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THE TINGLEY PROBLEM ON THE UNIT SPHERE OF COMPLEX $L^p[0,1]$SPACE

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Abstract. In this paper, we investigate the problem of extending isometric operators from unit sphere of complex L^p spaces ($1 < p < \infty$, $p \neq 2$) to general complex Banach spaces. By studying the isometric operators, we prove the Tingley problem on complex L^p spaces and provide a positive answer under some conditions. That is, it is proved that for a surjective isometry V_0 on any complex $L^p[0,1]$ unit sphere to any general complex Banach space E unit sphere, Under some conditions,V_0 can be extended to a linear isometry from the entire space $L^p[0,1]$ to E.

Keywords: surjective isometry; linear isometry; isometric extension; complex L^p space; 1-Lipshcitz mapping.
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1. INTRODUCTION

The classical Mazur-Ulam theorem [1] states that if X and Y are real normed spaces and $T : X \to Y$ is a surjective isometry, then T is affine. In particular, if $T(0) = 0$, then T is a linear isometry. However, local surjective isometries often lack desirable properties, leading to the following question raised by Tingley in 1987.

Tingley Problem [2]: Let X and Y be two normed spaces, and T_0 be a surjective isometry between the unit spheres $S(X)$ and $S(Y)$. Is there a linear isometry $T : X \to Y$ defined on the entire space such that $T|_{S(X)} = T_0$?

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The formulation of the Tingley problem has been a milestone in the study of isometries. Its significance lies in the fact that if the conclusion holds, then the geometric properties of a mapping on the unit sphere in a spatial context will determine its properties throughout the entire space. In 2011, L. Cheng and Y. Dong introduced the concept of Mazur-Ulam property related to the Tingley problem and the Mazur-Ulam theorem [3], making significant contributions to the study of isometric problems. The Mazur-Ulam property states that if a surjective isometry from the unit sphere of a normed space X to the unit sphere of an arbitrary normed space Y can be extended to a real linear isometry on the entire space, then the normed space X is said to have the Mazur-Ulam property [3] (MUP).

In recent years, researchers have primarily focused on the study of isometric extension problems in similar or different types of classical Banach spaces. Ding Guanggui and other scholars have summarized the conclusions regarding isometric problems in these spaces. In the case of similar type spaces, Ding Guanggui [4] proved in 2007 that two surjective isometries between the unit spheres of $l^{\infty}(\Gamma)$-type spaces can be extended to real linear isometries on the entire spaces. In 2011 and 2012, Tan Dongni provided affirmative answers to the Tingley problem on the F-spaces $(L_p(v), 0<p<1)$ and Tsirelson space, James space (see [5, 6]), respectively. In the case of different type spaces [7], Ding Guanggui initially proved that a surjective isometry from the unit sphere $S(E)$ of an arbitrary Banach space E to the unit sphere $S(\mathrm{C}[0,1])$ of the space of continuous functions on $[0,1]$ can be extended to a linear isometry on the entire space under certain conditions. Later, Wang Jianhua and Fang Xinian [8] made improvements by removing the additional conditions. Subsequently, Liu Rui [9] proved that the surjective isometry from the unit sphere of $L^{\infty}(\mu)$-type spaces to the unit sphere of an arbitrary Banach space, as well as Tan Dongni [10], proved that the isometry between the unit spheres of Banach spaces is linearly extendable when the Banach space is a locally GL space, thereby providing more general results. In 2012, Tan Dongni [11] studied $L^p(\mu,H)$ spaces (where H is a Hilbert space, $1<p<\infty$, and $p\neq 2$) and also provided exact conclusions.

Most of the research on the Tingley problem in real Banach spaces has been conducted by scholars (for further references, see [12, 13, 14, 15, 16, 17, 18]). While it is not always possible to extend a surjective isometry between the unit spheres of arbitrary complex Banach spaces to complex linear or conjugate linear operators on the entire space, this problem holds true for certain classical complex Banach spaces. In 2014, Yi Jijin, Wang Ruidong, and Wang Xiaoxiao [19] established that any surjective isometry from the unit sphere of a complex $l^p(\Gamma)$ space (where $1 < p < \infty$ and $p \neq 2$) to the unit sphere of a complex $l^p(\Delta)$ space can be extended to a real linear isometry on the entire space.

Building upon the inspiration from the aforementioned papers, this study considers the complex $L^p[0,1]$ spaces and aims to prove that any surjective isometry from the unit sphere of a complex $L^p[0,1]$ space (where $1 < p < \infty$ and $p \neq 2$) to the unit sphere of an arbitrary complex Banach space can always be linearly extended to the entire space.

2. Preliminary Knowledge

Definition 2.1. *Introduces the definition of the complex $L^p[0,1]$space as follows:*

$$L^p[0,1] := \left\{ f : [0,1] \to \mathbb{C} \,\middle|\, \int_{[0,1]} \left| f \right|^{\mathrm{p}} \mathrm{dt} < +\infty \right\}.$$

Here, $\mathrm{p} \in (1,\infty)$ and $\mathrm{p} \neq 2$. The norm on the space is defined as:

$$\|f\|_p = \left(\int_\Omega |f|^p dt \right)^{1/p}.$$

Definition 2.2. *States that for normed spaces* X *and* Y*, an operator* $V : \mathrm{X} \to \mathrm{Y}$ *is said to be a* $1-$*Lipschitz mapping if it satisfies:*

$$\|V(x) - V(y)\| \leq \|x - y\| \quad (\forall x, y \in X).$$

If $\|V(x) - V(y)\| = \|x - y\|$ *for all* $x, y \in \mathrm{X}$*, then* V *is called an isometric operator.*

Definition 2.3. *Defines the support of a function* $f \in S(L^p[0,1])$ *as* $\operatorname{supp} f := \{\omega \in \Omega \mid f(\omega) \neq 0\}$*. If* $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$*, then* $f \wedge g = 0$*.*

3. Related Lemmas and Results

Theorem 3.1. *Let E be a complex Banach space, V_0 be a surjective isometry from complex* $\mathrm{S}(L^p[0,1])$ *to complex* $\mathrm{S(E)}$*, where* $1 < \mathrm{p} < \infty$ *and* $\mathrm{p} \neq 2$*.* V_0 *satisfies* $V_0(a+x) - V_0(a) = V_0(b+x) - V_0(b)$ *for* $a, b, a+x, b+x \in \mathrm{S}(L^p[0,1])$ *with* $a \wedge x = 0$ *and* $b \wedge x = 0$*. And* $V_0(\lambda f) =$

$\lambda V_0(f)$ for all $f \in \mathrm{S}(L^p[0,1]), \lambda \in \mathbb{T}$, where $|\lambda| = 1$. Then V_0 can be linearly and isometrically extended to the whole space $L^p[0,1]$.

In order to prove this theorem, we need to introduce some notations and symbols, which will be used throughout this section.

Notations:

(1) The mapping V_0 : complex $\mathrm{S}(L^p[0,1]) \to$ complex S(E) is a surjective isometry.

(2) For positive real numbers α and β, we have $\alpha^p + \beta^p = 1$. For $f, g \in S(L^p[0,1])$, we denote $f \wedge g = 0$.And denote $\mathbb{T} = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$.

(3) We use the expressions

$$H_g(\alpha f, \beta g) = V_0^{-1}\left(\frac{V_0(\alpha f + \beta g) + V_0(\beta g - \alpha f)}{2\beta}\right),$$
$$H_f(\alpha f, \beta g) = V_0^{-1}\left(\frac{V_0(\alpha f + \beta g) + V_0(\alpha f - \beta g)}{2\alpha}\right).$$

to represent elements in complex $S(L^p[0,1])$.

The representation in (3) is justified because we have the following Lemma 3.2, which guarantees:

$$\left\|\frac{V_0(\alpha f + \beta g) + V_0(\beta g - \alpha f)}{2\beta}\right\| = 1.$$

The proof is mainly divided into two cases: $1 < \mathrm{p} < 2$ and $2 < \mathrm{p} < \infty$. For the case of $2 < \mathbf{p} < \infty$,the following inequality in $L^p[0,1]$ space, with $1 < \mathrm{p} < \infty$ and $\mathrm{p} \neq 2$, is known as the Clarkson inequality and is an important inequality in $L^p[0,1]$ space with $1 < \mathrm{p} < \infty$ and $\mathrm{p} \neq 2$.

Lemma 3.2. *For $S(L^p[0,1])$ $(1 < \mathrm{p} < \infty$, and $\mathrm{p} \neq 2)$, for any two elements f and g, the following hold:*

(i) $\|f+g\|^p + \|f-g\|^p \geq 2(\|f\|^p + \|g\|^p)$, $2 < \mathrm{p} < \infty$.

(ii) $\|f+g\|^p + \|f-g\|^p \leq 2(\|f\|^p + \|g\|^p)$, $1 < \mathrm{p} < 2$.

Moreover, equality holds in (i) and (ii) if and only if $f \perp g$.

This certification is detailed in the reference [12].

Lemma 3.3. *Suppose X and Y are complex Banach spaces, and let $\mathrm{V}_0 : S(X) \to S(Y)$ be a surjective isometry.Then, X is strictly convex if and only if Y is strictly convex. Moreover, for every $x \in \mathrm{S}(X)$ and $y \in \mathrm{S}(Y)$, we have $\mathrm{V}_0(-x) = -\mathrm{V}_0(x)$ and ${\mathrm{V}_0}^{-1}(-y) = -{\mathrm{V}_0}^{-1}(y)$.*

This certification is detailed in the reference [20]. It will not be proved here. From this theorem, we can deduce that in a strictly convex space, for any x, y, we have $\|x+y\| = \|\mathrm{V}_0(x)+\mathrm{V}_0(y)\|$ holds. Thus, in notation (3), we have $\|V_0(\alpha f+\beta g)+V_0(\beta g-\alpha f)\| = \|2\beta g\| = 2\beta$, which leads to:

$$\left\|\frac{V_0(\alpha f+\beta g)+V_0(\beta g-\alpha f)}{2\beta}\right\| = 1.$$

The following proof comes from the references [11] slightly modified. For completeness, we give the proof.

Lemma 3.4. *Let* E *be a complex Banach space, and* $V_0 : \mathrm{S}(L^p[0,1]) \to \mathrm{S}(\mathrm{E})$ *be a surjective isometry. Let* $f, g \in \mathrm{S}(L^p[0,1])$ *be complex-valued functions satisfying* $f \wedge g = 0$. *Let* α *and* β *be positive real numbers such that* $\alpha^p+\beta^p = 1$. *Suppose* V_0 *satisfies* $V_0(a+x)-V_0(a) = V_0(b+x)-V_0(b)$ *for* $a, b, a+x, b+x \in \mathrm{S}(L^p[0,1])$ *with* $a \wedge x = 0$ *and* $b \wedge x = 0$. *Then, the following hold:*

(i) $H_g(\alpha f, \beta g) = H_g(\beta f, \alpha g)$, $H_f(\alpha f, \beta g) = H_f(\beta f, \alpha g)$.

(ii) $H_g(\alpha f, \beta g) \perp H_f(\alpha f, \beta g)$.

(iii) $H_f(\alpha f, \beta \frac{g}{\lambda}) = H_f(\alpha f, \beta g)$, *where* $\lambda \in \mathbb{T}$ *and* $\mathbb{T}$ *is the unit sphere in the complex plane.*

Proof: Case 1: When $2 < \mathrm{p} < \infty$ and assuming $\beta \geq \alpha$. From the given conditions and Lemma 3.3, it follows that for any $w, u \in \mathrm{S}(L^p[0,1])$, we have $\|V_0(w) \pm V_0(u)\| = \|w \pm u\|$. Note that $f \wedge g = 0$. Now, let's consider:

$$\begin{aligned}
&\left\|H_f(\alpha g, \beta f)+H_g(\alpha f, \beta g)\right\| \\
=&\left\|V_0^{-1}\left(\frac{V_0(\alpha g+\beta f)+V_0(\beta f-\alpha g)}{2\beta}\right)+V_0^{-1}\left(\frac{V_0(\alpha f+\beta g)+V_0(\beta g-\alpha f)}{2\beta}\right)\right\| \\
=&\left\|\frac{V_0(\alpha g+\beta f)+V_0(\beta f-\alpha g)}{2\beta}+\frac{V_0(\alpha f+\beta g)+V_0(\beta g-\alpha f)}{2\beta}\right\| \\
=&\left\|\frac{V_0(\alpha g+\beta f)+V_0(\alpha f+\beta g)}{2\beta}+\frac{V_0(\beta f-\alpha g)+V_0(\beta g-\alpha f)}{2\beta}\right\| \\
\leq&\frac{\|V_0(\alpha g+\beta f)+V_0(\alpha f+\beta g)\|+\|V_0(\beta f-\alpha g)+V_0(\beta g-\alpha f)\|}{2\beta} \\
=&\frac{(\alpha+\beta)\|f+g\|+(\beta-\alpha)\|f+g\|}{2\beta} = \|f+g\|.
\end{aligned} \tag{3.2}$$

On the other hand, let's examine:

$$
\begin{aligned}
&\left\|H_f(\alpha g,\beta f)-H_g(\alpha f,\beta g)\right\| \\
&=\left\|V_0^{-1}\left(\frac{V_0(\alpha g+\beta f)+V_0(\beta f-\alpha g)}{2\beta}\right)-V_0^{-1}\left(\frac{V_0(\alpha f+\beta g)+V_0(\beta g-\alpha f)}{2\beta}\right)\right\| \\
&=\left\|\frac{V_0(\alpha g+\beta f)+V_0(\beta f-\alpha g)}{2\beta}-\frac{V_0(\alpha f+\beta g)+V_0(\beta g-\alpha f)}{2\beta}\right\| \\
&=\left\|\frac{V_0(\alpha g+\beta f)-V_0(\alpha f+\beta g)}{2\beta}+\frac{V_0(\beta f-\alpha g)-V_0(\beta g-\alpha f)}{2\beta}\right\| \\
&\leq\frac{\|V_0(\alpha g+\beta f)-V_0(\alpha f+\beta g)\|+\|V_0(\beta f-\alpha g)-V_0(\beta g-\alpha f)\|}{2\beta}=\|f-g\|.
\end{aligned}
\tag{3.3}
$$

By equations (3.2) and (3.3), together with Lemma 3.1 (i), we can deduce the following:

$$
\begin{aligned}
4&=2\left\|H_f(\alpha g,\beta f)\right\|^{\mathrm{p}}+2\left\|H_g(\alpha f,\beta g)\right\|^{\mathrm{p}} \\
&\leq\left\|H_f(\alpha g,\beta f)+H_g(\alpha f,\beta g)\right\|^{\mathrm{p}}+\left\|H_f(\alpha g,\beta f)-H_g(\alpha f,\beta g)\right\|^{\mathrm{p}} \\
&\leq\|f+g\|^{\mathrm{p}}+\|f-g\|^{\mathrm{p}}=4.
\end{aligned}
\tag{3.4}
$$

Hence, the equality holds in equation (3.4), as well as in equations (3.2) and (3.3).

Furthermore, since the complex $L^p[0,1]$ space is strictly convex, from Lemma 3.3, we can conclude that the space E is also strictly convex. Moreover, based on equations (3.2) and (3.3), we have:

$$(\beta-\alpha)\left(V_0(\alpha g+\beta f)+V_0(\alpha f+\beta g)\right)=(\beta+\alpha)\left(V_0(-\alpha g+\beta f)+V_0(-\alpha f+\beta g)\right),$$

$$(\beta+\alpha)\left(V_0(\alpha g+\beta f)-V_0(\alpha f+\beta g)\right)=(\beta-\alpha)\left(V_0(-\alpha g+\beta f)-V_0(-\alpha f+\beta g)\right).$$

Apologies for the repeated content. It seems there was an issue with the response. Let's continue with the proof:

By adding and subtracting the previous two equations, we have:

$$2\beta V_0(\alpha g+\beta f)-2\alpha V_0(\alpha f+\beta g)=2\beta V_0(-\alpha g+\beta f)+2\alpha V_0(-\alpha f+\beta g),$$

$$-2\alpha V_0(\alpha g+\beta f)+2\beta V_0(\alpha f+\beta g)=2\alpha V_0(-\alpha g+\beta f)+2\beta V_0(-\alpha f+\beta g).$$

From these equations, we can deduce:

$$H_g(\alpha f,\beta g)=H_g(\beta f,\alpha g),\quad H_f(\alpha f,\beta g)=H_f(\beta f,\alpha g).$$

Let's prove statement (ii): $H_g(\alpha f,\beta g)\perp H_f(\alpha f,\beta g)$.

From equation (3.4), we observe the following relation:

$$\begin{aligned}
&2\left(\left\|H_f(\alpha g,\beta f)\right\|^p+\left\|H_g(\alpha f,\beta g)\right\|^p\right)\\
=&\left\|H_f(\alpha g,\beta f)+H_g(\alpha f,\beta g)\right\|^p+\|H_f(\alpha g,\beta f)-H_g(\alpha f,\beta g)\|^p\\
=&4.
\end{aligned}$$

Based on Lemma 3.2 (i), this equation holds with equality if and only if $H_g(\alpha f,\beta g)\perp H_f(\alpha g,\beta f)$.

The above proof also applies to the case when $\beta\leq\alpha$. Therefore, the result holds.

Case 2: When $1<p<2$ and assuming $\beta\leq\alpha$, we can use the same approach as in Case 1 to obtain:

$$\left\|H_f(\alpha g,\beta f)+H_g(\alpha f,\beta g)\right\|\geq\|f+g\|,\quad \left\|H_f(\alpha g,\beta f)-H_g(\alpha f,\beta g)\right\|\geq\|f-g\|.$$

Similar to the steps in Case 1, we can verify statements (i) and (ii).

To prove statement (iii), we want to show that $H_f\left(\alpha f,\beta\frac{g}{\lambda}\right)=H_f(\alpha f,\beta g)$. Based on the definition of the operator H, we can rewrite this as:

$$V_0\left(\alpha f+\beta\frac{g}{\lambda}\right)+V_0\left(\alpha f-\beta\frac{g}{\lambda}\right)=V_0(\alpha f+\beta g)+V_0(\alpha f-\beta g). \tag{3.5}$$

Since $\alpha f+\beta\frac{g}{\lambda}-(\alpha f+\beta g)=\alpha f-\beta g-\left(\alpha f-\beta\frac{g}{\lambda}\right)$, and $f\wedge g=0$, we can apply the properties of V_0 to obtain:

$$V_0\left(\alpha f+\beta\frac{g}{\lambda}\right)-V_0(\alpha f+\beta g)=V_0(\alpha f-\beta g)-V_0\left(\alpha f-\beta\frac{g}{\lambda}\right).$$

Hence, equation (3.5) is proven. □

Lemma 3.5. *Suppose T_0 is a 1-Lipschitz mapping from $\mathrm{S}\left(L^p[0,1]\right)$ to $\mathrm{S}\left(L^p[0,1]\right)$, satisfying $T_0(\lambda f)=\lambda T_0(f)$ for all $f\in\mathrm{S}\left(L^p[0,1]\right)$ and $\lambda\in\mathbb{T}$, where $|\lambda|=1$. If $f\wedge g=0$ implies $T_0(f)\wedge T_0(g)=0$ for $f,g\in\mathrm{S}\left(L^p[0,1]\right)$, then T_0 can be extended to a linear isometry on the entire space.*

Proof: The theorem will be proven in three steps.

Step 1: Let $\forall f\in\mathrm{S}(L^p[0,1])$, where $f=\sum_{i=1}^n a_if_i$, with $f_i\wedge f_j=0$ for $i\neq j$, $a_i\in\mathbb{T}$, and $\{f_i\}_{i=1}^n\in\mathrm{S}(L^p[0,1])$. We have $\sum_{i=1}^n|a_i|^p=1$. The goal is to prove that for $\forall 1\leq i\leq n$, we have:

$$T_0(f)=\sum_{i=1}^n a_iT_0(f_i). \text{ (where } a_i\in\mathbb{T}\text{)}$$

Let's denote $A_i = \operatorname{supp}(T_0(f_i))$, which means $T_0(f)|_{A_i} = \{T_0(f)|_x : x \in A_i\}$.

Given that $T_0(f_i) \wedge T_0(f_j) = 0$ for $i \neq j$, by definition, we have $\operatorname{supp}(T_0(f_i)) \cap \operatorname{supp}(T_0(f_j)) = \emptyset$ holds. In other words, we have $A_i \cap A_j = \emptyset$ for $i \neq j$. It can be deduced that:

$$\sum_{i=1}^{n} \left\| T_0(f)|_{A_i} \right\|^p \leq 1.$$

Step 2: Next, we will mainly prove that $T_0(f)|_{A_i} = a_i T_0(f_i)$ (where $a_i \in \mathbb{T}$).

Let $a_i \in \mathbb{T}$, and we can write $a_i = |a_i| e^{i\theta_i}$, where $\operatorname{sign}(a_i) = e^{i\theta_i}$. Since T_0 is 1-Lipschitz and $T_0(\lambda f) = \lambda T_0(f)$ for $\lambda \in \mathbb{T}$, we have the following:

(i) $\|T_0(f) - \operatorname{sign}(a_i) T_0(f_i)\|^p = \left\|T_0(f) - e^{i\theta_i} T_0(f_i)\right\|^p = \left\|T_0(f) - T_0\left(e^{i\theta_i} f_i\right)\right\|^p \leq 1 - |a_i|^p + (1 - |a_i|)^p$.

(ii) On the other hand, we have $\|T_0(f) - \operatorname{sign}(a_i) T_0(f_i)\|^p = \left\|T_0(f) - e^{i\theta_i} T_0(f_i)\right\|^p$

$$\begin{aligned}
&= \left\| T_0(f)|_{A_i} + T_0(f)|_{\Omega \setminus A_i} - e^{i\theta_i} T_0(f_i) \right\|^p \\
&= \left\| T_0(f)|_{A_i} - T_0\left(e^{i\theta_i} f_i\right) \right\|^p + 1 - \left\| T_0(f)|_{A_i} \right\|^p \\
&\geq \left| \left\| T_0(f)|_{A_i} \right\| - \left\| T_0\left(-e^{i\theta_i} f_i\right) \right\| \right|^p + 1 - \left\| T_0(f)|_{A_i} \right\|^p.
\end{aligned}$$

Here, $\left\|T_0\left(-e^{i\theta_i} f_i\right)\right\| = 1$. Therefore, we have:

$$\left\| T_0(f) - e^{i\theta_i} T_0(f_i) \right\|^p \geq \left(1 - \left\| T_0(f)|_{A_i} \right\|\right)^p + 1 - \left\| T_0(f)|_{A_i} \right\|^p.$$

Combining (i) and (ii), we obtain:

$$(1 - |a_i|)^p + 1 - |a_i|^p \geq \left(1 - \left\| T_0(f)|_{A_i} \right\|\right)^p + 1 - \left\| T_0(f)|_{A_i} \right\|^p. \tag{3.6}$$

By observing that the form of equation (3.6) is consistent, let's assume the function $\varphi(t) = (1-t)^p + t^p$, where $t \in [0,1]$. It is easy to see that $\varphi(t)$ is a monotonically decreasing function for $t \in [0,1]$. Using equation (3.6), we have:

$$|a_i| \leq \left\| T_0(f)|_{A_i} \right\|.$$

Furthermore, from the conclusion obtained in Step 1, we have $\sum_{i=1}^{n} |a_i|^p = \sum_{i=1}^{n} \left\| T_0(f)|_{A_i} \right\|^p$ $= 1$. Thus, we can deduce that $|a_i| = \left\| T_0(f)|_{A_i} \right\|$. Therefore, we can easily obtain:

$$\left\| T_0(f)|_{A_i} - e^{i\theta_i} T_0(f_i) \right\|^p = \left(1 - \left\| T_0(f)|_{A_i} \right\|\right)^p.$$

Here, we have $1 = \left\|T_0\left(e^{i\theta_i} f_i\right)\right\| = \left\|e^{i\theta_i} T_0(f_i)\right\|$. Therefore,

$$\left\| T_0(f)|_{A_i} - e^{i\theta_i} T_0(f_i) \right\| = \left\| e^{i\theta_i} T_0(f_i) \right\| - \left\| T_0(f)|_{A_i} \right\|. \tag{3.7}$$

Given that the complex space $L^p[0,1]$ is strictly convex, equation (3.7) holds if and only if $\alpha > 0$ and $T_0(f)|_{A_i} = \alpha e^{i\theta_i} T_0(f_i)$. We will prove that $\alpha e^{i\theta_i} = a_i$.
Since $T_0(f)|_{A_i} = \alpha e^{i\theta_i} T_0(f_i)$ and $|a_i| = \left\| T_0(f)|_{A_i} \right\|$, we have:

$$\left\| T_0(f)|_{A_i} \right\| = \left| \alpha e^{i\theta_i} \right| \| T_0(f_i) \| = \left| \alpha e^{i\theta_i} \right| = |a_i|.$$

If $\alpha e^{i\theta_i} = -a_i$, then it contradicts the given condition. Hence, we conclude that $\alpha e^{i\theta_i} = a_i$. In conclusion, we have shown that $T_0(f)|_{A_i} = a_i T_0(f_i)$, which implies $V_0(f) = \sum_{i=1}^n a_i V_0(f_i)$ (where $a_i \in \mathbb{T}$).

Step 3: Let X denote the space of all simple functions in $L^p[0,1]$. We construct an operator on the space X as follows:

$$\mathrm{T}: L^p[0,1] \to L^p[0,1]$$

$$\mathrm{T}(f) = \begin{cases} \|f\| T_0\left(\frac{f}{\|f\|}\right), & \text{if } f \neq 0. \\ 0, & \text{if } f = 0. Here, f \in L^p[0,1]. \end{cases}$$

Let $L^p[0,1]$ be a complex function space. Consider any simple function in this space:

$$g = \sum_{i=1}^m a_i x_i \chi_{E_i}$$

where $E_i \cap E_j = \emptyset$ for $i \neq j$, $\{a_i\}_{i=1}^n \subset \mathbb{C}$, $\{x_i\}_{i=1}^n \subset L^p[0,1]$, and $a_i x_i \neq 0$.

(i) First, we prove that the operator T is isometric:

$$\|\mathrm{T}(g)\| = \| \|g\| T_0\left(\frac{g}{\|g\|}\right) \| = \|g\|.$$

(ii) We prove that the operator T is linear on the space X, using the definition of T and the properties of T_0 from Step 2:

$$\begin{aligned} \mathrm{T}(g) = \mathrm{T}\left(\sum_{i=1}^m a_i x_i \chi_{E_i}\right) &= \left\| \sum_{i=1}^m a_i x_i \chi_{E_i} \right\| T_0\left(\frac{\sum_{i=1}^m a_i x_i \chi_{E_i}}{\|\sum_{i=1}^m a_i x_i \chi_{E_i}\|}\right) \\ &= \left\| \sum_{i=1}^m a_i x_i \chi_{E_i} \right\| \frac{\sum_{i=1}^m a_i}{\|\sum_{i=1}^m a_i\|} T_0\left(\frac{x_i \chi_{E_i}}{\|x_i \chi_{E_i}\|}\right) \end{aligned}$$

$$= \sum_{i=1}^{m} a_i \|x_i\chi_{E_i}\| T_0\left(\frac{x_i\chi_{E_i}}{\|x_i\chi_{E_i}\|}\right) = \sum_{i=1}^{m} a_i \mathrm{T}(x_i\chi_{E_i}).$$

Thus, T is a linear isometry on the space X. Furthermore, since X is dense in $L^p[0,1]$, T being an isometry on the space X can be linearly extended to the entire space $L^p[0,1]$.

□

Lemma 3.5 is an important lemma in complex $L^p[0,1]$, and as a consequence, we obtain the following.

Corollary 3.6. *Let T_0 be a 1-Lipschitz mapping from $\mathrm{S}(L^p[0,1])$ to $\mathrm{S}(L^p[0,1])$, where $2 < p < \infty$, and $T_0(\lambda f) = \lambda T_0(f)$ for all $f \in \mathrm{S}(L^p[0,1])$ and $\lambda \in \mathbb{T}$, where $|\lambda| = 1$. If T_0 satisfies the condition*

$$-T_0(\mathrm{S}(L^p[0,1])) \subseteq T_0(\mathrm{S}(L^p[0,1])),$$

then T_0 can be isometrically extended to the entire complex $L^p[0,1]$ space.

Proof: Given the assumption on T_0 in the condition, for any $f \in \mathrm{S}(L^p[0,1])$, let $\tilde{f} \in \mathrm{S}(L^p[0,1])$ such that $T_0(\tilde{f}) = -T_0(f)$. Since T_0 is a 1-Lipschitz mapping, we have:

$$2 = \left\|T_0(\tilde{f}) - T_0(f)\right\| \leq \|\tilde{f} - f\| \leq \|\tilde{f}\| + \|f\| = 2.$$

Hence, $\left\|T_0(\tilde{f}) - T_0(f)\right\| = \|\tilde{f} - f\|$, which implies $\tilde{f} = -f$.
From $T_0(\tilde{f}) = -T_0(f)$, we can conclude that T_0 is an odd function, i.e., $T_0(-f) = -T_0(f)$ holds. Then, for any $f, g \in \mathrm{S}(L^p[0,1])$ (where $f \wedge g = 0$) satisfying condition (i) in Lemma 3.2, we examine the following expression:

$$\begin{aligned} 4 = 2(\|T_0(f)\|^p + 2\|T_0(g)\|^p) &\leq \|T_0(f) - T_0(g)\|^p + \|T_0(f) + T_0(g)\|^p \\ &= \|T_0(f) - T_0(g)\|^p + \|T_0(f) - T_0(-g)\|^p \\ &= \|f - g\|^p + \|f + g\|^p = 4. \end{aligned}$$

According to the equality condition in Lemma 3.2, we can see that equality holds if and only if $T_0(f) \wedge T_0(g) = 0$.

Therefore, by utilizing the conclusion in Lemma 3.5, we can deduce that T_0 can be uniquely linearly extended to the entire complex $L^p[0,1]$ space.

□

Theorem 3.7. *Let E be a complex Banach space, and let V_0 be a surjective isometry from* $\mathrm{S}(L^p[0,1])$ *to* $\mathrm{S}(\mathrm{E})$*, where* $2<p<\infty$*, satisfying* $V_0(\lambda f)=\lambda V_0(f)$ *for all* $f\in\mathrm{S}(L^p[0,1])$ *and* $\lambda\in\mathbb{T}$*, where* $|\lambda|=1$*. And* V_0 *satisfies* $V_0(a+x)-V_0(a)=V_0(b+x)-V_0(b)$ *for* $a,b,a+x,b+x\in\mathrm{S}(L^p[0,1])$ *with* $a\wedge x=0$ *and* $b\wedge x=0$*.Suppose there exist positive real numbers* α *and* β *such that* $\alpha^p+\beta^p=1$*. Let* f *be a fixed element on the unit sphere of* $\mathrm{S}(L^p[0,1])$*. Then, for any element* g *on the unit sphere that is orthogonal to* f*, we define the mapping:*

$$\Phi_f(g): g\mapsto H_g(\alpha f,\beta g).$$

Then, (i) the mapping $\Phi_f(g)$ *is a linear isometry. (ii) For all elements* g_1 *and* g_2 *in* $\mathrm{S}(L^p[0,1])$ *satisfying* $f\wedge g_1=0$ *and* $f\wedge g_2=0$*, we have* $H_f(\alpha f,\beta g_1)=H_f(\alpha f,\beta g_2)$*.*

Proof: (i) Let's first prove that $\Phi_f(\lambda g)=\lambda\Phi_f(g)$.

From the conditions and definitions, it is straightforward to see that $H_{\lambda g}(\alpha f,\beta\lambda g)=H_{\lambda g}(\alpha\lambda f,\beta\lambda g)=V_0^{-1}\left(\frac{V_0(\alpha\lambda f+\beta\lambda g)+V_0(\beta\lambda g-\alpha\lambda f)}{2\beta}\right)=\lambda H_g(\alpha f,\beta g)=\lambda\Phi_f(g)$. Therefore, we have shown that $\Phi_f(\lambda g)=\lambda\Phi_f(g)$.

Now, let $A_0=\operatorname{supp} f$. In fact, the defined mapping $\Phi_f(g)$ can be seen as a mapping from the unit sphere $\mathrm{S}(L^p[0,1]\backslash A_0)$ to the unit sphere $\mathrm{S}(L^p[0,1])$. According to Corollary 3.6, we only need to prove that this mapping is 1-Lipschitz and its range is symmetric.

To prove that $\Phi_f(g)$ is 1-Lipschitz: consider any $g_1,g_2\in\mathrm{S}(L^p[0,1])$ such that $f\wedge g_1=0$ and $f\wedge g_2=0$. We have:

$$\begin{aligned}
&\left\|\Phi_f(g_1)-\Phi_f(g_2)\right\|=\left\|H_{g_1}(\alpha f,\beta g_1)-H_{g_2}(\alpha f,\beta g_2)\right\|\\
=&\left\|V_0^{-1}\left(\frac{V_0(\alpha f+\beta g_1)+V_0(\beta g_1-\alpha f)}{2\beta}\right)-V_0^{-1}\left(\frac{V_0(\alpha f+\beta g_2)+V_0(\beta g_2-\alpha f)}{2\beta}\right)\right\|\\
=&\left\|\frac{V_0(\alpha f+\beta g_1)+V_0(\beta g_1-\alpha f)}{2\beta}-\frac{V_0(\alpha f+\beta g_2)+V_0(\beta g_2-\alpha f)}{2\beta}\right\|\\
\leq&\frac{1}{2\beta}\left(\beta\|g_1-g_2\|+\beta\|g_1-g_2\|\right)=\|g_1-g_2\|.
\end{aligned}\tag{3.8}$$

Therefore, we have shown that $\Phi_f(g)$ is 1-Lipschitz.

Next, we prove that the range of $\Phi_f(g)$ is symmetric:From the definition of $H_g(\alpha f,\beta g)$ and Lemma 3.3, we know that $H_{-g}(\alpha f,\beta(-g))=-H_g(\alpha f,\beta g)$. Therefore, by Corollary 3.6, $\Phi_f(g)$ is a linear isometry.

Now, we prove (ii). Since $\Phi_f(g)$ is an isometry, Equation (3.8) becomes an equality. Hence, we obtain:

$$V_0(\alpha f+\beta g_1)-V_0(\alpha f+\beta g_2)=V_0(\beta g_1-\alpha f)-V_0(\beta g_2-\alpha f).$$

Using Lemma 3.3, we have:

$$V_0(\alpha f+\beta g_1)+V_0(\alpha f-\beta g_1)=V_0(\alpha f+\beta g_2)+V_0(\alpha f-\beta g_2)$$

$$V_0^{-1}\left(\frac{V_0(\alpha f+\beta g_1)+V_0(\alpha f-\beta g_1)}{2\alpha}\right)=V_0^{-1}\left(\frac{V_0(\alpha f+\beta g_2)+V_0(\alpha f-\beta g_2)}{2\alpha}\right)$$

This implies that $H_f(\alpha f,\beta g_1)=H_f(\alpha f,\beta g_2)$. Thus, we have proved (ii). □

Theorem 3.8. *Let E be a complex Banach space, and let V_0 be a surjective isometry from* $\mathrm{S}(L^p[0,1])$ *to* $\mathrm{S}(\mathrm{E})$*, satisfying $V_0(\lambda f)=\lambda V_0(f)$ for all $f\in\mathrm{S}(L^p[0,1])$ and $\lambda\in\mathbb{T}$, where $|\lambda|=1$.And V_0 satisfies $V_0(a+x)-V_0(a)=V_0(b+x)-V_0(b)$ for $a,b,a+x,b+x\in\mathrm{S}(L^p[0,1])$ with $a\wedge x=0$ and $b\wedge x=0$. Suppose there exist positive real numbers α and β such that $\alpha^p+\beta^p=1$. Take a fixed element g on the unit sphere of $\mathrm{S}(L^p[0,1])$. Then, for any element f on the unit sphere that is orthogonal to g, we define the mapping:*

$$\Phi_g(f):f\mapsto H_f(\alpha f,\beta g).$$

Then, (i) the mapping $\Phi_g(f)$ is a linear isometry. (ii) For all elements f_1 and f_2 in $\mathrm{S}(L^p[0,1])$ satisfying $g\wedge f_1=0$ and $g\wedge f_2=0$, we have $H_g(\alpha f_1,\beta g)=H_g(\alpha f_2,\beta g)$.

Proof: The proof of Theorem 3.8 is similar to that of Theorem 3.7 and is omitted. □

To establish an important result in this section, we first introduce an important lemma that has been mentioned and proven in references [21, 22].

Lemma 3.9. *(Lamperti's Theorem): Let* U *be a linear isometry from $L^p(\Omega_1,\Sigma_1,\mu_1)$ to $L^p(\Omega_2,\Sigma_2,\mu_2)$, where $(1<\mathrm{p}<\infty$, and $\mathrm{p}\neq 2)$. Then, there exists a regular set isomorphism* T *from Σ_1 to Σ_2 and a function h defined on Ω_2 such that:*

$$Uf(t)=h(t)T_1(f), \tag{3.9}$$

where $f \in L^p(\Omega_1, \Sigma_1, \mu_1)$, T_1 is the linear transformation induced by T, and h satisfies:

$$\int_{TA} |h|^p d\mu_2 = \int_{TA} \frac{d\left(\mu_1 \circ T^{-1}\right)}{d\mu_2} d\mu_2 = \mu_1(A) \quad \forall A \in \Sigma_1.$$

Conversely, for any h and T *satisfying Equation (3.9), the corresponding U is an isometry.*

Theorem 3.10. *Let E be a complex Banach space, and let V_0 be a surjective isometry from* $\mathrm{S}(L^p[0,1])$ *to* $\mathrm{S}(\mathrm{E})$*, satisfying $V_0(\lambda f) = \lambda V_0(f)$ for all $f \in \mathrm{S}(L^p[0,1])$ and $\lambda \in \mathbb{T}$, where $|\lambda| = 1$.And V_0 satisfies $V_0(a+x) - V_0(a) = V_0(b+x) - V_0(b)$ for $a, b, a+x, b+x \in \mathrm{S}(L^p[0,1])$ with $a \wedge x = 0$ and $b \wedge x = 0$. Suppose there exist positive real numbers α and β such that $\alpha^p + \beta^p = 1$. For any measurable set $A \subset [0,1]$ with $0 < \mu(A) < 1$, define the mapping $\Phi_f(g) : g \mapsto H_g(\alpha f, \beta g)$ for all $f \in \mathrm{S}(L^p[0,1])$ such that $\mu(\operatorname{supp} f \cap A) = 0$. Then, the following equations hold:*

$$H_{\lambda \frac{\chi_A}{\mu(A)^{1/p}}}\left(\alpha\lambda \frac{\chi_A}{\mu(A)^{1/p}}, \beta f\right) = \lambda \frac{\chi_A}{\mu(A)^{1/p}} \quad , \quad H_{\lambda \frac{\chi_A}{\mu(A)^{1/p}}}\left(\alpha f, \beta\lambda \frac{\chi_A}{\mu(A)^{1/p}}\right) = \lambda \frac{\chi_A}{\mu(A)^{1/p}}.$$

Proof:

According to Lemma 3.4(iii), we have $H_{\lambda \frac{\chi_A}{\mu(A)^{1/p}}}\left(\alpha\lambda \frac{\chi_A}{\mu(A)^{1/p}}, \beta f\right) = \lambda H_{\frac{\chi_A}{\mu(A)^{1/p}}}\left(\alpha \frac{\chi_A}{\mu(A)^{1/p}}, \beta f\right)$ $= \lambda \frac{\chi_A}{\mu(A)^{1/p}}$. Therefore, it suffices to prove that $H_{\frac{\chi_A}{\mu(A)^{1/p}}}\left(\alpha \frac{\chi_A}{\mu(A)^{1/p}}, \beta f\right) = \frac{\chi_A}{\mu(A)^{1/p}}$, and similarly, it suffices to prove that $H_{\frac{\chi_A}{\mu(A)^{1/p}}}\left(\alpha f, \beta\lambda \frac{\chi_A}{\mu(A)^{1/p}}\right) = \frac{\chi_A}{\mu(A)^{1/p}}$.

Step 1: Let us first prove that for $\forall f, g \in S(L^p[0,1])$ such that $f \wedge g = 0$, the following statements hold:

$$\mu(\operatorname{supp} f \cap \operatorname{suppH}_f(\alpha f, \beta g)) > 0, \mu(\operatorname{supp} g \cap \operatorname{suppH}_g(\alpha f, \beta g)) > 0.$$

Indeed, without loss of generality, let's assume $\alpha \geq 2^{-1/p}$. Since V_0 is an isometry, we can examine the following:

(3.10)

$$\begin{aligned}
&\left\|H_f(\alpha f, \beta g) - f\right\| \\
&= \left\|V_0^{-1}\left(\frac{V_0(\alpha f + \beta g) + V_0(\alpha f - \beta g)}{2\alpha}\right) - f\right\| = \left\|\frac{V_0(\alpha f + \beta g) + V_0(\alpha f - \beta g)}{2\alpha} - V_0(f)\right\|, \\
&= \left\|\frac{V_0(\alpha f + \beta g) + V_0(\alpha f - \beta g)}{2\alpha} - \frac{\alpha V_0(f) + \alpha V_0(f)}{2\alpha}\right\|,
\end{aligned}$$

$$=\frac{1}{2\alpha}\|(1-\alpha)V_0(\alpha f+\beta g)+(1-\alpha)V_0(\alpha f-\beta g)+\alpha\,(V_0(\alpha f+\beta g)-V_0(f))+$$
$$\alpha\,(V_0(\alpha f-\beta g)-V_0(f))\,\|$$
$$\leq\frac{1}{2\alpha}\left[2(1-\alpha)+2\alpha\,((1-\alpha)^p+\beta^p)^{1/p}\right],$$
$$\leq 2^{1/p}-1+\left(\left(1-2^{-1/p}\right)^p+\frac{1}{2}\right)^{1/p}\leq 2^{1/p}.$$

(Where the function $\mathrm{h}(\alpha)=(1-\alpha)^p+1-\alpha^p$ is strictly decreasing on the interval $[0,1]$)

From this, it follows that $\mu(\operatorname{supp} f\cap \operatorname{supp}\mathrm{H}_f(\alpha f,\beta g))>0$. Furthermore, when $\alpha<2^{-1/p}$ and $\beta\geq 2^{-1/p}$, by Lemma 3.4 (i) which states $H_f(\alpha f,\beta g)=H_f(\beta f,\alpha g)$, we can still conclude that $\mu(\operatorname{supp} f\cap \operatorname{supp}\mathrm{H}_f(\alpha f,\beta g))>0$. Similarly, we can prove that $\mu(\operatorname{supp} g\cap \operatorname{supp}\mathrm{H}_g(\alpha f,\beta g))>0$.

Step 2: We will prove that for any measurable set $A\subset[0,1]$ and any function $f\in \mathrm{S}(L^p[0,1])$ satisfying $0\leq\mu(A)\leq 1$ and $\mu(\operatorname{supp} f\cap A)=0$, the following statements hold:

$$\operatorname{supp} H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}}\beta f)=A,\quad \operatorname{supp} H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha f,\beta\frac{\chi_A}{\mu(A)^{1/p}})=A.$$

To prove $\operatorname{supp} H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}},\beta f)=A$, let's first prove:

$$\mu(\operatorname{supp} H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}},\beta f)\cap A^C)=0.$$

Proof by contradiction: Let's assume that $\mu(\operatorname{supp} H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}},\beta f)\cap A^C)>0$. Then there exists a measurable set B, with $0<\mu(B)<1$, such that

$$B\subseteq \operatorname{supp} H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}},\beta f)\cap A^C.$$

With the chosen element $\frac{\chi_A}{\mu(A)^{1/p}}$, we know that $A\cap B=\emptyset$. Thus, we have $\frac{\chi_A}{\mu(A)^{1/p}}\wedge\frac{\chi_B}{\mu(B)^{1/p}}=0$ and $\left(\frac{\chi_A}{\mu(A)^{1/p}}\wedge f=0\right)$. Therefore, by applying Theorem 3.7(ii), we obtain

$$H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}},\beta\frac{\chi_B}{\mu(B)^{1/p}})=H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}},\beta f).$$

Therefore, $B\subseteq \operatorname{supp} H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}},\beta f)\cap A^C=\operatorname{supp} H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}},\beta\frac{\chi_B}{\mu(B)^{1/p}})\cap A^C$. Thus, we have $B\subseteq \operatorname{supp} H\frac{\chi_A}{\mu(A)^{1/p}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}},\beta\frac{\chi_B}{\mu(B)^{1/p}})$.

On the other hand, from Step 1, we know that

$$\mu(\operatorname{supp}\frac{\chi_B}{\mu(B)^{1/p}}\cap \operatorname{supp} H_{\frac{\chi_B}{\mu(B)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}},\beta\frac{\chi_B}{\mu(B)^{1/p}}))>0.$$

That is, $\mu\left(B\cap \operatorname{supp} H_{\frac{\chi_B}{\mu(B)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}},\beta\frac{\chi_B}{\mu(B)^{1/p}})\right)>0$.
Furthermore, since $B\subseteq \operatorname{supp} H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}},\beta\frac{\chi_B}{\mu(B)^{1/p}})$, we have

$$\mu\left(\operatorname{supp} H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}},\beta\frac{\chi_B}{\mu(B)^{1/p}})\cap \operatorname{supp} H_{\frac{\chi_B}{\mu(B)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}},\beta\frac{\chi_B}{\mu(B)^{1/p}})\right)>0. \tag{3.11}$$

However, according to Lemma 3.4 (ii), we know that $\frac{\chi_A}{\mu(A)^{1/p}},\frac{\chi_B}{\mu(B)^{1/p}}\in \mathrm{S}(L^p[0,1])$ and $\frac{\chi_A}{\mu(A)^{1/p}}\wedge \frac{\chi_B}{\mu(B)^{1/p}}=0$. Hence, we have:

$$H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}}\beta\frac{\chi_B}{\mu(B)^{1/p}})\perp H_{\frac{\chi_B}{\mu(B)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}}\beta\frac{\chi_B}{\mu(B)^{1/p}}).$$

This contradicts equation (3.11), and thus our assumption is not valid. Therefore, we conclude that

$$\mu(\operatorname{supp} H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}},\beta f)\cap A^C)=0.$$

Similarly, we can prove that $\mu(\operatorname{supp} H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha f,\beta\frac{\chi_A}{\mu(A)^{1/p}})\cap A^C)=0$. Next, we will prove that the following situation cannot occur:

$$\mu(\operatorname{supp} H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}},\beta f))<\mu(A).$$

Proof by contradiction: Assume that $\mu(\operatorname{supp} H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}},\beta f))<\mu(A)$.
Let $A_1=\operatorname{supp} H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}},\beta f)\cap A$ and $A_2=A\setminus A_1$. (Clearly, $A_1\cap A_2=\emptyset$.) Then, we have $\frac{\chi_A}{\mu(A)^{1/p}}=\frac{\chi_{A_1}+\chi_{A_2}}{\mu(A)^{1/p}}=\frac{\mu(A_1)^{1/p}}{\mu(A)^{1/p}}\frac{\chi_{A_1}}{\mu(A_1)^{1/p}}+\frac{\mu(A_2)^{1/p}}{\mu(A)^{1/p}}\frac{\chi_{A_2}}{\mu(A_2)^{1/p}}$.
According to Theorem 3.7 (i), the operator $H_g(\alpha f,\beta g)$ is a linear isometry. Therefore, we have:

$$H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}},\beta f)=\frac{\mu(A_1)^{1/p}}{\mu(A)^{1/p}}H_{\frac{\chi_{A_1}}{\mu(A_1)^{1/p}}}(\alpha\frac{\chi_{A_1}}{\mu(A_1)^{1/p}},\beta f)+\frac{\mu(A_2)^{1/p}}{\mu(A)^{1/p}}H_{\frac{\chi_{A_2}}{\mu(A_2)^{1/p}}}(\alpha\frac{\chi_{A_2}}{\mu(A_2)^{1/p}},\beta f). \tag{3.12}$$

Moreover, from the conclusion in Step 1, we have $\mu(\operatorname{supp} H\frac{\chi_{A_2}}{\mu(A_2)^{1/p}}(\alpha\frac{\chi_{A_2}}{\mu(A_2)^{1/p}},\beta f)\cap A_2)>0$ holds. Thus, we can deduce from equation (3.12):

$$\mu(\operatorname{supp} H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}},\beta f)\cap A_2)>0.$$

This contradicts the definition of A_2 as $A\setminus A_1$. Hence, our assumption is not valid.

Step 3: Finally, we prove that

$$H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}},\beta f)=\frac{\chi_A}{\mu(A)^{1/p}}\quad\text{and}\quad H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha f,\beta\frac{\chi_A}{\mu(A)^{1/p}})=\frac{\chi_A}{\mu(A)^{1/p}}.$$

By using Lemma 3.9 (Lamperti's theorem), for all linear isometries U on the space $L^p[0,1] \to L^p[0,1]$, there exists a unified representation:

$$Uf(t) = h(t)T_1(f),$$

where $f \in L^p[0,1]$ and T_1 is a linear transformation induced by T.

According to Theorem 3.7 (i), we know that $H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}}\beta f) = \Phi_f(\frac{\chi_A}{\mu(A)^{1/p}})$ is a linear isometry. By Lemma 3.9, let T be a regular set isomorphism associated with Φ_f, and T_1 be the linear transformation induced by T. Note that for any $A \subset [0,1]$ with $0 \leq \mu(A) \leq 1$, we have $T_1(\chi_A) = \chi_{T(A)}$ and $T(A) = A$ due to the properties of the linear transformation T_1. Then, we have

$$\Phi_f(\frac{\chi_A}{\mu(A)^{1/p}}) = hT_1(\frac{\chi_A}{\mu(A)^{1/p}}) = h\frac{\chi_A}{\mu(A)^{1/p}}. \tag{3.13}$$

Here, h satisfies the following equation:

$$\int_A |h|^p d\mu = \int_{TA} |h|^p d\mu = \int_A \frac{d(\mu \circ T^{-1})}{d\mu} d\mu = \mu(A).$$

Hence, we have $|h| = 1$ a.e.

Furthermore, since in Step 1 we have $\|H_f(\alpha f, \beta g) - f\| \leq 2^{1/p}$, we can conclude that

$$\|\Phi_f(\frac{\chi_A}{\mu(A)^{1/p}}) - \frac{\chi_A}{\mu(A)^{1/p}}\| = \|h\frac{\chi_A}{\mu(A)^{1/p}} - \frac{\chi_A}{\mu(A)^{1/p}}\| \leq 2^{1/p}.$$

Therefore, we have $h = 1$. Combining this with equation (3.13), we obtain $\Phi_f(\frac{\chi_A}{\mu(A)^{1/p}}) = H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha\frac{\chi_A}{\mu(A)^{1/p}}, \beta f) = \frac{\chi_A}{\mu(A)^{1/p}}$.

Similarly, we can show that $H_{\frac{\chi_A}{\mu(A)^{1/p}}}(\alpha f, \beta\frac{\chi_A}{\mu(A)^{1/p}}) = \frac{\chi_A}{\mu(A)^{1/p}}$.

Therefore, we have proven the conclusion. □

Theorem 3.11. *Let E be a complex Banach space, $V_0 : S(L^p[0,1]) \to S(E)$ be a surjective isometry, where $2 < p < \infty$.Satisfying $V_0(\lambda f) = \lambda V_0(f)$ for all $f \in \mathrm{S}(L^p[0,1])$ and $\lambda \in \mathbb{T}$, where $|\lambda| = 1$.And V_0 satisfies $V_0(a+x) - V_0(a) = V_0(b+x) - V_0(b)$ for $a, b, a+x, b+x \in \mathrm{S}(L^p[0,1])$ with $a \wedge x = 0$ and $b \wedge x = 0$.Then V_0 can be linearly and isometrically extended to the entire space $L^p[0,1]$.*

Proof: The proof is divided into two steps.

Step 1: Examining $V_0(f)=V_0\left(a_1\frac{\chi_{A_1}}{\mu(A_1)^{1/p}}+g\right)$

$$
\begin{aligned}
&=|a_1|\frac{V_0\left(|a_1|\frac{a_1}{|a_1|}\cdot\frac{\chi_{A_1}}{\mu(A_1)^{1/p}}+\|g\|\frac{g}{\|g\|}\right)+V_0\left(-\|g\|\frac{g}{\|g\|}+|a_1|\frac{a_1}{|a_1|}\cdot\frac{\chi_{A_1}}{\mu(A_1)^{1/p}}\right)}{2|a_1|}+\\
&\quad\|g\|\frac{V_0\left(|a_1|\frac{a_1}{|a_1|}\cdot\frac{\chi_{A_1}}{\mu(A_1)^{1/p}}+\|g\|\frac{g}{\|g\||}\right)+V_0\left(-|a_1|\frac{a_1}{|a_1|}\cdot\frac{\chi_{A_1}}{\mu(A_1)^{1/p}}+\|g\|\frac{g}{\|g\|}\right)}{2\|g\|},\\
&=a_1V_0\left(\frac{\chi_{A_1}}{\mu(A_1)^{1/p}}\right)+\|g\|V_0\left(\frac{g}{\|g\|}\right)
\end{aligned}
$$

Step 2: Let X denote the space of all simple functions in $L^p[0,1]$. Consider any simple function in the complex $L^p[0,1]$ space:

$$
g=\sum_{i=1}^{n}\lambda_i\frac{\chi_{A_i}}{\mu(A_i)^{1/p}}\in L^p[0,1]
$$

where $\{\lambda_i\}_{i=1}^n\subset\mathbb{C}$ and $\{A_i\}_{i=1}^n$ is a sequence of measurable sets satisfying $0\leq\mu(A_i)<1$ for $1\leq i\leq n$, and $\mu\left(A_i\cap A_j\right)=0$ for $i\neq j$. We construct an operator V on the space X as follows: $\mathrm{V}:L^p[0,1]\to E$

$$
\mathrm{V}(g)=\mathrm{V}\left(\sum_{i=1}^{n}\lambda_i\frac{\chi_{A_i}}{\mu(A_i)^{1/p}}\right)=\sum_{i=1}^{n}\lambda_iV_0\left(\frac{\chi_{A_i}}{\mu(A_i)^{1/p}}\right)g\in L^p[0,1].
$$

We will prove that the operator V is isometric:

$$
\begin{aligned}
\|\mathrm{V}(g)\|&=\left\|\mathrm{V}\left(\sum_{i=1}^{n}\lambda_i\frac{\chi_{A_i}}{\mu(A_i)^{1/p}}\right)\right\|=\left\|\sum_{i=1}^{n}|\lambda_i|V_0\left(\frac{\lambda_i}{|\lambda_i|}\frac{\chi_{A_i}}{\mu(A_i)^{1/p}}\right)\right\|,\\
&=\|g\|\left\|\sum_{i=1}^{n}\frac{|\lambda_i|}{\|g\|}V_0\left(\frac{\lambda_i}{|\lambda_i|}\frac{\chi_{A_i}}{\mu(A_i)^{1/p}}\right)\right\|,\\
&=\|g\|\left\|V_0\left(\sum_{i=1}^{n}\frac{\lambda_i}{\|g\|}\frac{\chi_{A_i}}{\mu(A_i)^{1/p}}\right)\right\|=\|g\|.
\end{aligned}
$$

Hence, V is a linear isometry on the space X. Since X is dense in $L^p[0,1]$ and both $L^p[0,1]$ and E are complete, V_0 can be linearly extended to the entire space $L^p[0,1]$. □

Next, we will consider the case of$1<\mathbf{p}<2$,The following Lemmas 3.12 and 3.13 are similar to the case of $2<p<\infty$ in the first part and their proof processes are omitted.

Lemma 3.12. *Let* E *be a complex Banach space,* V_0*:* $\mathrm{S}(L^p[0,1]) \to \mathrm{S}(\mathrm{E})$ *be a surjective isometry, and let* $f,g \in \mathrm{S}(L^p[0,1])$ *such that* $f \wedge g = 0$*. And suppose* V_0 *satisfies* $V_0(a+x) - V_0(a) = V_0(b+x) - V_0(b)$ *for* $a,b,a+x,b+x \in \mathrm{S}(L^p[0,1])$ *with* $a \wedge x = 0$ *and* $b \wedge x = 0$*. Moreover, let* α *and* β *be positive real numbers satisfying* $\alpha^p + \beta^p = 1$*. Then: (i)* $H_g(\alpha f, \beta g) = H_g(\beta f, \alpha g)$ *and* $H_f(\alpha f, \beta g) = H_f(\beta f, \alpha g)$*. (ii)* $H_g(\alpha f, \beta g) \perp H_f(\alpha f, \beta g)$*. (iii)* $H_g\left(\alpha \frac{f}{\lambda'}, \beta g\right) = H_g(\alpha f, \beta g)$*, where* $\lambda \in \mathbb{T}$ *and* $|\lambda| = 1$*.*

Lemma 3.13. *Let* T_0 *be a 1-Lipschitz mapping from* $\mathrm{S}(L^p[0,1])$ *to* $\mathrm{S}(L^p[0,1])$*, satisfying* $T_0(\lambda f) = \lambda T_0(f)$ *for all* $f \in \mathrm{S}(L^p[0,1])$ *and* $\lambda \in \mathbb{T}$*.If* $f \wedge g = 0$ *implies* $T_0(f) \wedge T_0(g) = 0$ *for* $f,g \in \mathrm{S}(L^p[0,1])$*, then* T_0 *can be extended to a linear isometry on the entire space.*

Lemma 3.14. *Let* E *be a complex Banach space, and* V_0*:* $\mathrm{S}(L^p[0,1]) \to \mathrm{S}(\mathrm{E})$ *be a surjective isometry, where* $1 < p < 2$*. And suppose* V_0 *satisfies* $V_0(a+x) - V_0(a) = V_0(b+x) - V_0(b)$ *for* $a,b,a+x,b+x \in \mathrm{S}(L^p[0,1])$ *with* $a \wedge x = 0$ *and* $b \wedge x = 0$*.Let* $\alpha_0, \beta_0, \gamma_0 > 0$ *satisfy* $\alpha_0^p + \beta_0^p + \gamma_0^p = 1$*. For any sequence of elements* $\{g_i\}_{i=1}^n \subset \mathrm{S}(L^p[0,1])$ *such that* $g_i \perp g_j$ *for* $i \neq j$ *and* $i,j = 0,1,2$*, we have:*

$$H_{g_1}(\alpha_0 g_0 + \gamma_0 g_2, \beta_0 g_1) \perp H_{g_2}(\alpha_0 g_0 - \gamma_0 g_1, \beta_0 g_2).$$

Proof:

Using a similar proof method as in Lemma 3.4 (ii), we can obtain the conclusion. □

Lemma 3.15. *Let* E *be a complex Banach space, and* V_0*:* $\mathrm{S}(L^p[0,1]) \to \mathrm{S}(\mathrm{E})$ *be a surjective isometry.And suppose* V_0 *satisfies* $V_0(a+x) - V_0(a) = V_0(b+x) - V_0(b)$ *for* $a,b,a+x,b+x \in \mathrm{S}(L^p[0,1])$ *with* $a \wedge x = 0$ *and* $b \wedge x = 0$*. In this case, when* $1 < p < 2$*, consider a fixed element* g_0*. Let* g_1 *and* g_2 *(where* $\{g_i\}_{i=1}^n \subset \mathrm{S}(L^p[0,1])$*) be elements orthogonal to* g_0*. Then, for any positive numbers* α *and* β *satisfying* $\alpha^p + \beta^p = 1$*, we have:*

$$H_{g_1}(\alpha g_0, \beta g_1) \perp H_{g_2}(\alpha g_0, \beta g_2).$$

Proof:

The conclusion obtained from Lemma 3.14 is $H_{g_1}(\alpha_0 g_0 + \gamma_0 g_2, \beta_0 g_1) \perp$

$H_{g_2}(\alpha_0 g_0 - \gamma_0 g_1, \beta_0 g_2)$, which can be written as:

$$\begin{aligned} & V_0^{-1}\left(\frac{V_0(\alpha_0 g_0 + \gamma_0 g_2 + \beta_0 g_1) + V_0(\beta_0 g_1 - \alpha_0 g_0 - \gamma_0 g_2)}{2\beta_0}\right) \\ = & V_0^{-1}\left(\frac{V_0(\alpha_0 g_0 - \gamma_0 g_1 + \beta_0 g_2) + V_0(\beta_0 g_2 - \alpha_0 g_0 + \gamma_0 g_1)}{2\beta_0}\right) \end{aligned}$$

By letting $\gamma_0 \to 0$, $\alpha_0 \to \alpha$, and $\beta_0 \to \beta$, we can deduce:

$$V_0^{-1}\left(\frac{V_0(\alpha g_0 + \beta g_1) + V_0(\beta g_1 - \alpha g_0)}{2\beta}\right) = V_0^{-1}\left(\frac{V_0(\alpha g_0 + \beta g_2) + V_0(\beta g_2 - \alpha g_0)}{2\beta}\right)$$

This implies that $H_{g_1}(\alpha g_0, \beta g_1) \perp H_{g_2}(\alpha g_0, \beta g_2)$. □

Theorem 3.16. *Let E be a complex Banach space, and V_0: $\mathrm{S}(L^p[0,1]) \to \mathrm{S(E)}$ be a surjective isometry, where $1 < p < 2$. Suppose $V_0(\lambda f) = \lambda V_0(f)$ for all $f \in \mathrm{S}(L^p[0,1])$ and $\lambda \in \mathbb{T}$.And suppose V_0 satisfies $V_0(a+x) - V_0(a) = V_0(b+x) - V_0(b)$ for $a, b, a+x, b+x \in \mathrm{S}(L^p[0,1])$ with $a \wedge x = 0$ and $b \wedge x = 0$.For positive real numbers α and β satisfying $\alpha^p + \beta^p = 1$, consider a fixed element f on the unit sphere $S(L^p[0,1])$. Then, for any element g orthogonal to f on the unit sphere, we define the mapping:*

$$\Phi_f(g): \quad g \mapsto H_g(\alpha f, \beta g)$$

Then: (i) The mapping $\Phi_f(g)$ is a linear isometry. (ii) For all g_1 and g_2 in $\mathrm{S}(L^p[0,1])$ satisfying $f \wedge g_1 = 0$ and $f \wedge g_2 = 0$, we have $H_f(\alpha f, \beta g_1) = H_f(\alpha f, \beta g_2)$.

Proof:

(i) To begin with, we prove that $\Phi_f(\lambda g) = \lambda \Phi_f(g)$ for any $\lambda \in \mathbb{C}$ and $\Phi_f(g)$ is 1-Lipschitz. The proof of these properties is similar to that of Theorem 3.7, and we omit it here.

Moreover, according to Lemma 3.15, for $1 < p < 2$, if we fix an g_0 in $\mathrm{S}(L^p[0,1])$, then for any g_1 and g_2 orthogonal to g_0 in $\mathrm{S}(L^p[0,1])$, when $g_1 \perp g_2$, we have $H_{g_1}(\alpha g_0, \beta g_1) \perp H_{g_2}(\alpha g_0, \beta g_2)$. In other words, $\Phi_{g_0}(g_1) \perp \Phi_{g_0}(g_2)$ holds. Since the conditions of Lemma 3.13 are satisfied, we can conclude that $\Phi_f(g)$ can be extended to the entire space as a linear isometry.

(ii) Similar to Theorem 3.7, we will not provide the proof here. □

Theorem 3.16 holds, and we can analogously prove that Theorem 3.8 remains valid for $1 < p <$

2. Following the same proof process as in the previous case with Theorems 3.9 and 3.10, we can obtain the final conclusion of this section.

Theorem 3.17. *Let E be a complex Banach space, and V_0: $S(L^p[0,1]) \to S(E)$ be a surjective isometry, where $1 < p < 2$.Suppose $V_0(\lambda f) = \lambda V_0(f)$ for all $f \in \mathrm{S}(L^p[0,1])$ and $\lambda \in \mathbb{T}$.And suppose V_0 satisfies $V_0(a+x) - V_0(a) = V_0(b+x) - V_0(b)$ for $a,b,a+x,b+x \in \mathrm{S}(L^p[0,1])$ with $a \wedge x = 0$ and $b \wedge x = 0$. Then, V_0 can be extended as a linear isometry to the entire space $L^p[0,1]$.*

Conflict of Interests

The author declares that there is no conflict of interests.

References

[1] S. Mazur, S. Ulam, Sur less transformations isométriques d'espaces vectoriels normés, C. R. Acad. Sci. Paris, 194 (1932), 946–948.

[2] D. Tingley, Isometries of the unit sphere, Geom. Dedicata. 22 (1987), 371–378. https://doi.org/10.1007/bf00147942.

[3] L. Cheng, Y. Dong, On a generalized Mazur-Ulam question: Extension of isometries between unit spheres of Banach spaces, J. Math. Anal. Appl. 377 (2011), 464–470. https://doi.org/10.1016/j.jmaa.2010.11.025.

[4] G.G. Ding, The isometric extension of the into mapping from a $\mathscr{L}^\infty(\Gamma)$-type space to some Banach space, Illinois J. Math. 51 (2007), 445–453. https://doi.org/10.1215/ijm/1258138423.

[5] D.N. Tan, Isometries of the unit spheres of the Tsirelson space T and the modified tsirelson space T_M, Houston J. Math. 38 (2012), 571–581.

[6] D.N. Tan, Some new properties and isometries on the unit spheres of generalized James spaces J_p, J. Math. Anal. Appl. 393 (2012), 457–469. https://doi.org/10.1016/j.jmaa.2012.03.024.

[7] G.G. Ding, On the extension of isometries between unit spheres of E and C(Ω), Acta Math. Sin. 19 (2003), 793–800. https://doi.org/10.1007/s10114-003-0240-z.

[8] X.N. Fang, J.H. Wang, On the extension of isometries between unit spheres of E and C(Ω), Acta Math. Sin. 22 (2006), 1819–1824.

[9] R. Liu, L. Zhang, On extension of isometries and approximate isometries between unit spheres, J. Math. Anal. Appl. 352 (2009), 749–761. https://doi.org/10.1016/j.jmaa.2008.11.034.

[10] D. Tan, X. Huang, R. Liu, Generalized-lush spaces and the Mazur-Ulam property, Stud. Math. 219 (2013), 139–153. https://doi.org/10.4064/sm219-2-4.

[11] D.N. Tan, Extension of isometries on the unit sphere of L^p spaces, Acta. Math. Sin. 28 (2011) 1197–1208. https://doi.org/10.1007/s10114-011-0302-6.

[12] X. Yang, On extension of isometries between unit spheres of $L_p(\nu,H)$ ($1 < p \neq 2$, H is a Hilbert space), J. Math. Anal. Appl. 323 (2006), 985–992. https://doi.org/10.1016/j.jmaa.2005.11.013.

[13] R. Wang, Isometries between normed spaces which are surjective on a sphere, Illinois J. Math. 53 (2009), 575–580. https://doi.org/10.1215/ijm/1266934793.

[14] G. Ding, The isometric extension problem in the unit spheres $l^p(\Gamma)(\mathrm{P} > 1)$ type spaces, Science in China, Sci. China Ser. A-Math. 46 (2003), 333–338.

[15] X.N. Fang, J.H. Wang, Extension of isometries between unit spheres of normed space E and $l^1(\Gamma)$, Acta Math. Sin. Chinese Ser. 51 (2008), 24–28.

[16] G.G. Ding, Extension of isometries on the unit spheres of AL-spaces. Sci. China Ser. A, 51 (2008), 1904–1918.

[17] S. Banach, Theorie des operations lineaires, Z Subwencji Funduszu Kultury Narodowej, Warszawa, 1932.

[18] L. Cheng, Y. Dong, On a generalized Mazur–Ulam question: Extension of isometries between unit spheres of Banach spaces, J. Math. Anal. Appl. 377 (2011), 464–470. https://doi.org/10.1016/j.jmaa.2010.11.025.

[19] J. Yi, R. Wang, X. Wang, Extension of isometries between the unit spheres of complex $l^p(\Gamma)(p > 1)$ spaces, Acta Math. Sci. 34 (2014), 1540–1550. https://doi.org/10.1016/s0252-9602(14)60102-8.

[20] X. Fang, J. Wang, Extension of isometries on the unit sphere of $l^p(\Gamma)$ space, Sci. China Math. 53 (2010), 1085–1096. https://doi.org/10.1007/s11425-010-0028-4.

[21] J. Lamperti, On the isometric of certain function spaces, Pac. J. Math. 8(1958), 459-466.

[22] R.J. Fleming, J.E. Jamison, Isometries on Banach spaces: function spaces, Chapman and Hall/CRC, New York, 2002. https://doi.org/10.1201/9781420026153.