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FIXED POINT THEOREM IN PARTIALLY ORDERED PARTIAL METRIC SPACES

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Abstract. In this paper, we introduce the existence fixed point theorem in the context of partial metric space endowed with partial ordering. Our results generalize and extend some recent results of Ran and Reurings (2004) and Nieto and Rodríguez-López (2005) to partial metric spaces.

Keywords: fixed point; partially ordered set; partial metric space.

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1. Introduction

In 1922, the Banach contraction principle was introduced by S. Banach as follows:

Theorem 1.1. [1] *Let (X,d) be a complete metric space and a self-mapping $f: X \to X$. If f is contraction mapping i.e. there exist $\kappa \in [0,1)$ such that for all $x,y \in X$, $d(f(x), f(y)) \leq \kappa d(x,y)$ then f has a unique fixed point in X.*

Banach's contraction principle is one of the fundamental and useful tools in mathematics. A number of authors have defined contractive type mapping [2] on a complete metric space X

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which is a generalization of the Banach contraction principle [3]. Recently, Ran and Reurings [4] initiated the trend of weaken the contraction condition by considering single-valued mapping on partially ordered metric space. O'Regan and Petruşel [5] called the contraction's condition on Ran and Reurings by the Banach-Caccioppoli type principle. The main result in [4] is the following theorem.

Theorem 1.2. *Let $(X,\preccurlyeq)$ be a partially ordered set such that every pair $x,y\in X$ has an upper and lower bound. Let d be a metric on X such that (X,d) is a complete metric space. Let $f:X\to X$ be a continuous monotone (i.e. either non-decreasing or non-increasing) mapping. Suppose that the following conditions hold:*

(1) there exist $\kappa\in(0,1)$ with

$$d(f(x),f(y))\leq \kappa d(x,y),$$

for all $x\succcurlyeq y$,

(2) there exist $x_0\in X$ with $x_0\preccurlyeq f(x_0)$ or $x_0\succcurlyeq f(x_0)$, then f has a unique fixed point $x^*\in X$ and for each $x\in X$,

$$\lim_{n\to\infty} f^n(x)=x^*.$$

After the result of Ran and Reurings, the study on fixed point theory developments in partially ordered sets has been constantly growing. Several authors considered the problem of the existence (and uniqueness) of a fixed point for contraction mapping on a partially ordered set. In 2005, J.J Nieto and R Rodríguez-López [6] present a new extension of the Banach contractive mapping theorem to partially ordered sets by removing the upper bound and lower bound hypothesis. This results in the following theorem.

Theorem 1.3. *Let $(X,\preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $f:X\to X$ be a continuous and non-decreasing mapping such that there exists $\kappa\in[0,1)$ with*

$$d(f(x),f(y))\leq \kappa d(x,y),$$

for all $x\succcurlyeq y$. If there exists $x_0\in X$ with $x_0\preccurlyeq f(x_0)$, then f has a fixed point.

Nieto and Rodríguez-López [6] also present a new extension of the Banach contractive mapping theorem to partially ordered sets that allow to consider of discontinuous functions.

Theorem 1.4. *Let $(X,\preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Assume that X satisfies: "If a non-decreasing sequence $x_n \to x$ in X, then $x_n \preccurlyeq x$ for $n \in \mathbb{N}$". Let $f : X \to X$ be monotone non-decreasing mapping such that there exists $\kappa \in [0,1)$ with*

$$d(f(x),f(y)) \leq \kappa d(x,y),$$

for all $x \succcurlyeq y$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq f(x_0)$, then f has a fixed point.

Consider the hypothesis "every x,y in X has a lower bound and an upper bound" on Theorem 1.2. Without this condition as in Theorem 1.3 and Theorem 1.4, the existence of a fixed point is still guaranteed but not for the uniqueness. Nieto and Rodríguez-López [6], prove that a weaker condition, namely the existence of a lower bound or upper bound, guarantees the uniqueness of a fixed point of f. This result is presented in the following theorem.

Theorem 1.5. *Assume that X satisfies:*
Every $x,y \in X$ has a lower bound or an upper bound. (1.1)
Adding condition (1.1) to the hypothesis of Theorem 1.3 (resp. Theorem 1.4), we obtain the uniqueness of the fixed point of f.

Furthermore, from Theorem 1.3 and Theorem 1.4 the following theorem is obtained.

Theorem 1.6. *Let $(X,\preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $f : X \to X$ be monotone non-decreasing mapping such that there exists $\kappa \in [0,1)$ with*

$$d(f(x),f(y)) \leq \kappa d(x,y),$$

for all $x \succcurlyeq y$. Assume that either f is continuous or X is such that: "If a non-increasing sequence $x_n \to x$ in X, then $x \preccurlyeq x_n$, for $n \in \mathbb{N}$". If there exists $x_0 \in X$ with $x_0 \succcurlyeq f(x_0)$, then f has a fixed point.

Motivated by these works, we are going to combine the techniques employed by Ran and Reurings [4], Nieto and Rodríguez-López [6], and Matthews [7] in generalizing and extending Theorem 1.2-1.6.

2. PRELIMINARIES

Firstly, we recall some definitions and some properties of partial metric space and partially ordered sets. The notation $\mathbb{R}^+, \mathbb{N}$ denotes the set of all positive real numbers, the set of all positive integer numbers, respectively.

In the development of metric space study, Matthews [7] introduced the notion of partial metric space as a generalization of metric space. The most remarkable property in a partial metric space is that the self-distance need not be zero.

Definition 2.1. [7] Let X be a nonempty set, and let $p : X \times X \to [0,\infty)$ be a function satisfying the following:

(P1). $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$

(P2). $p(x,x) \leq p(x,y)$

(P3). $p(x,y) = p(y,x)$

(P4). $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$,

for all $x,y,z \in X$ and the pair (X,p) is called a partial metric space.

Remark 2.1. [8] *In partial metric space* (X,p),

(1) If $p(x,y) = 0$ *then* $x = y$*, but if* $x = y$ *then* $p(x,y)$ *may not be zero.*

(2) $p(x,y) > 0$ *for all* $x,y \in X, x \neq y$.

Example 2.1. [7, 8] $X = \mathbb{R}^+$, *and define* $p(x,y) = \max\{x,y\}$, *for all* $x,y \in X$, *pair* (X,p) *is partial metric space.*

Definition 2.2. [8, 9, 10] In partial metric space (X,p),

(1) A sequence (x_n) is said to converge to a point $x \in X$ if and only if $\lim_{n\to\infty} p(x_n,x) = p(x,x)$.

(2) A sequence (x_n) is called Cauchy sequence if and only if $\lim_{n\to\infty} p(x_n,x_m)$ is finite.

(3) If every Cauchy sequence (x_n) converges to a point $x \in X$ such that $\lim_{n\to\infty} p(x_n,x_m) = p(x,x)$ then (X,p) is known as complete partial metric space.

(4) A mapping $f : X \to X$is said to be continuous at $x_0 \in X$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(B(x_0,\delta)) \subset B(f(x_0),\varepsilon).$$

Romaguera [11] defined the notion of 0-Cauchy sequence (x_n) in partial metric spaces and also introduced the concept of 0-completeness in the same class of spaces.

Definition 2.3. [11, 12] A sequence (x_n) in partial metric space (X,p) is called 0-Cauchy if $\lim_{n\to\infty} p(x_n,x_m) = 0$. A partial metric space (X,p) is said to be 0-complete if every 0-Cauchy sequence in X converges, with respect to topology τ_p to a point $x \in X$ such that $p(x,x) = 0$. In this case, p is said to be a 0-complete partial metric on X.

Observe that each 0-Cauchy sequence is also a Cauchy sequence in a partial metric space. In particular, we note that each complete partial metric space is a 0-complete partial metric on X. However, the converse is not true.

Matthews [7] gave a modified version of the Banach contraction principle in partial metric spaces.

Theorem 2.1. *Let (X,p) be a complete partial metric space and $f : X \to X$ be a mapping such that there exists $\kappa \in [0,1)$ with*

$$p(f(x), f(y)) \leq \kappa p(x,y),$$

for all $x,y \in X$ then there exists a unique $x^* \in X$ such that $x^* = f(x^*)$, furthermore $p(x^*,x^*) = 0$.

Definition 2.4. [13, 14] A partial order is a binary relation $\preccurlyeq$ over a set X which satisfies the following conditions:

(1) $x \preccurlyeq x$ (reflexity)
(2) If $x \preccurlyeq y$ and $y \preccurlyeq x$ then $x = y$ (antisymmetry)
(3) If $x \preccurlyeq y$ and $y \preccurlyeq z$ then $x \preccurlyeq z$ (transitivity),

for all $x,y,z \in X$. A set with partial order $\preccurlyeq$ is called a partially ordered set.

Definition 2.5. [13, 14, 15] Let $(X,\preccurlyeq)$ be a partially ordered set and $x,y \in X$. Element x and y are said to be comparable elements of X if either $x \preccurlyeq y$ or $y \preccurlyeq x$.

Definition 2.6. [14] Let $(X,\preccurlyeq)$ be a partially ordered set. If $S \subset X$ and $y \in X$.

(1) Element y is called lower bound of S, if $y \preccurlyeq x$ for each $x \in S$.
(2) Element y is called upper bound of S, if $x \preccurlyeq y$ for each $x \in S$.

Remark 2.2. [6] *Let $(X, \preccurlyeq)$ be a partially ordered set. Every pair of elements X has a lower bound or an upper bound if and only if for every $x, y \in X$, there exist $z \in X$ which is comparable to x and y.*

Indeed, if every pair of elements X has a lower bound or an upper bound, given $x, y \in X$, the upper or lower bound of x and y is comparable both to x and y. Conversely, if $x, y \in X$ and $z \in X$ is comparable with them, we have the following possibilities:

(1) If $x \preccurlyeq z, y \preccurlyeq z$ or $z \preccurlyeq x, z \preccurlyeq y$, then every pair of elements X has a lower bound or an upper bound.

(2) If $x \preccurlyeq z \preccurlyeq y$ or $y \preccurlyeq z \preccurlyeq x$, then x and y are comparable and every pair of elements X has a lower bound or an upper bound.

Definition 2.7. [6, 15] Let $(X, \preccurlyeq)$ be a partially ordered set and $f : X \to X$. We say that f is monotone non-decreasing on X if for every $x, y \in X$ with $x \preccurlyeq y$, then $f(x) \preccurlyeq f(y)$. And we say that f is monotone non-increasing on X if for every $x, y \in X$ with $x \preccurlyeq y$, then $f(y) \preccurlyeq f(x)$.

3. Main Results

We begin with the following theorem that gives the existence of a fixed point in partially ordered partial metric space.

Theorem 3.1. *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, p) is a complete partial metric space. Let $f : X \to X$ be a continuous and non-decreasing mapping such that there exists $\kappa \in [0, 1)$ with*

$$p(f(x), f(y)) \leq \kappa p(x, y), \tag{3.1}$$

for all $x \succcurlyeq y$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq f(x_0)$, then f has a fixed point, that is x^*. Furthermore, $p(x^*, x^*) = 0$.

Proof. Let $x_0 \in X$ be such that $x_0 \preccurlyeq f(x_0)$ or $x_0 \succcurlyeq f(x_0)$. Since f non-decreasing then we have either

$$x_0 \preccurlyeq f(x_0) \preccurlyeq f(f(x_0)) = f^2(x_0) \preccurlyeq f(f^2(x_0)) = f^3(x_0) \preccurlyeq \ldots \preccurlyeq f^n(x_0) \preccurlyeq f^{n+1}(x_0) \preccurlyeq \ldots$$

Therefore from (3.1) it follows that

$$\begin{aligned} p(f^{n+1}(x_0), f^n(x_0)) &\leq \kappa p(f^n(x_0), f^{n-1}(x_0)) \\ &\leq \kappa\kappa p(f^{n-1}(x_0), f^{n-2}(x_0)) = \kappa^2 p(f^{n-1}(x_0), f^{n-2}(x_0)) \\ &\leq \kappa\kappa^2 p(f^{n-2}(x_0), f^{n-3}(x_0)) = \kappa^3 p(f^{n-2}(x_0), f^{n-3}(x_0)) \\ &\vdots \\ &\leq \kappa^n p(f(x_0), x_0). \end{aligned}$$

where $n \in \mathbb{N}$. Furthermore, by P4 we have

$$\begin{aligned} &p(f^{n+k+1}(x_0), f^n(x_0)) \\ &\leq p(f^{n+k+1}(x_0), f^{n+k}(x_0)) + p(f^{n+k}(x_0), f^n(x_0)) - p(f^{n+k}(x_0), f^{n+k}(x_0)) \\ &\leq \kappa^{n+k} p(f(x_0), x_0) + p(f^{n+k}(x_0), f^n(x_0)). \end{aligned}$$

Similarly,

$$\begin{aligned} p(f^{n+k+1}(x_0), f^{n+k}(x_0)) &\leq p(f^{n+k+1}(x_0), f^n(x_0)) + p(f^n(x_0), f^{n+k}(x_0)) - p(f^n(x_0), f^n(x_0)) \\ &\leq \kappa^{n+k} p(f(x_0), x_0) + p(f^n(x_0), f^n(x_0)). \end{aligned}$$

where $n, k \in \mathbb{N}$. Thus, for every $n, k \in \mathbb{N}$ we have

$$\begin{aligned} p(f^{n+k+1}(x_0), f^n(x_0)) &\leq (\kappa^{n+k} + \kappa^{n+k-1} + \ldots + \kappa^n) p(f(x_0), x_0) + p(f^n(x_0), f^n(x_0)) \\ &\leq \frac{\kappa^n(1-\kappa^{n+1})}{1-\kappa} p(f(x_0), x_0) + \kappa^n p(x_0, x_0) \\ &= \kappa^n \left(\frac{1-\kappa^{n+1}}{1-\kappa} p(f(x_0), x_0) + p(x_0, x_0) \right) \\ &\leq \kappa^n \left(\frac{1}{1-\kappa} p(f(x_0), x_0) + p(x_0, x_0) \right) \end{aligned}$$

which means $f^n(x_0)$ is Cauchy Sequence such that for $m, n \in \mathbb{N}$ we have

$$\lim_{m,n\to\infty} p(f^n(x_0), f^m(x_0)) = 0.$$

Since X is a complete metric space, so there exist $x^* \in X$ such that $f^n(x_0)$ converges to x^*, we have

$$\lim_{n\to\infty} p(f^n(x_0), x^*) = 0.$$

Furthermore, we will prove that $f(x^*) = x^*$. For any $n \in \mathbb{N}$ we have

$$\begin{aligned} p(f(x^*),x^*) &\leq p(f(x^*), f^{n+1}(x_0)) + p(f^{n+1}(x_0), f(x^*)) - p(f^{n+1}(x_0), f^{n+1}(x_0)) \\ &\leq p(f(x^*), f(f^n(x_0)) + p(f(f^n(x_0)), f(x^*)). \end{aligned} \tag{3.2}$$

Taking $n \to \infty$ in the above inequality and by the continuity of f we have

$$p(f(x^*),x^*) \leq p(f(x^*), f(x^*)).$$

By (P2) we have $p(f(x^*), f(x^*)) \leq p(f(x^*),x^*)$, so, we obtain $p(f(x^*),x^*) = p(f(x^*), f(x^*))$. From (3.1) we have $p(f(x^*),x^*) \leq p(f(x^*), f(x^*)) \leq \kappa p(x^*,x^*)$. Since $\kappa \in [0,1)$, then

$$p(f(x^*),x^*) \leq p(x^*,x^*).$$

On the other side, by (P2) we have $p(x^*,x^*) \leq p(f(x^*),x^*)$, so we obtain $p(f(x^*),x^*) = p(x^*,x^*)$. Finally, by (P1) we have $x^* = f(x^*)$. On other word, x^* is a fixed point of f. Furthermore, by (P1) and (P2) we obtain

$$p(x^*,x^*) = 0.$$

This complete the proof.

Theorem 3.1 is still valid for f not necessarily continuous, assuming an additional hypothesis on X. This result is the following theorem.

Theorem 3.2. *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a partial metric p in X such that (X, p) is a complete partial metric space. Assume that X satisfies: If a non-decreasing sequence $x_n \to x$ in X, then $x_n \preccurlyeq x$ for $n \in \mathbb{N}$. Let $f : X \to X$ be monotone non-decreasing mapping such that there exists $\kappa \in [0,1)$ with*

$$p(f(x), f(y)) \leq \kappa p(x,y),$$

for all $x \succcurlyeq y$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq f(x_0)$, then f has a fixed point, namely x^*. Furthermore, $p(x^*,x^*) = 0$.

Proof. The proof is following Theorem 3.1's proof. We only have to check that $f(x^*) = x^*$. Since $f^n(x_0)$ converges to x^*, we have $f^n(x_0) \preccurlyeq x^*$. Thus, we have

$$p(f^n(x_0), f(x^*)) \leq \kappa p(f^{n-1}(x_0),x^*) \to 0, \tag{3.3}$$

then $\lim_{n\to\infty} p(f^n(x_0), f(x^*)) = 0$. Furthermore,

$$\begin{aligned} p(x^*, f(x^*)) &\leq p(x^*, f^n(x_0)) + p(f^n(x_0), f(x^*)) - p(f^n(x_0), f^n(x_0)) \\ &\leq p(x^*, f^n(x_0)) + p(f^n(x_0), f(x^*)). \end{aligned} \tag{3.4}$$

Taking $n \to \infty$ in (3.4) and using (3.3), the right-hand side tends to 0. Consequently, we have $p(x^*, f(x^*)) = 0$, hence $x^* = f(x^*)$. So, $p(x^*, x^*) = 0$. This completes the proof.

From Theorem 3.1 and Theorem 3.2 we know that the existence of fixed point is guaranteed but not for the uniqueness. The following theorem gives not only the existence but also the uniqueness of a fixed point in partially ordered partial metric space.

Theorem 3.3. *Let $(X, \preccurlyeq)$ be a partially ordered set such that every pair $x, y \in X$ has an upper and lower bound. Let p be a partial metric on X such that (X, p) is a complete partial metric space. Let $f : X \to X$ be a continuous non-decreasing mapping. Suppose that the following conditions hold:*

(1) there exist $\kappa \in (0,1)$ with

$$p(f(x), f(y)) \leq \kappa p(x,y),$$

for all $x \succcurlyeq y$,

(2) there exist $x_0 \in X$ with $x_0 \preccurlyeq f(x_0)$ or $x_0 \succcurlyeq f(x_0)$,

then f has a unique fixed point $x^* \in X$. Furthermore, $p(x^*, x^*) = 0$ and for each $x \in X$,

$$\lim_{n\to\infty} p(f^n(x), x^*) = p(x^*, x^*).$$

Proof. For the existence of a fixed point is following Theorem 3.1's proof. Now, we will show that x^* is the unique fixed point of f. We will do this by showing that

$$\lim_{n\to\infty} p(f^n(x), x^*) = 0,$$

for every $x \in X$. Let $x \in X$ be arbitrary, x_1 be an upper bound of x and x_0, and x_2 be a lower bound of x and x_0, thus

$$x_2 \preccurlyeq x \preccurlyeq x_1, and x_2 \preccurlyeq x_0 \preccurlyeq x_1.$$

So, for $n \in \mathbb{N}$ we have

$$f^n(x_2) \preccurlyeq f^n(x) \preccurlyeq f^n(x_1), and f^n(x_2) \preccurlyeq f^n(x_0) \preccurlyeq f^n(x_1). \tag{3.5}$$

For $x_0 \preccurlyeq x_1$ then we have $f^n(x_0) \preccurlyeq f^n(x_1)$. From (3.1) we have

$$p(f^n(x_0), f^n(x_1)) \leq \kappa^n p(x_0, x_1).$$

Since the right-hand side tends to 0 if $n \to \infty$, so we have

$$\lim_{n\to\infty} p(f^n(x_0), f^n(x_1)) = 0.$$

Consequently, $\lim_{n\to\infty} p(f^n(x_1), x^*) = \lim_{n\to\infty} p(f^n(x_0), x^*) = 0$. Furthermore, we obtained

$$\lim_{n\to\infty} p(f^n(x_1), x^*) = p(f^n(x_2), x^*) = 0. \tag{3.6}$$

From (3.5) and (3.6) we have

$$\lim_{n\to\infty} p(f^n(x), x^*) = 0,$$

for every $x \in X$. This complete the proof.

Theorem 3.3 is essentially Theorem 1.2 [4] when we generalized metric d to partial metric p. Theorem 3.3 shows the existence of a lower bound and upper bound of every pair element has an important role in the uniqueness of a fixed point. When we remove this condition, the hypothesis in Theorem 3.3 does not guarantee the uniqueness of the fixed point. To guarantee the uniqueness of the fixed point, following Nieto and Rodríguez-López, we can give a new hypothesis that is weaker than the previous condition. It is sufficient to consider that X is such that:

Every pair of elements has a lower bound or an upper bound. (3.7)

From Remark 2.2, condition (3.7) is equivalent to: for every $x, y \in X$, there exists $z \in X$ which is comparable to x and y. So, by condition (3.7) the uniqueness of the fixed point in Theorem 3.1 and Theorem 3.2 is still guaranteed.

Theorem 3.4. *Adding conditions: "every pair of elements has a lower bound or an upper bound" to the hypothesis of Theorem 3.1 (rep. Theorem 3.2), we still obtain the uniqueness of the fixed point of f.*

Proof. Assume that $x \in X$ is another fixed point of f, we prove that $p(x,x^*) = 0$, where

$$\lim_{n\to\infty} p(f^n(x_0),x^*) = 0.$$

Suppose x and x^* are comparable. Since, both x and x^* are fixed point of f then we have $f^n(x) = x$ and $f^n(x^*) = x^*$, for every $n = 0,1,2,\ldots$. Since x is comparable to x^* then $f^n(x)$ comparable to $f^n(x^*)$ for every n, and we obtain

$$p(x,x^*) = p(f^n(x), f^n(x^*)) \leq \kappa p(x,x^*).$$

Hence, $p(x,x^*) = 0$. Furthermore, if x is not comparable to x^*, there exists either an upper or a lower bound of x and x^*. It means there exists $z \in X$ comparable to x and y. Consequently, $f^n(z)$ comparable to $f^n(x) = x$ and $f^n(x^*) = x^*$, for all n, and we have

$$\begin{aligned} p(x,x^*) &\leq p(f^n(x), f^n(z)) + p(f^n(z), f^n(x^*)) - p(f^n(z), f^n(z)) \\ &\leq p(f^n(x), f^n(z)) + p(f^n(z), f^n(x^*)) \\ &\leq \kappa^n p(x,z) + \kappa^n p(z,x^*) \\ &\leq \kappa^n \left(p(x,z) + p(z,x^*)\right). \end{aligned}$$

Taking the limits as $n \to \infty$ in the above inequality yields $p(x,x^*) = 0$. This completes the proof.

From Theorem 3.4 it can be seen that either upper bound or lower bound are essential things that need to be considered in showing the uniqueness of a fixed point in a partially ordered metric space. For the next, from Theorem 3.2 and Theorem 3.3 we have the following theorem.

Theorem 3.5. *Let $(X,\preccurlyeq)$ be a partially ordered set and suppose that there exists a partial metric p in X such that (X,p) is a complete partial metric space. Let $f : X \to X$ be monotone non-decreasing mapping such that there exists $\kappa \in [0,1)$ with*

$$p(f(x), f(y)) \leq \kappa p(x,y),$$

for all $x \succcurlyeq y$. Assume that either "f is continuous" or "X is such that if a non-increasing sequence $x_n \to x$ in X", then $x \preccurlyeq x_n$, for $n \in \mathbb{N}$. If there exists $x_0 \in X$ with $x_0 \succcurlyeq f(x_0)$, then f has a fixed point, namely x^*. Furthermore, $p(x^*,x^*) = 0$.

Proof. The proof is similar to the procedure followed in the proof of the previous theorems. Since $x_0 \succcurlyeq f(x_0)$ and f non-decreasing

$$x_0 \succcurlyeq f(x_0) \succcurlyeq f(f(x_0)) = f^2(x_0) \succcurlyeq f(f^2(x_0)) = f^3(x_0) \succcurlyeq \ldots \succcurlyeq f^n(x_0) \succcurlyeq f^{n+1}(x_0) \succcurlyeq \ldots$$

then, $p(f^n(x_0), f^{n+1}(x_0)) \leq \kappa^n p(x_0, f(x_0))$, for all n. This is show that $f^n(x_0)$ is Cauchy sequences, so there exists $x^* \in X$ such that

$$\lim_{n\to\infty} p(f^n(x_0), x^*) = 0.$$

We will show that x^* is fixed point of f.

Consider the condition: if f is continuous mapping thus the same reasoning of Theorem 3.1 is valid. On the other side, if a non-increasing sequence $x_n \to x$ in X, then $x \preccurlyeq x_n$, for $n \in \mathbb{N}$ and using condition $\lim_{n\to\infty} p(f^n(x_0), x^*) = 0$ then $x^* \preccurlyeq f^n(x_0)$. Analogues with the proof of Theorem 3.2 then we obtain $p(x^*, f(x^*)) = 0$, hence $f(x^*) = x^*$. So, $p(x^*, x^*) = 0$. This completes the proof.

Theorem 3.6. *Adding condition "every pair of elements has a lower bound or an upper bound" to the hypotheses of Theorem 3.5, we obtain the uniqueness of the fixed point of f.*

Proof. It follows on a similar line as Theorem 3.4.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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