= d(Tx_n,Tx_{n-1}) \leq \psi d(x_n,x_{n-1}) \\
&= \psi d(Tx_{n-1},Tx_{n-2}) \\
&\leq \psi^2 d(x_{n-1},x_{n-2}) \\
&\ldots \\
&\leq \psi^n (d(x_1,x_0)) \longrightarrow 0 \quad as \quad n \longrightarrow \infty,
\end{aligned}
$$

From Case (i)-Case (iii), we get

$$d(x_{n+1},x_n) \longrightarrow 0 \quad as \quad n \longrightarrow \infty, \tag{3.2}$$

form (3.1) and (3.2), we deduce that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0 = \lim_{n\to\infty} d(x_{n+1},x_n) \tag{3.3}$$

Now, we show $\lim_{n\to\infty} d(x_n,x_{n+2}) = 0 = \lim_{n\to\infty} d(x_{n+2},x_n)$

$$d(x_n,x_{n+2}) = d(Tx_{n-1},Tx_{n+1}) \leq \tau d(Tx_{n-1},Tx_{n+1}) \leq \psi(M(x_{n-1},x_{n+1})), \quad for \quad all \quad n \leq 1.$$

where

$$\begin{aligned}
&M(x_n,x_{n-1})\\
&=max\{d(x_{n-1},x_{n+1}),\frac{d(x_{n-1},Tx_{n-1})d(x_{n-1},Tx_{n+1})}{1+d(x_{n-1},Tx_{n+1})+d(x_{n+1},Tx_{n-1})},d(x_{n-1},Tx_{n-1}),d(x_{n+1},Tx_{n+1})\}\\
&=max\{d(x_{n-1},x_{n+1}),\frac{d(x_{n-1},x_n)d(x_{n-1},x_{n+2})}{1+d(x_{n-1},x_{n+2})+d(x_{n+1},x_n)},d(x_{n-1},x_n),d(x_{n+1},x_{n+2})\}\\
&=max\{d(x_{n-1},x_{n+1}),d(x_{n-1},x_n),d(x_{n+1},x_{n+2})\}.
\end{aligned}$$

We consider three different cases: Case (i) if $M(x_n,x_{n-1})=d(x_{n-1},x_{n+1})$, we get

$$d(x_n,x_{n+2})\leq\psi d(x_{n-1},x_{n+1})\leq\psi^{n-1}d(x_0,x_2)\longrightarrow 0\quad as\quad n\longrightarrow\infty,$$

Case (ii) if $M(x_{n-1},x_{n+1})=d(x_{n-1},x_n)$, we get

$$d(x_n,x_{n+2})\leq\psi d(x_{n-1},x_n)\leq\psi^{n-1}d(x_0,x_1)\longrightarrow 0\quad as\quad n\longrightarrow\infty,$$

Case (iii) if $M(x_{n-1},x_{n+1})=d(x_{n+1},x_{n+2})$, we get

$$d(x_n,x_{n+2})\leq\psi^{n+1}(d(x_0,x_1))\longrightarrow 0 asn\longrightarrow\infty,$$

From Case (i)-Case (iii), we get

$$d(x_n,x_{n+2})\longrightarrow 0\quad as\quad n\longrightarrow\infty, \tag{3.4}$$

$$d(x_{n+2},x_n)=d(Tx_{n+1},Tx_{n-1})\leq\tau(d(Tx_{n+1},Tx_{n-1}))\leq\psi(M(x_{n+1},x_{n-1})),\quad forall\quad n\geq 1,$$

where

$$\begin{aligned}
&M(x_{n+1},x_{n-1})\\
&=max\{d(x_{n+1},x_{n-1}),\frac{d(x_{n+1},Tx_{n+1})d(x_{n+1},Tx_{n-1})}{1+d(x_{n+1},Tx_{n-1})+d(x_{n-1},Tx_{n+1})},d(x_{n+1},Tx_{n+1}),d(x_{n-1},Tx_{n-1})\}\\
&=max\{d(x_{n+1},x_{n-1}),\frac{d(x_{n+1},x_{n+2})d(x_{n+1},x_n)}{1+d(x_{n+1},x_n)+d(x_{n-1},x_{n+2})},d(x_{n+1},x_{n+2}),d(x_{n-1},x_n)\}\\
&=max\{d(x_{n+1},x_{n-1}),d(x_{n+1},x_{n+2}),d(x_{n-1},x_n)\}
\end{aligned}$$

We consider three different cases:

Case (i) If $M(x_{n+1},x_{n-1})=d(x_{n+1},x_{n-1})$, we get

$$d(x_{n+2},x_n)\leq\psi d(x_{n+1},x_{n-1})\leq\psi^{n+1}d(x_2,x_0)\longrightarrow 0\quad as\quad n\longrightarrow\infty,$$

Case (ii) If $M(x_{n+1},x_{n-1}) = d(x_{n+1},x_{n+2})$, we get

$$d(x_{n+2},x_n) \le \psi d(x_{n+1},x_{n+2}) \le \psi^{n+1} d(x_0,x_1) \longrightarrow 0 \quad as \quad n \longrightarrow \infty,$$

Case (iii) If $M(x_{n+1},x_{n-1}) = d(x_{n-1},x_n)$, we get

$$d(x_{n+2},x_n) \le \psi d(x_{n-1},x_n) \le \psi^{n-1}(d(x_0,x_1)) \longrightarrow 0 \quad as \quad n \longrightarrow \infty,$$

From Case (i)-Case (iii), we get

$$d(x_{n+2},x_n) \longrightarrow 0 \quad as \quad n \longrightarrow \infty, \tag{3.5}$$

from (3.4) and (3.5), we deduce that

$$\lim_{n\to\infty} d(x_n,x_{n+2}) = 0 = \lim_{n\to\infty} d(x_{n+2},x_n)$$

Step 2: We shall prove that $\{x_n\}$ is a rectangular quasi b-Cauchy sequence, that is,

$$\lim_{n\to\infty} d(x_n,x_{n+p}) = 0 = \lim_{n\to\infty} d(x_{n+p},x_n) \quad forall \quad p \in \mathbb{N}$$

Case (i) Suppose that for some $n,m \in \mathbb{N}$ with $m > n$, we have $x_n = x_m$,

$$\begin{aligned} d(x_n,x_{n+1}) &= d(x_n,Tx_n) \\ &= d(x_m,Tx_m) \\ &= d(x_m,x_{m+1}) \\ &\le \psi^{m-n}(d(x_n,x_{n+1})) \\ &\le s\psi(d(x_n,x_{n+1})) < d(x_n,x_{n+1}), \end{aligned}$$

which is a contradiction. Case (ii) Suppose that for some $n,m \in \mathbb{N}$ with $n > m$, we have $x_n = x_m$,

$$\begin{aligned} d(x_{m+1},x_m) &= d(Tx_m,x_m) \\ &= d(Tx_n,x_n) \\ &= d(x_{n+1},x_n) \\ &\le \psi^{n-m}(d(x_{m+1},x_m)) \\ &\le s\psi(d(x_{m+1},x_m)) < d(x_{m+1},x_m), \end{aligned}$$

which is a contradiction.

Therefore, from Case (i) and Case (ii) $x_n \neq x_m$, for $m \neq n$.

The case $p=1$ and $p=2$ is proved. Now we take $p \geq 3$; arbitrary, we distinguish four different cases:

Case (i) Let $p=2m$, where $m \geq 2$. By the rectangular inequality, we get

$$\begin{aligned}
& d(x_n, x_{n+2m}) \\
& \leq s\left[d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+2m})\right] \\
& \leq sd(x_n, x_{n+2}) + sd(x_{n+2}, x_{n+3}) + s^2\left[d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+2m})\right] \\
& = sd(x_n, x_{n+2}) + sd(x_{n+2}, x_{n+3}) + s^2 d(x_{n+3}, x_{n+4}) + s^2 d(x_{n+4}, x_{n+5}) + s^2 d(x_{n+5}, x_{n+2m}) \\
& \ldots \\
& \leq sd(x_n, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) + s^4 d(x_{n+3}, x_{n+4}) + s^5 d(x_{n+4}, x_{n+5}) + \ldots. \\
& \quad + s^{2m} d(x_{n+2m-1}, x_{n+2m}) \\
& = sd(x_n, x_{n+2}) + \sum_{k=n+2}^{n+2m-1} s^{k-n+1} d(x_k, x_{k+1}) \\
& \leq sd(x_n, x_{n+2}) + \sum_{k=n+2}^{n+2m-1} s^k \psi^k(d(x_0, x_1)) \\
& \leq sd(x_n, x_{n+2}) + \sum_{k=n+2}^{\infty} s^k \psi^k(d(x_0, x_1)).
\end{aligned}$$

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0$$

and

$$\sum_{k=n+2}^{\infty} s^k \psi^k(d(x_0, x_1)) \to 0 \quad as \quad n \to \infty.$$

Therefore,

$$\lim_{n,m\to\infty} d(x_n, x_{n+2m}) = 0.$$

Case (ii) Let $p=2m+1$, where $m \geq 1$. By the rectangular inequality, we get

$$\begin{aligned}
& d(x_n, x_{n+2m+1}) \\
& \leq s\left[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m+1})\right]
\end{aligned}$$

$$\leq sd(x_n,x_{n+1})+sd(x_{n+1},x_{n+2})+s^2\left[d(x_{n+2},x_{n+3})+d(x_{n+3},x_{n+4})+d(x_{n+4},x_{n+2m+1})\right]$$

$$= sd(x_n,x_{n+1})+sd(x_{n+1},x_{n+2})+s^2d(x_{n+2},x_{n+3})+s^2d(x_{n+3},x_{n+4})+s^2d(x_{n+4},x_{n+2m+1})$$

$$\leq \ldots$$

$$\leq sd(x_n,x_{n+1})+s^2d(x_{n+1},x_{n+2})+s^3d(x_{n+2},x_{n+3})+s^4d(x_{n+3},x_{n+4})+\ldots$$

$$\ldots.+s^{2m+1}d(x_{n+2m},x_{n+2m+1})$$

$$= \sum_{k=n}^{n+2m} s^{k-n+1}d(x_k,x_{k+1})$$

$$= \sum_{k=n}^{n+2m} s^{k-n+1}\psi^k(d(x_0,x_1))$$

$$\leq \sum_{k=n}^{n+2m} s^k\psi^k(d(x_0,x_1))$$

$$\leq \sum_{k=n}^{\infty} s^k\psi^k(d(x_0,x_1)) \to 0 \quad as \quad n\to\infty.$$

Thus, we obtain

$$\lim_{n,m\to\infty} d(x_n,x_{n+2m+1})=0.$$

Case (iii) Let $p=2m$, where $m\geq 2$. By the rectangular inequality, we get

$$d(x_{n+2m},x_n)$$

$$\leq s\left[d(x_{n+2m},x_{n+2m-2})+d(x_{n+2m-2},x_{n+2m-3})+d(x_{n+2m-3},x_n)\right]$$

$$\leq sd(x_{n+2m},x_{n+2m-2})+sd(x_{n+2m-2},x_{n+2m-3})+s^2(d(x_{n+2m-3},x_{n+2m-4})+$$

$$d(x_{n+2m-4},x_{n+2m-5})+d(x_{n+2m-5},x_n))$$

$$= sd(x_{n+2m},x_{n+2m-2})+sd(x_{n+2m-2},x_{n+2m-3})+s^2d(x_{n+2m-3},x_{n+2m-4})+$$

$$s^2d(x_{n+2m-4},x_{n+2m-5})+s^2d(x_{n+2m-5},x_n)$$

$$\leq \ldots$$

$$\leq sd(x_{n+2m},x_{n+2m-2})+s^{n+2m-2}d(x_{n+2m-2},x_{n+2m-3})+s^{n+2m-3}d(x_{n+2m-3},x_{n+2m-4})+$$

$$s^{n+2m-4}d(x_{n+2m-4},x_{n+2m-5})+\ldots.+s^{n-1}d(x_{n-1},x_n)$$

$$= sd(x_{n+2m}, x_{n+2m-2}) + \sum_{k=n-1}^{n+2m-1} s^k d(x_k, x_{k+1})$$

$$\leq sd(x_{n+2m}, x_{n+2m-2}) + \sum_{k=n-1}^{n+2m-1} s^k \psi^k(d(x_2, x_0))$$

$$\leq sd(x_{n+2m}, x_{n+2m-2}) + \sum_{k=n-1}^{\infty} s^k \psi^k(d(x_2, x_0))$$

Since

$$\lim_{m,n\to\infty} d(x_{n+2m}, x_{n+2m-2}) = 0$$

and

$$\sum_{k=n-1}^{\infty} s^k \psi^k(d(x_2, x_0)) \to 0 \quad as \quad n \to \infty.$$

Therefore, $\lim_{n,m\to\infty} d(x_{n+2m}, x_n) = 0$.

Case (iv) Let $p = 2m+1$, where $m \geq 1$. By the rectangular inequality, we get

$$d(x_{n+2m+1}, x_n)$$

$$\leq s\left[d(x_{n+2m+1}, x_{n+2m}) + d(x_{n+2m}, x_{n+2m-1}) + d(x_{n+2m-1}, x_n)\right]$$

$$\leq sd(x_{n+2m+1}, x_{n+2m}) + sd(x_{n+2m}, x_{n+2m-1}) + s^2(d(x_{n+2m-1}, x_{n+2m-2}) +$$

$$d(x_{n+2m-2}, x_{n+2m-3}) + d(x_{n+2m-3}, x_n))$$

$$=\leq sd(x_{n+2m}, x_{n+2m}) + sd(x_{n+2m}, x_{n+2m-1}) + s^2 d(x_{n+2m-1}, x_{n+2m-2})$$

$$+ s^2 d(x_{n+2m-2}, x_{n+2m-3}) + s^2 d(x_{n+2m-3}, x_n)$$

$$\leq \ldots$$

$$\leq s^{n+2m+1} d(x_{n+2m+1}, x_{n+2m}) + s^{n+2m} d(x_{n+2m}, x_{n+2m-1}) + s^{n+2m-1} d(x_{n+2m-1}, x_{n+2m-2}) + ..$$

$$.. + s^{n+1} d(x_{n+1}, x_n)$$

$$= \sum_{k=n+1}^{n+2m} s^k d(x_{k+1}, x_k) = \sum_{k=n+1}^{n+2m} s^{k-n+1} d(x_1, x_0)$$

$$\leq \sum_{k=n+1}^{n+2m} s^k \psi^k(d(x_1, x_0))$$

$$\leq \sum_{k=n+1}^{\infty} s^k \psi^k(d(x_1, x_0)) \to 0 \quad as \quad n \to \infty.$$

Thus, we obtain

$$\lim_{m,n\to\infty} d(x_{n+2m+1},x_n) = 0.$$

Finally, from Case (i)-Case (iv), we get

$$\lim_{n,m\to\infty} d(x_n,x_{n+p}) = 0 = \lim_{n,m\to\infty} d(x_{n+p},x_n) \quad for\ all \quad p \geq 3.$$

Thus, $\{x_n\}$ is rectangular quasi b-Cauchy sequence in (X,d).

Since X is a complete rectangular quasi b-metric space, there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u, \lim_{n\to\infty} d(x_n,u) = 0 = \lim_{n\to\infty} d(u,x_n.) \tag{3.6}$$

Now, we shaw that u is a fixed point of T. Since T is continuous, from (3-6), we have

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n) = T(\lim_{n\to\infty} x_n) = Tu, \quad which\ gives \quad u = Tu. \tag{3.7}$$

Thus, u is a fixed point of T. □

Now, we state the following fixed point theorem by removing the continuity assumption of T from Theorem 1.

Theorem 3.8. *Let (X,d) be an complete rectangular quasi b-metric space and $T : X \to X$ be generalized τ,ψ-contraction mapping. Suppose that*

(i) *there exist $x_n \in X$ such that $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \geq 0$*

(ii) *if $\{x_n\}$ is a sequence in X such that $x_n \to x \in X$ as $n \to \infty$ Then T has a unique fixed point.*

Proof. Following the proof of Theorems 1, we know that the sequence x_n defined by $x_{n+1} = Tx_n$ for all $n \geq 0$ is rectangular quasi b- converges to a point u in X. It is sufficient to show that T admits a fixed point. By the rectangular inequality of rectangular quasi b-metric space, property of ψ, and (ii), we have

$$\begin{aligned} d(u,Tu) &\leq sd(u,x_n) + sd(x_n,x_{n+1}) + sd(x_{n+1},Tu) \\ &= sd(u,x_n) + sd(x_n,x_{n+1}) + sd(Tx_n,Tu) \\ &\leq sd(u,x_n) + sd(x_n,x_{n+1}) + s\tau d(Tx_n,Tu) \\ &\leq sd(u,x_n) + sd(x_n,x_{n+1}) + s\psi(M(x_n,u)) \end{aligned}$$

where

$$M\left(x_n,u\right)=\max\left\{d(x_n,u),\frac{d(x_n,Tx_n)d(x_n,Tu)}{1+d(x_n,Tu)+d(u,Tx_n)},d(x_n,Tx_n),d(u,Tu)\right\}$$
$$=max\left\{d(x_n,u),\frac{d(x_n,x_{n+1})d(x_n,Tu)}{1+d(x_n,Tu)+d(u,x_{n+1})},d(x_n,x_{n+1}),d(u,Tu)\right\}$$
$$=max\left\{d(x_n,u),d(x_n,x_{n+1}),d(u,Tu)\right\}.$$

We consider three different cases: Case (i) if $M\left(x_n,x_{n-1}\right)=d\left(x_n,x_{n-1}\right)$, then by (..), we get

$$d\left(u,Tu\right)\leq sd(u,x_n)+sd(x_n,x_{n+1})+s\psi\left(d(x_n,u)\right)$$
$$<sd(u,x_{n+1})+sd(x_n,x_{n+1})+d(x_{n+1},u).$$

Letting $n\to\infty$ the above inequality, we get that $d(u,Tu)\leq 0$.

Case (ii) if $M(x_n,u)=d(x_n,x_{n+1})$, we get

$$d\left(u,Tu\right)\leq sd(u,x_n)+sd(x_n,x_{n+1})+s\psi(d(x_n,x_{n+1})$$
$$<sd(u,x_n)+sd(x_n,x_{n+1})+d(x_n,x_{n+1}).$$

Letting $n\to\infty$ the above inequality, we get that $d(u,Tu)\leq 0$.

Case (iii) if $M\left(x_n,u\right)=d\left(u,Tu\right)$, we get

$$d\left(u,Tu\right)\leq sd(u,x_n)+sd(x_n,x_{n+1})+s\psi\left(d(u,Tu)\right)$$
$$<sd(u,x_n)+sd(x_n,x_{n+1})+d(u,Tu).$$

Letting $n\to\infty$ the above inequality, we get that $0\leq 0$ it is general truth.

Clearly $d(u,Tu)\leq 0$,

from Case (i)-Case (iii), we can obtain $d(u,Tu)=0$.

Also,

$$d(Tu,u)\leq sd(Tu,x_n)+sd(x_n,x_{n+1})+sd(x_{n+1},Tu)$$
$$=sd(Tu,Tx_{n-1})+sd(x_n,x_{n+1})+sd(x_{n+1},u)$$
$$\leq s\tau d(Tu,Tx_{n-1})+sd(x_n,x_{n+1})+sd(x_{n+1},u)$$
$$\leq s\psi\left(M(u,x_{n-1})+sd(x_n,x_{n+1})+sd(x_n,u),\right.$$

where

$$\begin{aligned} M(u,x_{n-1}) &= \max\left\{d(u,x_{n-1}),\frac{d(u,Tu)d(u,Tx_{n-1})}{1+d(u,Tx_{n-1})+d(x_{n-1},Tu)},d(u,Tu),d(x_{n-1},Tx_{n-1})\right\} \\ &= max\left\{d(u,x_{n-1}),\frac{d(u,Tu)d(u,x_n)}{1+d(u,x_n)+d(x_{n-1},Tu)},d(u,Tu),d(x_{n-1},x_n)\right\} \\ &= max\{d(u,x_{n-1}),d(x_n,x_{n+1}),d(u,Tu)\}. \end{aligned}$$

We consider three different cases:

Case (i) if $M(u,x_{n-1}) = d(u,x_{n-1})$, we get

$$\begin{aligned} d(Tu,u) &\leq s\psi(d(u,x_{n-1})) + sd(x_n,x_{n-1}) + sd(x_{n+1},u) \\ &< d(u,x_{n-1}) + sd(x_n,x_{n+1}) + sd(x_{n+1},u). \end{aligned}$$

Letting $n \to \infty$ in the above equality, we get that

$d(Tu,u) \leq 0$.

Case (ii) if $M(u,x_{n-1}) = d(u,Tu)$, we get

$$\begin{aligned} d(Tu,u) &\leq s\psi(d(x_n,x_{n+1})) + sd(x_n,x_{n+1}) + sd(x_{n+1},u) \\ &< sd(x_n,x_{n+1}) + sd(x_n,x_{n+1}) + sd(x_{n+1},u). \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get that

$d(u,Tu) \leq 0$.

Case (iii) if $M(u,x_{n-1}) = d(u,Tu)$, we get

$$\begin{aligned} d(Tu,u) &\leq s\psi(d(u,Tu)) + sd(x_n,x_{n+1}) + sd(x_{n+1},u)) \\ &< sd(u,Tu) + sd(x_n,x_{n+1}) + d(x_{n+1},u). \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get that $d(Tu,u) \leq 0$.

Clearly $d(u,Tu) \geq 0$.

From Case (i)-Case (iii), we can obtain $d(Tu,u) = 0$.

From (3.8) and (3.9), it follows that $d(u,Tu) = 0 = d(Tu,u)$. So that, $Tu = u$.

Thus, u is a fixed point of T. □

Example 3.9. Let $X=\{0,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5}\}$. We define d on X as follows
$d(0,\frac{1}{2})=d(\frac{1}{2},\frac{1}{3})=d(\frac{1}{4},\frac{1}{5})=0.3;d(0,\frac{1}{3})=d(\frac{1}{3},\frac{1}{2})=d(\frac{1}{5},\frac{1}{4})=d(\frac{1}{3},\frac{1}{4})=0.1;$
$d(0,\frac{1}{4})=d(\frac{1}{4},\frac{1}{2})=d(\frac{1}{3},\frac{1}{5})=0.6;d(0,\frac{1}{5})=d(\frac{1}{2},\frac{1}{4})=d(\frac{1}{5},\frac{1}{3})=0.4;$
$d(\frac{1}{2},0)=d(\frac{1}{4},0)=d(\frac{1}{2},\frac{1}{5})=1.05;d(\frac{1}{3},0)=d(\frac{1}{5},0)=d(\frac{1}{5},\frac{1}{2})=d(\frac{1}{4},\frac{1}{3})=0.5;$
$d(0,0)=d(\frac{1}{2},\frac{1}{2})=d(\frac{1}{3},\frac{1}{3})=d(\frac{1}{4},\frac{1}{4})=d(\frac{1}{5},\frac{1}{5})=0;$
Then (X,d) is a rectangular quasi b-metric space with coefficient $s=\frac{3}{2}\geq 1$. We define $T:X\to X$, $\psi:\mathbb{R}^+\to\mathbb{R}^+$ by $Tx=\frac{1}{3}$. $\psi(t)=\frac{1}{2}t$ for all $t\in\mathbb{R}^+$ and $\theta>1$. We show that T is a generalized (θ,ψ)-contraction mapping. Let $x,y\in X$, we have $Tx=\frac{1}{3}$ and $Ty=\frac{1}{3}$. Then $\theta d(Tx,Ty)=0\leq\psi(M(x,y))$. Note that for $s=\frac{3}{2}$ and $\psi(t)=\frac{1}{2}t$, we have $\sum_{n=1}^{\infty}s^n\psi^n(t)=t\sum_{n=1}^{\infty}(\frac{3}{4})^n<\infty$,and $\frac{3}{2}\psi(t)<t$ for all $t>0$.
Hence all the conditions of Theorem 2 are satisfied, and so $\frac{1}{3}$ is the fixed point of T.

Conflict of Interests

The authors declare that there is no conflict of interests.

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