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SOME NEW FIXED POINTS RESULTS FOR A CLASS OF SET-VALUED MAPPINGS IN METRIC SPACES

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Abstract. In this article, a wider class of set-valued mappings is introduced, and a fixed point theorem for this new mapping in a metric space is proved. Then, we derive a number of implications from our main finding. We also present two non-trivial examples to support our primary theorem. Moreover, we look into fixed point set stability for set-valued mappings and well-posedness. Finally, we present an application to integral inclusion problem.

Keywords: metric space; set-valued mappings; fixed points; stability.

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1. Introduction and Preliminaries

One of the most significant results in metric fixed point theory is the Banach contraction theorem (BCT). In many branches of science and technology, it has been extensively employed. Numerous mathematicians have expanded and generalized the BCT in different ways and settings (cf. [4, 11, 15, 17, 14, 18] and reference thereof). The BCT was extended to set-valued mappings by Nadler Jr. [11] in 1969 (see Theorem 1.1). Nadler's conclusion was expanded upon in 1972 by Ciric [3] to a larger class of set-valued mappings (see Theorem 1.2). This

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study examines a broader class of set-valued mappings and derives certain fixed point conclusions. Some illustrative examples support our findings. We also investigate fixed point set stability for set-valued mappings.

We recollect some notations, definitions, and findings from the literature [3, 11, 17]. Throughout this paper, $(\mathscr{M},d)$ denotes a metric space, $CB(\mathscr{M})$ the collection of all nonempty closed and bounded subsets of M and $C(\mathscr{M})$ the collection of all nonempty compact subsets of $\mathscr{M}$. As seen below, the Hausdorff metric $\mathscr{H}$ induced by d is

$$\mathscr{H}(A,B) = \max\left\{\sup_{\alpha\in A}\mathscr{D}(\alpha,B),\ \sup_{\beta\in B}\mathscr{D}(\beta,A)\right\},$$

for all $A,B\in CB(\mathscr{M})$, where $\mathscr{D}(\alpha,B)=\inf_{\beta\in B} d(\alpha,\beta)$. Let $\xi : M\to CB(\mathscr{M})$ be a set-valued (or multi-valued) mapping. A point $z\in M$ is said to be a fixed point of ξ if $z\in\xi(z)$ and strict fixed point of ξ if $\{z\}=\xi(z)$. $F(\xi)$ and $SF(\xi)$ stand for the set of all fixed points and set of strict fixed point of ξ, respectively.

Theorem 1.1. [11]. *Consider $(\mathscr{M},d)$ is a complete metric space and $\xi : M\to CB(\mathscr{M})$ a set-valued mapping such that for all $\alpha,\beta\in M$,*

$$\mathscr{H}(\xi(\alpha),\xi(\beta)) \le k\, d(\alpha,\beta), \tag{1.1}$$

where $k\in[0,1)$. Then ξ has a fixed point.

Theorem 1.2. [3]. *Suppose $(\mathscr{M},d)$ is a complete metric space and $\xi : M\to CB(\mathscr{M})$ a set-valued mapping such that for all $\alpha,\beta\in M$,*

$$\mathscr{H}(\xi(\alpha),\xi(\beta)) \le k\, m(\alpha,\beta), \tag{1.2}$$

where k as in Theorem 1.1 *and*

$$m(\alpha,\beta) = \max\left\{d(\alpha,\beta),\mathscr{D}(\alpha,\xi(\alpha)),\mathscr{D}(\beta,\xi(\beta)),\frac{\mathscr{D}(\alpha,\xi(\beta))+\mathscr{D}(\beta,\xi(\alpha))}{2}\right\}.$$

Then ξ has a fixed point.

Definition 1.3. [14]. Let $\eta : [0,\infty)\to[0,\infty)$ is such that $\eta(t)<t$ for all $t>0$, and $\limsup_{s\to t^+}\eta(s)< t$.

2. MAIN RESULTS

Theorem 2.1. *Suppose* $(\mathscr{M},d)$ *is a complete metric space and* $\xi : M \to CB(\mathscr{M})$ *a set-valued mapping such that for all* $\alpha,\beta \in M$,

$$\frac{1}{2}\mathscr{D}(\alpha,\xi(\alpha)) \leq d(\alpha,\beta) \text{ implies } \mathscr{H}(\xi(\alpha),\xi(\beta)) \leq \eta(m(\alpha,\beta)), \tag{2.1}$$

where $m(\alpha,\beta)$ *is as in Theorem* 1.2 *and* η *is as in Definition* 1.3.

Then ξ has a fixed point.

Proof. Let $\alpha_1 \in \mathscr{M}$ and $\alpha_2 \in \xi(\alpha_1)$. Since $\frac{1}{2}\mathscr{D}(\alpha_1,\xi(\alpha_1)) \leq \mathscr{D}(\alpha_1,\xi(\alpha_1)) \leq d(\alpha_1,\alpha_2)$ by (2.1), we have

$$\begin{aligned}
\mathscr{D}(\alpha_2,\xi(\alpha_2)) &\leq \mathscr{H}(\xi(\alpha_1),\xi(\alpha_2)) \\
&\leq \eta\left(\max\left\{d(\alpha_1,\alpha_2),\mathscr{D}(\alpha_1,\xi(\alpha_1)),\mathscr{D}(\alpha_2,\xi(\alpha_2)),\frac{\mathscr{D}(\alpha_1,\xi(\alpha_2))+0}{2}\right\}\right) \\
&< \max\left\{d(\alpha_1,\alpha_2),\mathscr{D}(\alpha_2,\xi(\alpha_2)),\frac{d(\alpha_1,\alpha_2)+\mathscr{D}(\alpha_2,\xi(\alpha_2))}{2}\right\}.
\end{aligned}$$

This implies $\mathscr{D}(\alpha_2,\xi(\alpha_2)) < d(\alpha_1,\alpha_2)$. So there exists $\alpha_3 \in \xi(\alpha_2)$ such that

$$d(\alpha_2,\alpha_3) < d(\alpha_1,\alpha_2).$$

Continuing this way, we can construct a sequence (α_n) in $\mathscr{M}$ such that $\alpha_{n+1} \in \xi(\alpha_n)$ and $d_{n+1} < d_n$, where $d_n := d(\alpha_n,\alpha_{n+1})$. This implies that

$$d_{n+1} \leq \eta(d_n) < d_n.$$

It is evident that the sequences (d_n) and $(\eta(d_n))$ are bounded below and monotone decreasing. Therefore both the real sequences converge. A standard argument shows that $\lim_{n\to\infty} d_n = 0$ and the sequence (α_n) is Cauchy. The completeness of $\mathscr{M}$ implies that (α_n) converges to some point in $z \in \mathscr{M}$. Now, we show that

$$\text{either } \frac{1}{2}d(\alpha_n,\alpha_{n+1}) \leq d(\alpha_n,z) \text{ or } \frac{1}{2}d(\alpha_{n+1},\alpha_{n+2}) \leq d(\alpha_{n+1},z), \tag{2.2}$$

for each $n \in \mathbb{N}$. By inference and contradiction, we assume that

$$\frac{1}{2}d(\alpha_n,\alpha_{n+1}) > d(\alpha_n,z) \text{ and } \frac{1}{2}d(\alpha_{n+1},\alpha_{n+2}) > d(\alpha_{n+1},z)$$

for each $n \in \mathbb{N}$. As a result of the triangle inequality, we have

$$\begin{aligned} d(\alpha_n, \alpha_{n+1}) \leq d(\alpha_n, z) + d(z, \alpha_{n+1}) \\ &< \frac{1}{2} d(\alpha_n, \alpha_{n+1}) + \frac{1}{2} d(\alpha_{n+1}, \alpha_{n+2}) \\ &< \frac{1}{2} d(\alpha_n, \alpha_{n+1}) + \frac{1}{2} d(\alpha_n, \alpha_{n+1}) = d(\alpha_n, \alpha_{n+1}). \end{aligned}$$

This contradicts itself. The inequality (2.2) is valid for $n \in \mathbb{N}$. In the first scenario, $\frac{1}{2}\mathscr{D}(\alpha_n, \xi(\alpha_n)) \leq \frac{1}{2} d(\alpha_n, \alpha_{n+1}) \leq d(\alpha_n, z)$ by (2.1), we have

$$\mathscr{D}(\alpha_{n+1}, \xi(z)) \leq \mathscr{H}(\xi(\alpha_n), \xi(z)) \leq \eta\left(m(\alpha_n, z)\right).$$

We obtain by adding $n \to \infty$,

$$\mathscr{D}(z, \xi(z)) \leq \lim_{n\to\infty} \eta(m(\alpha_n, z)).$$

Also $\lim_{n\to\infty} m(\alpha_n, z) = \mathscr{D}(z, \xi(z))$. Let $\lambda = \mathscr{D}(z, \xi(z))$. Then by $\limsup_{s\to t^+} \eta(s) < t$ for all $t > 0$, we obtain

$$\lambda \leq \lim_{n\to\infty} \eta(m(\alpha_n, z)) \leq \lim_{\delta\to+0} \sup_{s\in(\lambda,\lambda+\delta)} \eta(s) < \lambda.$$

Therefore, unless $\mathscr{D}(z, \xi(z)) = 0$, is a contradiction. This suggests that $z \in \xi(z)$. In the other scenario, we can conclude that $z \in \xi(z)$. □

If we replace $m(\alpha, \beta) = \max\{d(\alpha, \beta), d(\alpha, \xi(\alpha)), d(\beta, \xi(\beta))\}$ in Theorem 2.1 then we get the following result.

Corollary 2.2. *Assuming that $(\mathscr{M}, d)$ is a complete metric space and $\xi : M \to CB(\mathscr{M})$ a set-valued mapping such that*

$$\frac{1}{2}\mathscr{D}(\alpha, \xi(\alpha)) \leq d(\alpha, \beta) \textit{ implies } \mathscr{H}(\xi(\alpha), \xi(\beta)) \leq \eta(\max\{d(\alpha, \beta), d(\alpha, \xi(\alpha)), d(\beta, \xi(\beta))\}).$$

for all $\alpha, \beta \in \mathscr{M}$, where η is as in Definition 1.3. *Then ξ has a fixed point.*

Similarly, if we replace $m(\alpha, \beta) = d(\alpha, \beta)$ in Theorem 2.1 then we get the following result.

Corollary 2.3. *Assuming that* $(\mathscr{M},d)$ *is a complete metric space and* $\xi : M \to CB(\mathscr{M})$ *a set-valued mapping such that*

$$\frac{1}{2}\mathscr{D}(\alpha,\xi(\alpha)) \le d(\alpha,\beta) \text{ implies } \mathscr{H}(\xi(\alpha),\xi(\beta)) \le \eta(d(\alpha,\beta)),$$

for all $\alpha,\beta \in \mathscr{M}$*, where* η *is as in Definition* 1.3*. Then* ξ *has a fixed point.*

Corollary 2.4. *Let* $(\mathscr{M},d)$ *be a complete metric space and* $\xi : M \to CB(\mathscr{M})$ *a set-valued mapping such that*

$$\mathscr{H}(\xi(\alpha),\xi(\beta)) \le \eta(m(\alpha,\beta)),$$

for all $\alpha,\beta \in \mathscr{M}$*,* η *is as in Definition* 1.3*. Then* ξ *has a fixed point.*

Corollary 2.5. *Theorem* 1.2*.*

Proof. It comes from Corollary 2.4, when we take $\eta(t) = kt$ with $k \in [0,1)$. ☐

Example 2.6. Let $\mathscr{M} = \{1,2,3,4\}$ and d is the metric on $\mathscr{M}$ defined by

$$d(\alpha,\alpha) = 0, d(\alpha,\beta) = d(\beta,\alpha),\ d(1,2) = d(1,3) = d(1,4) = 1,$$
$$d(2,3) = d(2,4) = d(3,4) = \frac{3}{2}.$$

Then $(\mathscr{M},d)$ is a complete metric space. Define $\eta : [0,\infty) \to [0,\infty)$ and $\xi : M \to CB(\mathscr{M})$ by

$$\eta(t) = \begin{cases} \frac{t^2}{2}, & \text{if } t \le 1, \\ t - \frac{1}{4}, & \text{otherwise;} \end{cases} \qquad \xi(\alpha) = \begin{cases} \{1\}, & \text{if } \alpha \in \{1,2,4\}, \\ \{2,4\}, & \text{if } \alpha = 3. \end{cases}$$

We consider the followings cases.

Case 1: $\alpha,\beta \in \{1,2,4\}$. Then

$$\mathscr{H}(\xi(\alpha),\xi(\beta)) = 0 \le \eta(d(\alpha,\beta)).$$

Case 2: $\alpha = 1,\ \beta = 3$. Then

$$\mathscr{H}(\xi(1),\xi(3)) = \mathscr{H}(\{1\},\{2,4\}) = 1 < \frac{5}{4} = \eta(\mathscr{D}(3,\xi(3))).$$

Case 3: $\alpha = 2,\ \beta = 3$. Then

$$\mathscr{H}(\xi(2),\xi(3)) = 1 < \frac{5}{4} = \eta(d(2,3)).$$

Case 4: $\alpha = 3,\ \beta = 4$. Then

$$\mathscr{H}(\xi(3),\xi(4)) = 1 < \frac{5}{4} = \eta(d(3,4)).$$

Thus in all the cases, $\mathscr{H}(\xi(\alpha),\xi(\beta)) \leq \eta(m(\alpha,\beta))$, and (2.1) is satisfied. Further, all the conditions of Theorem 2.1 are satisfied and $1 \in \xi(1) \subset M$ is a fixed point of ξ.

Example 2.7. Let $\mathscr{M} = [-2,6]$ and d the usual metric $\mathscr{M}$. Then $(\mathscr{M},d)$ is a complete metric space. Define $\eta : [0,\infty) \to [0,\infty)$ and $\xi : M \to CB(\mathscr{M})$ by

$$\eta(t) = \begin{cases} \frac{t^2}{2}, & \text{if } t \leq 1, \\ t - \frac{1}{3}, & \text{otherwise;} \end{cases} \qquad \xi(\alpha) = \begin{cases} \left\{\frac{\alpha}{3}, 0\right\}, & \text{if } \alpha < 0, \\ \left[0, \frac{\alpha}{3}\right], & \text{if } \alpha \geq 0. \end{cases}$$

We consider the followings cases.

Case 1: $\alpha,\beta < 0$. Then

$$\begin{aligned} \mathscr{H}(\xi(\alpha),\xi(\beta)) &= H\left(\left\{\frac{\alpha}{3},0\right\},\left\{\frac{\beta}{3},0\right\}\right) \\ &= \max\left\{\left|\frac{\alpha}{3}-\frac{\beta}{3}\right|,\left|\frac{\alpha}{3}\right|,\left|\frac{\beta}{3}\right|\right\} \\ &\leq \eta\left(\max\left\{d(\alpha,\beta),\mathscr{D}(\alpha,\xi(\alpha)),\mathscr{D}(\beta,\xi(\beta))\right\}\right). \end{aligned}$$

Case 2: $\alpha < 0,\ \beta > 0$. Then

$$\begin{aligned} \mathscr{H}(\xi(\alpha),\xi(\beta)) &= H\left(\left\{\frac{\alpha}{3},0\right\},\left[0,\frac{\beta}{3}\right]\right) \\ &= \max\left\{\left|\frac{\alpha}{3}\right|,\left|\frac{\beta}{3}\right|\right\} \\ &\leq \eta(\max\{\mathscr{D}(\alpha,\xi(\alpha)),\mathscr{D}(\beta,\xi(\beta))\}). \end{aligned}$$

Case 3: $\alpha,\beta \geq 0$. Then

$$\begin{aligned} \mathscr{H}(\xi(\alpha),\xi(\beta)) &= H\left(\left[0,\frac{\alpha}{3}\right],\left[0,\frac{\beta}{3}\right]\right) \\ &= \left|\frac{\alpha}{3}-\frac{\beta}{3}\right| \leq |\alpha-\beta|. \end{aligned}$$

Thus in all the cases, $\mathscr{H}(\xi(\alpha),\xi(\beta)) \leq \eta(m(\alpha,\beta))$, and (2.1) is satisfied. Further, all the conditions of Theorem 2.1 are satisfied and $0 \in \xi(0) \subset \mathscr{M}$ is a fixed point of ξ.

3. Stability of Fixed Point Sets and Well-Posedness

The idea of stability is connected to a system's limiting characteristics. The stability of fixed points describes the relationship between a set of mappings' convergence and their fixed points [1, 2, 5, 6, 8, 9, 10, 12, 16]. The stability of the fixed point sets of the set-valued mappings is covered in this section. We start with the next lemma.

Lemma 3.1. [11]. *If $(\mathscr{M},d)$ is metric space and $B \in C(X)$ then for every $\alpha \in X$ there exists $\beta \in B$ such that $d(\alpha,\beta) = \mathscr{D}(\alpha,B)$.*

Theorem 3.2. *Let $(\mathscr{M},d)$ be a complete metric space. Assume that $\xi_j : M \to C(\mathscr{M})$ $(j \in \{1,2\})$ are two set-valued mappings satisfying* (2.1) *and $\sum_{k=1}^{\infty} \eta^k(t) < \infty$ for all $t > 0$. Then*

- **(a):** $F(\xi_j) \neq \emptyset$ $(j \in \{1,2\})$.
- **(b):** $\mathscr{H}(F(\xi_1),F(\xi_2)) \leq \Psi(L)$, *where* $L = \sup_{\alpha \in M} \mathscr{H}(\xi_1(\alpha),\xi_2(\alpha))$ *and* $\Psi(L) = \sum_{k=1}^{\infty} \eta^k(L)$.

Proof. Theorem 2.1 guaranties $F(\xi_j) \neq \emptyset$ $(j \in \{1,2\})$ and (a) is proved. Next, suppose $z_1 \in F(\xi_1)$, that is, $z_1 \in \xi_1(z_1)$. By Lemma 3.1 there exists a $z_2 \in \xi_2(z_1)$ such that

$$d(z_1,z_2) = \mathscr{D}(z_1,\xi_2(z_1)). \tag{3.1}$$

Again by Lemma 3.1 there exists a $z_3 \in \xi_2(z_2)$ such that

$$d(z_2,z_3) = \mathscr{D}(z_2,\xi_2(z_2)).$$

Continuing this way and following proof of Theorem 2.1, we are able to create a sequence (z_n) so that

$$z_{n+1} \in \xi_2(z_n) \text{ and } d(z_{n+1},z_{n+2}) \leq \eta(d(z_n,z_{n+1})) \leq \cdots \leq \eta^n(d(z_1,z_2)). \tag{3.2}$$

We establish that the sequence (z_n) is Cauchy by means of the Theorem 2.1's proof. Consequently, it converges to a point $w \in M$. It may also be demonstrated that w is a fixed point of ξ. Now, using (3.1) and the specification of L,

$$d(z_1,z_2) = \mathscr{D}(z_1,\xi_2(z_1)) \leq \mathscr{H}(\xi_1(z_1),\xi_2(z_2)) \leq M = \sup_{\alpha \in M} \mathscr{H}(\xi_1(\alpha),\xi_2(\alpha)). \tag{3.3}$$

Using the triangle inequality and (3.2), we have

$$d(z_1,w) \le \sum_{k=1}^{n+1} d(z_i,z_{i+1}) + d(z_{n+2},w) \le \sum_{k=1}^{n} \eta^k(d(z_1,z_2)) + d(z_{n+2},w).$$

Making $n \to \infty$ and using (3.3), we get

$$d(z_1,w) \le \sum_{k=1}^{\infty} \eta^k(d(z_1,z_2)) + d(z_{n+2},w) \le \sum_{k=1}^{\infty} \eta^k(L) = \Psi(L).$$

Therefore for given $z_1 \in F(\xi_1)$, we have $w \in F(\xi_2)$ such that $d(z_1,w) \le \Psi(L)$. Similarly, we can prove that for given $w_1 \in F(\xi_2)$, we have $u \in F(\xi_1)$ such $d(w_1,u) \le \Psi(L)$. Combining above two, gives (b). □

Lemma 3.3. *Assume that $(\mathscr{M},d)$ is a complete metric space and $\xi_n : M \to CB(\mathscr{M})$ $(n \in \mathbb{N})$ be a sequence of set-valued mappings. Let (ξ_n) converges uniformly $\xi : M \to CB(\mathscr{M})$ and for $n \in \mathbb{N}$ each ξ_n satisfies all the conditions of Theorem* 2.1. *Then ξ also satisfies* (2.1) *and has a fixed point in M.*

Proof. Choose $\alpha \in M$ and $\beta \in \xi(\alpha)$ arbitrarily. Since each ξ_n for $n \in \mathbb{N}$ satisfies (2.1), we have

$$\frac{1}{2}\mathscr{D}(\alpha,\xi_n(\alpha)) \le d(\alpha,\beta) \text{ implies } \mathscr{H}(\xi_n(\alpha),\xi_n(\beta)) \le \eta(m_n(\alpha,\beta))$$

for all $\alpha,\beta \in M$, where

$$m_n(\alpha,\beta) = \max\left\{d(\alpha,\beta), \mathscr{D}(\alpha,\xi_n(\alpha)), \mathscr{D}(\beta,\xi_n(\beta)), \frac{\mathscr{D}(\alpha,\xi_n(\beta)) + \mathscr{D}(\beta,\xi_n(\alpha))}{2}\right\}.$$

For $\xi_n \to \xi$ uniformly, making $n \to \infty$ and arguing same as in the proof of Theorem 2.1, we get

$$\frac{1}{2}\mathscr{D}(\alpha,\xi(\alpha))) \le d(\alpha,\beta) \text{ implies } \mathscr{H}(\xi(\alpha),\xi(\beta)) \le \eta(m(\alpha,\beta))$$

for all $\alpha,\beta \in M$, where $m(\alpha,\beta)$ is as in Theorem 2.1. So, ξ satisfies (3.1). Since X is complete and ξ satisfies (3.1), ξ has a fixed point in M. □

Theorem 3.4. *Suppose $(\mathscr{M},d)$ is a complete metric space. Let $\xi_n : M \to CB(\mathscr{M})$ $(n \in \mathbb{N})$ be a sequence of set-valued mappings. Let ξ_n converges uniformly $\xi : M \to CB(\mathscr{M})$. Suppose for $n \in \mathbb{N}$ each ξ_n satisfies all the conditions of Theorem* 2.1. *Then $F(\xi_n) \neq \emptyset$ for all $n \in \mathbb{N}$ and $F(\xi) \neq \emptyset$.*

Moreover, if $\lim_{t\to 0}\Psi(t) = 0$, where $\Psi(t) = \sum_{k=1}^{\infty} \eta^k(t)$ then $\lim_{n\to\infty} \mathscr{H}(F(\xi_n),F(\xi)) = 0$.

Proof. By Lemma 3.3, $F(\xi_n) \neq \emptyset$ for all $n \in \mathbb{N}$ and $F(\xi) \neq \emptyset$. Suppose $L_n = \sup_{\alpha \in M} \mathscr{H}(\xi_n(\alpha), \xi(\alpha))$. For (ξ_n) is uniformly convergent to ξ, we get

$$\lim_{n\to\infty} \sup_{\alpha \in M} \mathscr{H}(\xi_n(\alpha), \xi(\alpha)) = 0.$$

From Theorem 3.2, we have

$$\mathscr{H}(F_n(\xi), F(\xi)) \leq \Psi(L_n) \text{ for all } n \in \mathbb{N}.$$

Further, $\lim_{t\to 0} \Psi(t) = 0$ implies

$$\lim_{n\to\infty} \mathscr{H}(F_n(\xi), F(\xi)) \leq \lim_{n\to\infty} \Psi(L_n) = 0.$$

Therefore sets of fixed points of ξ_n are stable. □

Now we show that fixed point problem is well-posed. We begin with the following definitions.

Definition 3.5. [13]. Assume that $(\mathscr{M}, d)$ is a metric space and $\xi : \mathscr{M} \to CB(M)$ a set-valued mappings. We say fixed point problem is well-posed for ξ with respect to $\mathscr{D}$ if

(i): $SF(\xi) = \{z\}$;

(ii): for any (α_n) in $\mathscr{M}$ with $\lim_{n\to\infty} \mathscr{D}(\alpha_n, \xi(\alpha_n)) = 0$, we have $\lim_{n\to\infty} d(\alpha_n, z) = 0$.

Definition 3.6. [13]. Assume that $(\mathscr{M}, d)$ is a metric space and $\xi : \mathscr{M} \to CB(M)$ a set-valued mappings. We say fixed point problem is well-posed for ξ with respect to $\mathscr{H}$ if

(i): $SF(\xi) = \{z\}$;

(ii): for any (α_n) in $\mathscr{M}$ with $\lim_{n\to\infty} \mathscr{H}(\alpha_n, \xi(\alpha_n)) = 0$, we have $\lim_{n\to\infty} d(\alpha_n, z) = 0$.

It is easy to prove that if $F(\xi) = SF(\xi)$ and fixed point problem is well-posed for ξ with respect to $\mathscr{D}$ then it is well-posed with respect to $\mathscr{H}$.

Theorem 3.7. *Suppose all the conditions of Corollary* 2.2 *are satisfied with* $SF(\xi) \neq \emptyset$. *Then*

(a): $F(\xi) = SF(\xi) = \{z\}$.

(b): *The fixed point problem is well-posed for* ξ *with respect to* $\mathscr{H}$.

Proof. **(a)** Let $u \in SF\xi$ and $z \in F(\xi)$ such that $u \neq z$. Then $0 = \frac{1}{2}\mathscr{D}(u,\xi(u)) < d(u,z)$. Thus by (2.1), we have

$$\begin{aligned} \mathscr{H}(\xi(u),\xi(z)) &\leq \varphi(\max\{d(u,z),\mathscr{D}(u,\xi(u)),\mathscr{D}(z,\xi(z))\}) \\ &= \varphi(d(u,z)) < d(u,z). \end{aligned}$$

Therefore, we have

$$d(u,z) = \mathscr{D}(z,\xi(u)) \leq \mathscr{H}(\xi(u),\xi(z)) < d(u,z),$$

which is a contradiction unless $u = z$.

(b) Suppose (α_n) is a sequence in $\mathscr{M}$ such that $\lim_{n\to\infty} \mathscr{D}(\alpha_n,\xi(\alpha_n)) = 0$. We prove that $\lim_{n\to\infty} d(\alpha_n,z) = 0$. Assume by contraction $\lim_{n\to\infty} d(\alpha_n,z) \neq 0$. Then there exists $\varepsilon > 0$ such that $\varepsilon < d(\alpha_n,z)$ for each $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \mathscr{D}(\alpha_n,\xi(\alpha_n)) = 0$, there exists $n(\varepsilon) \in \mathbb{N}$ such that $\mathscr{D}(\alpha_n,\xi(\alpha_n)) < \varepsilon$ for each $n > n(\varepsilon)$. Now it is evident that $\frac{1}{2}\mathscr{D}(\alpha_n,\xi(\alpha_n)) < \varepsilon < d(\alpha_n,z)$. Using (2.1) for each , we get

$$\begin{aligned} d(\alpha_n,z) &= \mathscr{D}(\alpha_n,\xi(z)) \\ &\leq \mathscr{D}(\alpha_n,\xi(\alpha_n)) + \mathscr{H}(\xi(\alpha_n),\xi(z)) \\ &\leq \mathscr{D}(\alpha_n,\xi(\alpha_n)) + \varphi(\max\{d(\alpha_n,z),\mathscr{D}(\alpha_n,\xi(\alpha_n)),\mathscr{D}(z,\xi(z))\}) \\ &= \mathscr{D}(\alpha_n,\xi(\alpha_n)) + \varphi(\max\{d(\alpha_n,z),\mathscr{D}(\alpha_n,\xi(\alpha_n))\}). \end{aligned}$$

Now on taking $n \to \infty$ and using property of φ, we get

$$\varepsilon < d(\alpha_n,z) \leq \varphi(d(\alpha_n,z)) < \varepsilon,$$

a contradiction. Therefore $\lim_{n\to\infty} d(\alpha_n,z) = 0$, and the fixed point problem is well-posed for ξ with respect to $\mathscr{H}$. □

4. An Application to Integral Inclusion Problem

Now, we discuss an application of Theorem 2.1 to integral inclusion of Volterra-type. Let the space of all continuous real valued functions be denoted by $\mathscr{X} = \mathscr{K}([u,v],\mathbb{R})$ and

$$\rho(\alpha,\beta) = \sup_{p\in[u,v]} |\alpha(p) - \beta(p)|.$$

We consider the integral inclusion

$$\alpha(p) \in f(p) + \int_u^p \mathscr{G}(p,q,\alpha(q))ds, \ p \in [u,v], \tag{4.1}$$

where $\mathscr{G} : [u,v] \times [u,v] \times \mathbb{R} \to CB(\mathbb{R})$. Assuming that $\mathscr{G}_\alpha(p,q) = \mathscr{G}(p,q,\alpha(q))$ is a lower semi-continuous and for $f \in \mathscr{X}$, where $(p,q) \in [u,v] \times [u,v], \ \alpha \in \mathbb{R}$.

Define $P : \mathscr{X} \to C(\mathscr{X})$ as follows

$$P(\alpha(p)) = \left\{ \alpha(p) \in \mathscr{X} \ : \ \alpha(p) \in f(p) + \int_u^p \mathscr{G}(p,q,\alpha(q))ds, \ p \in [u,v] \right\} \tag{4.2}$$

for all $\alpha \in \mathscr{X}$.

A continuous mapping $f : \mathscr{X} \to \mathscr{X}$ is a selection for P if $f(\alpha) \in P(\alpha)$. The Michael's selection theorem [7] ensures existence of continuous operator $k_\alpha : [u,v] \times [u,v] \to \mathbb{R}$ such that $k_\alpha(p,q) \in \mathscr{G}(p,q,\alpha(q))$ for $p, \ q \in [u,v]$ and $\alpha \in \mathscr{X}$. Thus

$$f(p) + \int_u^p k_\alpha(p,q)ds \in P(\alpha(p)) \text{ and } P(\alpha(p)) \neq \emptyset.$$

Theorem 4.1. *Suppose $P : \mathscr{X} \to C(\mathscr{X})$ is defined by* (4.2) *such that*

$$\frac{1}{2} d(\alpha(q), P(\alpha(q))) \leq d(\alpha(q), \beta(q))$$

implies

$$\mathscr{H}\left(\mathscr{G}(p,q,\alpha(q)), \mathscr{G}(p,q,\beta(q))\right) \leq \varphi(m(\alpha(q),\beta(q))) \tag{4.3}$$

for all $q, p \in [u,v], \ \alpha, \beta \in \mathscr{X}$, where

$$\begin{aligned} m(\alpha(q),\beta(q)) &= \max\{d(\alpha(q),\beta(q)), d(\alpha(q),P(\alpha(q))), d(\beta(q),P(\beta(q))), \\ &\quad \frac{1}{2}[d(\alpha(q),P(\beta(q))) + d(\beta(q),P(\alpha(q)))]\}. \end{aligned}$$

Then the integral inclusion (4.1) *has a solution.*

Proof. First, we show that P satisfies condition (2.1) of Theorem 2.1. Let $\alpha \in \mathscr{X}$ and $\alpha(p) \in P(\alpha)$. By Michael's selection theorem, we get $k_\alpha(p,q) \in \mathscr{G}_\alpha(p,q)$ for $p,q \in [u,v]$ such that

$$\alpha(p) = f(p) + \int_u^p k_\alpha(p,q)ds.$$

For $\beta \in P(\alpha)$,

$$\frac{1}{2}d(\alpha(q),P(\alpha(q))) \leq d(\alpha(q),\beta(q)).$$

By (4.3), there exists $r(p,q) \in \mathscr{G}_{\beta}(p,q)$ such that

$$|k_{\alpha}(p,q)-r(p,q)| \leq \varphi\left(m(\alpha(q),\beta(q))\right),$$

for $p,q \in [u,v]$. For $p,q \in [u,v]$ define a set-valued operator Q by

$$Q(p,q) \;=\; \mathscr{G}_{\beta}(p,q) \cap \left\{\beta \in \mathbb{R} : |k_{\alpha}(p,q)-\beta| \leq \varphi\left(m(\alpha(q),\beta(q))\right)\right\}.$$

The lower semi-continuity of Q gives a continuous mapping $k_{\beta}(p,q) \in Q(p,q)$ for $p,q \in [u,v]$ such that

$$\beta(p) = f(p) + \int_{u}^{p} k_{\beta}(p,q)ds \;\in P(\beta(q)) \;\text{ for }\; p \in [u,v].$$

For any $p \in [u,v]$, we get

$$\begin{aligned} d(\alpha(p),\beta(p)) &= \sup_{p\in[u,v]} \left|\int_{u}^{p} k_{\alpha}(p,q)ds - \int_{u}^{p} k_{\beta}(p,q)ds\right| \leq \int_{u}^{p} \sup_{p\in[u,v]} |k_{\alpha}(p,q)-k_{\beta}(p,q)|ds \\ &\leq \varphi(m(\alpha(q),\beta(q))). \end{aligned}$$

If we interchange the role of α and β then we get

$$\mathscr{H}(P(\alpha),P(\beta)) \leq \varphi(m(\alpha(q),\beta(q))).$$

Therefore all the conditions of Theorem 2.1 are satisfied. Thus the inclusion problem (4.1) has a solution in $\mathscr{X}$. □

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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