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STRONG CONVERGENCE THEOREMS FOR A COMMON FIXED POINT OF A FINITE FAMILY OF LIPSCHITZ HEMICONTRACTIVE-TYPE MULTIVALUED MAPPINGS

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Abstract. Let K be a non-empty, closed and convex subset of a real Hilbert space H. Let $T_i : K \to CB(K), i = 1, 2, \ldots, N$, be a finite family of Lipschitz hemicontractive-type mappings with Lipschitz constants $L_i, i = 1, 2, \ldots, N$, respectively. It is our purpose, in this paper, to introduce a Halpern type algorithm which converges strongly to a common fixed point of a finite family of Lipschitz hemicontractive-type multivalued mappings under certain mild conditions. There is no compactness assumption on either the domain set or on the mappings T_i considered.

Keywords: Fixed points of mappings; hemicontractive mappings, pseudocontractive mappings; strong convergence.

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1. Introduction

Let E be a nonempty real normed linear space. A subset K of E is called proximinal if for each $x \in E$ there exists $k \in K$ such that

$$\|x-k\| = \inf\{\|x-y\| : y \in K\} = d(x,K).$$

It is known that every closed convex subset of a uniformly convex Banach space is proximinal. In fact, if K is a closed and convex subset of a uniformly convex Banach space E, then for any $x \in E$ there exists a unique point $u_x \in K$ such that (see, e.g., [12], [11], [18] and [19])

$$\|x-u_x\| = \inf\{\|x-y\| : y \in K\} = d(x,K).$$

We will denote the family of all nonempty proximinal subsets of E by $P(E)$, the family of all nonempty closed, bounded and convex subsets of E by $CBC(E)$, the family of all nonempty closed and bounded subsets of E by $CB(E)$ and the family of all nonempty subsets of E by 2^E for a nonempty real normed space E.

Let D be the Hausdorff metric induced by the metric d on E, that is, for every $A,B \in CB(E)$,

$$D(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}.$$

Let $T : D(T) \subseteq E \to 2^E$ be a multivalued mapping on E. A point $x \in D(T)$ is called a *fixed point of* T if $x \in Tx$. The set $F(T) = \{x \in D(T) : x \in Tx\}$ is called a fixed point set of T. A multivalued mapping $T : D(T) \subseteq E \to CB(E)$ is called *L-Lipschitzian* if there exists $L \geq 0$ such that for all $x,y \in D(T)$, we have

$$D(Tx,Ty) \leq L\|x-y\|. \tag{1}$$

In (1), if $L \in [0,1)$, T is said to be a *contraction,* while T is *nonexpansive* if $L = 1$.

A mapping $T : D(T) \subset E \to CB(E)$ is said to be *hemicontractive-type* in the terminology of Hicks and Cubicek [21], if $F(T) \neq \emptyset$ and for all $p \in F(T),\ x \in D(T)$

$$D^2(Tx,Tp) \leq \|x-p\|^2 + \|x-u\|^2, \forall\, u \in Tx, \tag{2}$$

where $D^2(Tx,Tp) = [D(Tx,Tp)]^2$. A mapping $T : D(T) \subset E \to CB(E)$ is said to be *demicontractive-type*, if $F(T) \neq \emptyset$ and for all $p \in F(T),\ x \in D(T)$ there exists $k \in [0,1)$ such that

$$D^2(Tx,Tp) \leq \|x-p\|^2 + k\|x-u\|^2, \forall\, u \in Tx. \tag{3}$$

If in (3), we have $k = 0$, then T is called *quasi-nonexpansive-type* mapping.

Note that the class of quasi- nonexpansive type mappings is contained in a class of demicontractive-type mappings while the class of demicontractive-type mappings is contained in a class of hemicontractive-type mappings. As the following examples show, the inclusions are strict. We first give an example of a hemicontractive-type mapping which is not demicontractive-type.

Example 1.1 *Let $T : \mathbb{R} \to CB(\mathbb{R})$ be given by*

$$Tx = \begin{cases} [-\sqrt{2}x, 0] & x \in [0,\infty] \\ [0, -\sqrt{2}x], & x \in [-\infty, 0]. \end{cases}$$

Then, $F(T) = \{0\}$ and for any $x \in \mathbb{R}$,

$$\begin{aligned} D(Tx,T0)^2 &= |\sqrt{2}x - 0|^2 \\ &= |x-0|^2 + |x-0|^2. \end{aligned}$$

But, $d(x,Tx)^2 = |x-0|^2$. Thus,

$$D(Tx,T0)^2 = |x-0|^2 + d(x,Tx)^2 \leq |x-0|^2 + |x-u|^2,\ \forall u \in Tx.$$

So, T is hemicontractive-type but not demicontractive-type mapping. To see this take $x = 1$ and $u = 0$.

A demicontractive-type mapping may not be quasi nonexpansive-type.

Example 1.2 *Let $T : [0,\infty) \to CB(\mathbb{R})$ be given by*

$$Tx = \left[-\frac{4}{3}, -x\right].$$

Then, $F(T) = \{0\}$ and T is demicontractive-type, but not quasi nonexpansive-type mapping.

A mapping $T : K \to CB(E)$ is said to be *k-strictly pseudocontractive-type* mapping if there exists $k \in [0,1)$ such that

$$D^2(Tx,Ty) \leq \|x-y\|^2 + k\|x-y-(u-v)\|^2, \forall\, u \in Tx, v \in Ty. \tag{4}$$

In (4), if $k = 0$, then T reduces to a nonexpansive-type mapping.
A mapping $T : K \to CB(E)$ is said to be *pseudocontractive-type* mapping if

$$D^2(Tx,Ty) \leq \|x-y\|^2 + \|x-y-(u-v)\|^2, \forall\, u \in Tx, v \in Ty. \tag{5}$$

From the definitions, we observe that every multivalued nonexpansive-type mapping is k-strictly pseudocontractive-type and every k-strictly pseudocontractive-type mapping is pseudo-contractive-type mapping. However, the converses may not hold, as can be seen from the following examples.

Example 1.3 *Let* $T : [0,1] \to CB(\mathbb{R})$ *be given by* $Tx = \left\{0, 4 - \frac{4}{3}x\right\}$.
Then we have

$$\begin{aligned} D(Tx,Ty) &= \max\left\{\sup_{a\in Tx} d(a,Ty), \sup_{b\in Ty} d(b,Tx)\right\} \\ &= \max\left\{\min\left\{|4-\frac{4}{3}x|, \frac{4}{3}|x-y|\right\}, \min\left\{|4-\frac{4}{3}y|, \frac{4}{3}|x-y|\right\}\right\} \\ &= \frac{4}{3}|x-y|. \end{aligned}$$

Hence,

$$D^2(Tx,Ty) = |x-y|^2 + \frac{7}{9}|x-y|^2.$$

Obviously, T is not nonexpansive-type. To show that it is k- strictly pseudocontractive-type, with out loss of generality assume that $x < y$.
We will take four cases.
<u>*Case 1:*</u> *Let* $u = 0$ *and* $v = 0$. *Then* $|x-y-(u-v)| = |x-y|$ *and hence*

$$D^2(Tx,Ty) \leq |x-y|^2 + \frac{7}{9}|x-y-(u-v)|^2.$$

Case 2: Let $u = 4 - \frac{4}{3}x$ and $v = 0$. Then $x - y - (4 - \frac{4}{3}x) < x - y \leq 0$. Thus
$\left|x - y - (4 - \frac{4}{3}x - 0)\right|^2 = \left|x - y - (4 - \frac{4}{3}x)\right|^2 \geq |x-y|^2$. This gives us

$$D^2(Tx,Ty) \leq |x-y|^2 + \frac{7}{9}|x-y-(u-v)|^2.$$

Case 3: Let $u = 0$ and $v = 4 - \frac{4}{3}x$. Then $x - y \in [-1,0]$ and $(4 - \frac{4}{3}y) \geq 2(y-x)$. Thus, since $x - y + (4 - \frac{4}{3}y) \geq x - y + 2(y-x) \geq y - x \geq 0$, we get that
$\left|x - y - (0 - (4 - \frac{4}{3}y))\right|^2 = \left|x - y + (4 - \frac{4}{3}y)\right|^2 \geq |x-y|^2$. This implies that

$$D^2(Tx,Ty) \leq |x-y|^2 + \frac{7}{9}|x-y-(u-v)|^2.$$

Case 4: Let $u = 4 - \frac{4}{3}x$ and $v = 4 - \frac{4}{3}y$. Then
$|x - y - (-\frac{4}{3}(x-y))|^2 = (1 + \frac{4}{3})^2|x-y|^2 \geq |x-y|^2$. Thus,

$$D^2(Tx,Ty) \leq |x-y|^2 + \frac{7}{9}|x-y-(u-v)|^2.$$

Therefore, T is k-strictly pseudocontractive-type mapping.

The following mapping is shown to be pseudocontractive-type but not k- strictly pseudocontractive-type mapping (see; [26]).

Example 1.4 *Let $T : [0,\infty] \to CB(\mathbb{R})$ be given by*

$$Tx = \begin{cases} \{2\}, & x = 0; \\ \{0,x\}, & x \neq 0. \end{cases}$$

It is well known that nonexpansive-type mappings are quasi-nonexpansive-type, though the converse may not hold.

Example 1.5 *Let $T : [0,\infty) \to CB(\mathbb{R})$ be given by*

$$Tx = \begin{cases} 0, & x \leq 1; \\ \left[x - \frac{1}{3}, x - \frac{1}{4}\right], & x > 1. \end{cases}$$

Then, $F(T) = \{0\}$ and

$$D(Tx,T0) \leq |x-0|,$$

and hence T is quasi nonexpansive-type. Taking $x = 2$ and $y = 1$, it can be seen that T is not nonexpansive-type mapping.

From the definitions it is also clear that the class of k- strictly pseudocontractive-type mappings is properly contained in a class of demicontractive-type mappings, while the class of pseudocontractive-type mappings is properly contained in a class of hemicontractive-type mappings.

Example 1.6 *Let $T : [0,\infty) \to CB(\mathbb{R})$ be given by $Tx = \left[-3x, -\frac{5}{2}x\right]$.*
Now, $d(x,Tx)^2 = \left|x - \left(-\frac{5}{2}x\right)\right|^2 = \frac{49}{4}|x-0|^2$ and $F(T) = \{0\}$. In addition,

$$\begin{aligned} D(Tx,T0)^2 &= |x-0|^2 + 8|x-0|^2 \\ &= |x-0|^2 + \frac{32}{49}d(x,Tx)^2 \\ &\leq |x-0|^2 + \frac{32}{49}|x-u|^2, \ \forall u \in Tx. \end{aligned}$$

So, T is demicontractive-type but not k- strictly pseudocontractive-type mapping. To see this take $x = 1$, $y = 2, u = -\frac{5}{2}$ and $v = -6$.

Example 1.7 *Let $T : \mathbb{R} \to CB(\mathbb{R})$ be given by*

$$Tx = \begin{cases} [-\sqrt{2}x, 0] & x \in [0,\infty] \\ [0, -\sqrt{2}x], & x \in [-\infty, 0]. \end{cases}$$

Then, $F(T) = \{0\}$ and for any x,

$$\begin{aligned} D(Tx,T0)^2 &= |\sqrt{2}x - 0|^2 \\ &= |x-0|^2 + |x-0|^2. \end{aligned}$$

But, $d(x,Tx)^2 = |x-0|^2$. Thus,

$$D(Tx,T0)^2 = |x-0|^2 + d(x,Tx)^2 \leq |x-0|^2 + |x-u|^2, \ \forall u \in Tx.$$

So, T is hemicontractive-type but not psuedocontractive-type mapping. To see this take $x = 1$, $y = 2$, $u = -1$ and $v = -\frac{1}{2}$.

Remark 1.1 *Example 1.4 shows that the set of fixed points of a hemicontractive-type mapping may not be closed.*

Following the introduction of the study of fixed points for multi-valued nonexpansive mappings using the Hausdorff metric by Markin [6] (see also [7]), the theory has developed greatly with applications in control theory, convex optimization, differential inclusion and economics (see, for example, [8] and references therein). Currently, several schemes have been given on the approximation of fixed points of multi-valued nonexpansive mappings (see for example [9], [10], [11], [12] and [13], and the references therein) and their generalizations (see e.g., [14]).

In 2005, Sastry and Babu [12] introduced Mann and Ishikawa schemes for multivalued mappings and proved the following result.

Theorem 1.1 *Let H be a real Hilbert space, K be a nonempty, compact and convex subset of H, and $T : K \to P(K)$ be a multivalued nonexpansive mapping with nonempty fixed point set. For $x_0 \in K$ let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n z_n, z_n \in Tx_n, \|z_n - p\| = d(p, Tx_n), \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n u_n, u_n \in Ty_n, \|u_n - p\| = d(p, Ty_n), \end{cases} \tag{6}$$

where $p \in F(T)$ and $\{\alpha_n\}, \{\beta_n\}$ are real sequences which satisfy the following conditions: $[i.]$ $0 \le \alpha_n, \beta_n < 1$, $[ii.]$ $\lim_{n\to\infty} \beta_n = 0$ and $[iii.]$ $\sum_{n=1}^{\infty} \alpha_n\beta_n = 0$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

In 2007, Panyanak [11] extended the above result of Sastry and Babu [12] to uniformly convex real Banach spaces. He proved the following result. Before we state his theorem, we need the following definition.

Definition 1.1 [25] *A mapping $T : K \to CB(K)$ is said to satisfy condition (I) if there exists a strictly increasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that $d(x, T(x)) \ge f(d(x, F(T))), \forall\, x \in D$.*

Theorem 1.2 *Let E be a uniformly convex real Banach space. Let K be a nonempty, closed, bounded and convex subset of E, and $T : K \to P(K)$ be a multivalued nonexpansive mapping that satisfies condition (I). Assume that $[i.]$ $0 \le \alpha_n < 1$, $[ii.]$ $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose that $F(T)$ is a* *nonempty proximinal subset of K. Let $\{x_n\}$ be defined by*

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n, \end{cases} \tag{7}$$

$\alpha_n \in [a,b], 0 < a < b < 1, n \geq 0$, where $y_n \in Tx_n$ is such that $\|y_n - u_n\| = d(u_n, Tx_n)$, and $u_n \in F(T)$ is such that $\|x_n - u_n\| = d(x_n, F(T))$. Then, $\{x_n\}$ converges strongly to a fixed point of T.

The scheme of Sastry and Babu [12] requires knowing points $p \in F(T)$. This seems inappropriate because, if a fixed point is already known there is no need to construct a scheme to search for it. Panyanak's [11] scheme also seems to have a similar difficulty. In 2008, Song and Wang [13] proved the following theorem.

Theorem 1.3 *Let K be a nonempty, compact and convex subset of a uniformly convex real Banach space E. Let $T : K \to CB(K)$ be a multivalued nonexpansive mapping with $F(T) \neq \emptyset$ satisfying $T(p) = p$ for all $p \in F(T)$. Assume that $[i.]$ $0 \leq \alpha_n, \beta_n < 1$, $[ii.]$ $\beta_n, \gamma_n \to 0$, $[iii.]$ $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$. Then, the sequence $\{x_n\}$ defined by*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n z_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n u_n, \end{cases} \tag{8}$$

where $z_n \in Tx_n, u_n \in Ty_n$, are such that $\|z_n - u_n\| = D(Tx_n, Ty_n) + \gamma_n$ and $\|z_{n+1} - u_n\| \leq D(Tx_{n+1}, Ty_n) + \gamma_n$. Then, the sequence in (8) converges strongly to a fixed point of T.

Recently, Shahzad and Zegeye [15] showed their concerns on the work of Song and Wang [13]. In particular, they pointed out that the assumption "$Tp = \{p\}$ for any $p \in F(T)$" in [13] is quite strong. They observed that if E is a normed linear space and $T : D(T) \subset E \to P(E)$ is any multivalued mapping then the mapping $P_T : D(T) \to P(E)$ defined for each x by

$$P_T(x) = \{y \in Tx : d(x, Tx) = \|x - y\|\}, \tag{9}$$

has the property that $P_T(p) = \{p\}$ for all $p \in F(T)$. Using this idea they removed the strong condition "$T(p) = \{p\}$ for all $p \in F(T)$" and extended and improved the results of Song and Wang [13] to multivalued quasi-nonexpansive mappings. The assumption that K is compact is dispensed with. Also, in an attempt to remove the restriction $Tp = \{p\}, \forall p \in F(T)$ in Theorem 1.3, they introduced a new iteration scheme as follows: Let K be a nonempty closed convex subset of a real Banach space E. Let $\alpha_n, \beta_n \in [0,1]$. Choose $x_0 \in K$ and define $\{x_n\}$ as follows:

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n z_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n u_n, \end{cases} \tag{10}$$

where $z_n \in P_T x_n, u_n \in P_T y_n$. Then, they proved the following result.

Theorem 1.4 [15] *Let E be a uniformly convex real Banach space, K be a nonempty, closed and convex subset of E, and $T : K \to P(K)$ be a multivalued mapping with $F(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be the iterates defined by (10). Assume that T satisfies condition (I) and $\alpha_n, \beta_n \in [0,1]$. Then, $\{x_n\}$ converges strongly to a fixed point of T.*

We note that Song and Wang [13] imposed the assumption that K is compact while Shahzad and Zegeye [15] imposed the condition that the mapping T satisfies condition (I).

It is our purpose in this paper to introduce an iterative scheme which converges strongly to a common point of the fixed point set of a finite family of Lipschitz hemicintractive-type mappings under some mild conditions. As consequence, we obtain a convergent sequence to a common point of the fixed point set of a finite family of k-strictly pseudocontractive-type mappings which extend results in the literature that rely on either compactness of K or T or Condition (I) for strong convergence to common fixed points.

2. Preliminaries

Definition 2.1 *Let E be a Banach space. Let $T : D(T) \subseteq E \to 2^E$ be a multivalued mapping. $I - T$ is said to be demiclosed at zero, if for any sequence $\{x_n\} \subseteq D(T)$ such that $\{x_n\}$ converges weakly to p and $D(x_n, Tx_n) \to 0$, then $p \in Tp$.*

Lemma 2.1 [16] *Let H be a real Hilbert space. Then, the following equations hold:*

(1) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \forall t \in [0,1]$,

(2) Given any x, y in H, $\|x-y\|^2 = \|x-z\|^2 + \|z-y\|^2 + 2\langle x-z, z-y\rangle$.

Lemma 2.2 [4] *Let H be a real Hilbert space. Then, the following equation holds: If $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup z \in H$, then*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z - y\|^2, \forall y \in H.$$

Lemma 2.3 [20] *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $T : K \to CBC(K)$ be a multivalued mapping and $P_T(x) = \{y \in Tx : \|x - y\| = d(x, Tx)\}$. Then, for any $x \in K, x_0 \in P_T(x)$ if and only if $\langle z - x_0, x - x_0\rangle \le 0, \forall z \in Tx$.*

Lemma 2.4 [21] *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$, for all $i \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \le a_{m_k+1}, \text{ and } a_k \le a_{m_k+1}.$$

In fact, $m_k := \max\{j \le k : a_j < a_{j+1}\}$.

Lemma 2.5 [22] *Let K be a metric space. Let $T : K \to P(K)$ be a multivalued mapping. Then, the following are equivalent: (i) $x \in Tx$, (ii) $P_T x = \{x\}$ and (iii) $x \in F(P_T)$. Moreover, $F(T) = F(P_T)$.*

Lemma 2.6 *Let H be a real Hilbert space. Then,*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

Lemma 2.7 [3] *Let H be a Hilbert space. Let K be a nonempty closed and convex subset of H. Let $T : K \to CB(K)$ be k-strictly pseudocontractive-type multivalued mapping. Then T is L-Lipschitz mapping.*

Lemma 2.8 [1] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n\delta_n, n \ge n_0,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\delta_n\} \subset R$ satisfying the following conditions:
$\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\limsup_{n\to\infty} \delta_n \le 0$. *Then,* $\lim_{n\to\infty} a_n = 0$.

3. Main results

Theorem 3.1 *Let K be a non-empty, closed and convex subset of a real Hilbert space H. Let $T_i : K \to CB(K), i = 1,2,\dots,N$, be a finite family of Lipschitz hemicontractive-type mappings with Lipschitz constants $L_i, i = 1,2,\dots,N$, respectively. Assume that $I - T_i, i = 1,\dots,N$ are demiclosed at zero and $\mathscr{F} = \cap_{i=1}^N F(T_i)$ is non-empty, closed and convex with $T_i(p) = \{p\}$, $\forall p \in F(T)$ and for each $i = 1,2,\dots,N$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 = w \in K$ by*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n u_n,\ u_n \in T_n x_n, \\ z_n = \gamma_n w_n + (1-\gamma_n)x_n,\ w_n \in T_n y_n, \\ x_{n+1} = \alpha_n w + (1-\alpha_n)z_n,\ n \ge 1, \end{cases} \tag{11}$$

where $T_n := T_n(\mod N)$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ satisfy the following conditions:

i. $0 \le \alpha_n \le c < 1$, $\forall n \ge 1$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

ii. $0 < \alpha \le \gamma_n \le \beta_n \le \beta < \dfrac{1}{\sqrt{4L^2+1}+1}$, $\forall n \ge 1$, *for* $L := \max\{L_i : 1,2,\dots,N\}$.

Then, $\{x_n\}$ converges strongly to some point p in $\mathscr{F}$ nearest to w.

Proof. Let $p = P_{\mathscr{F}}(w)$. Now, using (1) of Lemma 2.1,

$$\begin{aligned} \|x_{n+1}-p\|^2 &= \|\alpha_n(w-p)+(1-\alpha_n)(z_n-p)\|^2 \\ &\le \alpha_n\|w-p\|^2+(1-\alpha_n)\|z_n-p\|^2 \\ &= \alpha_n\|w-p\|^2+(1-\alpha_n)\|\gamma_n(w_n-p)+(1-\gamma_n)(x_n-p)\|^2 \\ &= \alpha_n\|w-p\|^2+(1-\alpha_n)\gamma_n\|w_n-p\|^2+(1-\alpha_n)(1-\gamma_n)\|x_n-p\|^2 \\ &\quad -(1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n-x_n\|^2 \\ &= \alpha_n\|w-p\|^2+(1-\alpha_n)(1-\gamma_n)\|x_n-p\|^2+(1-\alpha_n)\gamma_n\|w_n-p\|^2 \\ &\quad -(1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n-x_n\|^2 \\ &\le \alpha_n\|w-p\|^2+(1-\alpha_n)(1-\gamma_n)\|x_n-p\|^2+(1-\alpha_n)\gamma_n D(T_ny_n,T_np)^2 \\ &\quad -(1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n-x_n\|^2 \end{aligned}$$

$$\begin{aligned} &\leq \alpha_n\|w-p\|^2+(1-\alpha_n)(1-\gamma_n)\|x_n-p\|^2+(1-\alpha_n)\gamma_n \\ &\quad \left[\|y_n-p\|^2+\|y_n-w_n\|^2\right]-(1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n-x_n\|^2. \end{aligned}$$

Thus,

$$\begin{aligned} \|x_{n+1}-p\|^2 &\leq \alpha_n\|w-p\|^2+(1-\alpha_n)(1-\gamma_n)\|x_n-p\|^2+(1-\alpha_n) \\ &\quad \times \gamma_n\|y_n-p\|^2+(1-\alpha_n)\gamma_n\|y_n-w_n\|^2-(1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n-x_n\|^2. \end{aligned} \tag{12}$$

On the other hand, from (11) and the fact that $\|u_n-w_n\| \leq 2D(T_nx_n, T_ny_n)$ we have

$$\begin{aligned} \|y_n-w_n\|^2 &= \|(1-\beta_n)(x_n-w_n)+\beta_n(u_n-w_n)\|^2 \\ &= (1-\beta_n)\|x_n-w_n\|^2+\beta_n\|u_n-w_n\|^2-\beta_n(1-\beta_n)\|x_n-u_n\|^2 \\ &\leq (1-\beta_n)\|x_n-w_n\|^2+\beta_n 4D^2(T_nx_n,T_ny_n)-\beta_n(1-\beta_n)\|x_n-u_n\|^2 \\ &\leq (1-\beta_n)\|x_n-w_n\|^2+\beta_n 4L^2\|x_n-y_n\|^2-\beta_n(1-\beta_n)\|x_n-u_n\|^2 \\ &\leq (1-\beta_n)\|x_n-w_n\|^2+4L^2\beta_n^3\|x_n-u_n\|^2-\beta_n(1-\beta_n)\|x_n-u_n\|^2. \end{aligned}$$

Hence,

$$\|y_n-w_n\|^2 \leq (1-\beta_n)\|x_n-w_n\|^2-\beta_n(1-\beta_n-4L^2\beta_n^2)\|x_n-u_n\|^2. \tag{13}$$

Again,

$$\begin{aligned} \|y_n-p\|^2 &= \|(1-\beta_n)x_n+\beta_nu_n-p)\|^2 \\ &= \|(1-\beta_n)(x_n-p)+\beta_n(u_n-p)\|^2 \\ &= (1-\beta_n)\|x_n-p\|^2+\beta_n\|u_n-p\|^2-\beta_n(1-\beta_n)\|x_n-u_n\|^2 \\ &\leq (1-\beta_n)\|x_n-p\|^2+\beta_nD^2(T_nx_n,T_np)-\beta_n(1-\beta_n)\|x_n-u_n\|^2 \\ &\leq (1-\beta_n)\|x_n-p\|^2+\beta_n\left[\|x_n-p\|^2+\|x_n-u_n\|^2\right] \\ &\quad -\beta_n(1-\beta_n)\|x_n-u_n\|^2. \end{aligned}$$

Thus,

$$\|y_n-p\|^2 \leq \|x_n-p\|^2+\beta^2\|x_n-u_n\|^2. \tag{14}$$

Now substituting (14), (13) into (12),

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \alpha_n\|w-p\|^2+(1-\alpha_n)(1-\gamma_n)\|x_n-p\|^2+(1-\alpha_n)\gamma_n||x_n-p||^2 \\
&+ (1-\alpha_n)\gamma_n\beta_n^2\|x_n-u_n\|^2+(1-\alpha_n)\gamma_n(1-\beta_n)\|x_n-w_n\|^2 \\
&- \beta_n(1-\alpha_n)\gamma_n(1-\beta_n-4L^2\beta_n^2)\|u_n-x_n\|^2 \\
&\quad -(1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n-x_n\|^2.
\end{aligned}$$

which reduces to

$$\begin{aligned}
(15)\quad \|x_{n+1}-p\|^2 \leq \alpha_n\|w-p\|^2+(1-\alpha_n)\|x_n-p\|^2-\beta_n(1-\alpha_n) \\
\times\gamma_n(1-2\beta_n-4L^2\beta_n^2)\|u_n-x_n\|^2+(1-\alpha_n)\gamma_n(\gamma_n-\beta_n)\|x_n-w_n\|^2.
\end{aligned}$$

From the hypothesis (ii) in (11) we have that

$$1-2\beta_n-4L^2\beta_n^2 \geq 1-2\beta-4L^2\beta^2, \tag{16}$$

$$\gamma_n \leq \beta_n. \tag{17}$$

Using (16) and (17) in (15) we get that

$$\|x_{n+1}-p\|^2 \leq (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|w-p\|^2. \tag{18}$$

Thus, by induction

$$\|x_{n+1}-p\|^2 \leq \max\{\|x_1-p\|^2, \|w-p\|^2\},\ \forall n \geq 1.$$

This implies that $\{x_n\},\{y_n\}$ and $\{z_n\}$ are all bounded. Furthermore, from (11), Lemma 2.6 and (15) we get that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(1-\alpha_n)(\gamma_n w_n+(1-\gamma_n)x_n)+\alpha_n w-p\|^2\\
&= \|(1-\alpha_n)((\gamma_n w_n+(1-\gamma_n)x_n)-p)+\alpha_n(w-p)\|^2\\
&\leq (1-\alpha_n)\|\gamma_n w_n+(1-\gamma_n)x_n-p\|^2+2\alpha_n\langle w-p,x_{n+1}-p\rangle\\
&= (1-\alpha_n)\left[\gamma_n\|w_n-p\|^2+(1-\gamma_n)\|x_n-p\|^2-\gamma_n(1-\gamma_n)\|x_n-w_n\|^2\right]\\
&\quad +2\alpha_n\langle w-p,x_{n+1}-p\rangle\\
&\leq (1-\alpha_n)\left[\gamma_n D(T_n y_n,T_n p)^2+(1-\gamma_n)\|x_n-p\|^2-\gamma_n(1-\gamma_n)\|x_n-w_n\|^2\right]\\
&\quad +2\alpha_n\langle w-p,x_{n+1}-p\rangle\\
&\leq (1-\alpha_n)\left[\gamma_n(\|y_n-p\|^2+\|y_n-w_n\|^2)+(1-\gamma_n)\|x_n-p\|^2\right]\\
&\quad -(1-\alpha_n)\gamma_n(1-\gamma_n)\|x_n-w_n\|^2+2\alpha_n\langle w-p,x_{n+1}-p\rangle,
\end{aligned} \tag{19}$$

which implies

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (1-\alpha_n)\gamma_n\|x_n-p\|^2+(1-\alpha_n)\gamma_n\beta_n^2\|x_n-u_n\|^2+(1-\alpha_n)\gamma_n\\
&\quad \times\left[(1-\beta_n)\|x_n-w_n\|^2-\beta_n(1-\beta_n-4L^2\beta_n^2)\|x_n-u_n\|^2\right]\\
&\quad -(1-\alpha_n)\gamma_n(1-\gamma_n)\|w_n-x_n\|^2+2\alpha_n\langle w-p,x_{n+1}-p\rangle\\
&= (1-\alpha_n)\|x_n-p\|^2-(1-\alpha_n)\gamma_n\beta_n(1-2\beta_n-4L^2\beta_n^2)\|x_n-u_n\|^2\\
&\quad +2\alpha_n\langle w-p,x_{n+1}-p\rangle+(1-\alpha_n)\gamma_n(\gamma_n-\beta_n)\|x_n-w_n\|^2.
\end{aligned}$$

That is, we get that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (1-\alpha_n)\|x_n-p\|^2-(1-\alpha_n)\gamma_n\beta_n(1-2\beta_n-4L^2\beta_n^2)\\
&\quad \times\|x_n-u_n\|^2+2\alpha_n\langle w-p,x_{n+1}-p\rangle,
\end{aligned} \tag{20}$$

and

$$\begin{aligned} \|x_{n+1}-p\|^2 &\leq (1-\alpha_n)\|x_n-p\|^2-(1-c)\alpha^2(1-2\beta-4L^2\beta^2) \\ &\quad \times\|x_n-u_n\|^2+2\alpha_n\langle w-p,x_{n+1}-p\rangle. \end{aligned} \tag{21}$$

Now we consider the following two cases:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n-p\|\}$ is non-increasing, $\forall n \geq n_0$. Then, we get that $\{\|x_n-p\|\}$ is convergent. So, from (21) and the fact that $\alpha_n \to 0$, we have that

$$(1-c)\alpha^2(1-2\beta-4L^2\beta^2)\|x_n-u_n\|^2 \leq (1-\alpha_n)\|x_n-p\|^2-\|x_{n+1}-p\|^2,$$

which gives that

$$x_n-u_n \to 0. \tag{22}$$

Now, from (11) and (22) we get

$$y_n-x_n=\beta_n(u_n-x_n) \to 0,$$

and hence we get that

$$\begin{aligned} \|z_n-x_n\| &= \gamma_n\|w_n-x_n\|=\gamma_n\|w_n-u_n+u_n-x_n\| \\ &\leq \gamma_n\|w_n-u_n\|+\gamma_n\|u_n-x_n\| \\ &\leq \gamma_n 2D(T_ny_n,T_nx_n)+\gamma_n\|u_n-x_n\| \\ &\leq \gamma_n 2L\|y_n-x_n\|+\gamma_n\|u_n-x_n\| \to 0, \end{aligned} \tag{23}$$

and by (11), (23), the fact that $\|w-z_n\|$ is bounded and $\alpha_n \to 0$, we have

$$\begin{aligned} \|x_{n+1}-x_n\| &= \|x_{n+1}-z_n+z_n-x_n\| \\ &\leq \|x_{n+1}-z_n\|+\|z_n-x_n\| \\ &= \alpha_n\|w-z_n\|+\|z_n-x_n\| \to 0. \end{aligned} \tag{24}$$

But then, since, $\|x_{n+i}-x_n\| \leq \|x_{n+i}-x_{n+i-1}\|+\ldots+\|x_{n+1}-x_n\|$, we get that

$$\|x_{n+i}-x_n\| \to 0,\ \forall i=1,2,\ldots,N. \tag{25}$$

Now, since $T_{n+i}x_n$ and $T_{n+i}x_{n+i}$ are closed and bounded there exist $u_n^* \in T_{n+i}x_n$ and $u_{n+i}^* \in T_{n+i}x_{n+i}$ such that $\|x_n - u_n^*\| = d(x_n, T_{n+i}x_n)$ and $\|x_{n+i} - u_{n+i}^*\| = d(x_{n+i}, T_{n+i}x_{n+i})$. Now, by (22) and (25)

$$\begin{aligned}
d(x_n, T_{n+i}x_n) &= \|x_n - u_n^*\| \\
&\leq \|x_n - x_{n+i}\| + \|x_{n+i} - u_n^*\| \\
&\leq \|x_n - x_{n+i}\| + \|x_{n+i} - u_{n+i}^*\| + \|u_{n+i}^* - u_n^*\| \\
&\leq \|x_n - x_{n+i}\| + \|x_{n+i} - u_{n+i}^*\| + 2D(T_{n+i}x_{n+i}, T_{n+i}x_n) \\
&\leq \|x_n - x_{n+i}\| + \|x_{n+i} - u_{n+i}^*\| + 2L\|x_n - x_{n+i}\| \to 0. \qquad (26)
\end{aligned}$$

Now, since $\{\|x_n - p\|\}$ converges, there exists a subsequence $\{x_{n_j+1}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle w - p, x_{n+1} - p\rangle = \lim_{j\to\infty} \langle w - p, x_{n_j+1} - p\rangle,$$

and $x_{n_j+1} \rightharpoonup z$, for some $z \in K$. Now, from (24) we get $x_{n_j} \rightharpoonup z$. Hence, from (26) and since T_i, $\forall i = 1, \ldots, N$ are demiclosed by assumption, we get that $z \in F(T_i)$, $\forall i = 1, \ldots, N$, i.e., $z \in \mathscr{F}$. Therefore, since by assumption $\mathscr{F}$ is closed and convex, Lemma 2.3 implies that

$$\begin{aligned}
\limsup_{n\to\infty} \langle w - p, x_{n+1} - p\rangle &= \lim_{j\to\infty} \langle w - p, x_{n_j+1} - p\rangle \\
&= \langle w - p, z - p\rangle \leq 0. \qquad (27)
\end{aligned}$$

Now, from (21) we have that

$$\|x_{n+1} - p\|^2 \leq (1 - \alpha_n)\|x_n - p\|^2 + 2\alpha_n \langle w - p, x_{n+1} - p\rangle. \qquad (28)$$

It then follows from (28), (27) and Lemma 2.8 that $\|x_n - p\| \to 0$, i.e., $x_n \to p$.

Case 2 Suppose there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\|x_{n_k} - p\| < \|x_{n_k+1} - p\|, \ \forall k \in \mathbb{N}.$$

Thus, by Lemma 2.4, there is a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, $\|x_{m_k} - p\| \leq \|x_{m_k+1} - p\|$ and $\|x_k - p\| \leq \|x_{m_k+1} - p\|$, $\forall k \in \mathbb{N}$. Now, from (21) and the fact that $\alpha_n \to 0$

we get that $x_{m_k} - u_{m_k} \to 0$, when $u_{m_k} \in T_i x_{m_k}$, $\forall i = 1, \ldots, N$. Hence as in Case 1, $x_{m_k+1} - x_{m_k} \to 0$ and that

$$\limsup_{n\to\infty} \langle w - p, x_{m_k+1} - p\rangle \leq 0. \tag{29}$$

From (21) we have that

$$\|x_{m_k+1} - p\|^2 \leq (1-\alpha_{m_k})\|x_{m_k} - p\|^2 + 2\alpha_{m_k}\langle w - p, x_{m_k+1} - p\rangle, \tag{30}$$

and since $\|x_{m_k} - p\| \leq \|x_{m_k+1} - p\|$, (30) implies that

$$\begin{aligned} \alpha_{m_k}\|x_{m_k} - p\|^2 &\leq \|x_{m_k} - p\|^2 - \|x_{m_k+1} - p\|^2 + 2\alpha_{m_k}\langle w - p, x_{m_k+1} - p\rangle \\ &\leq 2\alpha_{m_k}\langle w - p, x_{m_k+1} - p\rangle \end{aligned}$$

So, from (29) we get that $\|x_{m_k} - p\|^2 \leq 2\alpha_{m_k}\langle w - p, x_{m_k+1} - p\rangle \leq 0$. Hence, $x_{m_k} \to p$ which implies from (30) that $\|x_{m_k+1} - p\| \to 0$ as $k \to \infty$. But, $\|x_k - p\| \leq \|x_{m_k+1} - p\|$, $\forall k \in \mathbb{N}$. Therefore, $\{x_n\}$ converges strongly to an element p in $\mathscr{F}$ nearest to w.

Remark 3.2 *We note that, since every pseudocontractive-type mapping with $F(T) \neq \emptyset$ is hemicontractive-type the above theorem holds for a finite family of pseudocontractive-type mappings.*

Lemma 3.3 *Again, since every quasi-nonexpansive type is a demicontractive-type and every demicontractive-type mapping is hemicontractive-type the above theorem also holds for a finite family of quasi-nonexpansive type and demicontractive-type mappings.*

If, in Theorem 3.1, we consider a single hemicontractive-type mapping we get the following corollary.

Corollary 3.4 *Let H be a real Hilbert space and K be a non-empty, closed and convex subset of H. Let $T : K \to CB(K)$, be Lipschitz hemicontractive-type mapping with Lipschitz constant L. Assume that $I - T$ is demiclosed at zero and $F(T)$ is non-empty, closed and convex with*

$T(p)=\{p\}$, $\forall p\in F(T)$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1=w\in K$ by

$$
\begin{cases}
y_n=(1-\beta_n)x_n+\beta_n u_n,\ u_n\in Tx_n,\\
z_n=\gamma_n w_n+(1-\gamma_n)x_n,\ w_n\in Ty_n,\\
x_{n+1}=\alpha_n w+(1-\alpha_n)z_n,\ n\geq 1,
\end{cases}
\tag{31}
$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}\subset(0,1)$ satisfy the following conditions:

i. $0\leq\alpha_n\leq c<1$, $\forall n\geq 1$ *such that* $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$,

ii. $0<\alpha\leq\gamma_n\leq\beta_n\leq\beta<\dfrac{1}{\sqrt{4L^2+1}+1}$.

Then, $\{x_n\}$ converges strongly to some point p in $\mathscr{F}$ nearest to w.

Proof. *Put $T_i:=T$, $\forall i=1,\ldots,N$ in (11) and the scheme reduces to (31). Now, as in (20) and (21)*

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (1-\alpha_n)\|x_n-p\|^2-(1-\alpha_n)\gamma_n\beta_n(1-2\beta_n-4L^2\beta_n^2)\\
&\quad\times\|x_n-u_n\|^2+2\alpha_n\langle w-p,x_{n+1}-p\rangle,\ u_n\in Tx_n\\
&\leq (1-\alpha_n)\|x_n-p\|^2-(1-c)\alpha^2(1-2\beta-4L^2\beta^2)\|x_n-u_n\|^2\\
&\quad+2\alpha_n\langle w-p,x_{n+1}-p\rangle\\
&\leq (1-\alpha_n)\|x_n-p\|^2+2\alpha_n\langle w-p,x_{n+1}-p\rangle.
\end{aligned}
$$

The rest of the proof is as in Theorem 3.1.

If, in Theorem 3.1 we assume that $P_{T_i}, i=1,\ldots,N$ are Lipschitz hemicontractive-type mappings, then by Lemma 2.5, the requirement that $T_i(p)=\{p\}$ may not be needed. Thus, we obtain the following corollary.

Corollary 3.5 *Let H be a real Hilbert space and K be a non-empty, closed and convex subset of H. Let $T_i:K\to CB(K), i=1,2,\ldots,N$, be a finite family of multivalued mappings. Let $P_{T_i}, i=1,2,\ldots,N$, be Lipschitz hemicontractive-type mappings with Lipschitz constants $L_i, i=1,2,\ldots,N$, respectively. Assume that $I-P_{T_i}, i=1,\ldots,N$ are demiclosed and $\mathscr{F}=\cap_{i=1}^{N}F(T_i)$ is non-empty, closed and convex. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1=w\in K$*

by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n u_n,\ u_n \in P_{T_n}x_n, \\ z_n = \gamma_n w_n + (1-\gamma_n)x_n,\ w_n \in P_{T_n}y_n, \\ x_{n+1} = \alpha_n w + (1-\alpha_n)z_n,\ n \geq 1 \end{cases} \tag{32}$$

where $T_n := T_n(\mod N)$ *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ *satisfy the following conditions:*

i. $0 \leq \alpha_n \leq c < 1$, $\forall n \geq 1$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

ii. $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \frac{1}{\sqrt{4L^2+1}+1}$, $\forall n \geq 1$ *for* $L := \max\{L_i : 1,2,\ldots,N\}$.

Then, $\{x_n\}$ *converges strongly to some point* p *in* $\mathscr{F}$ *nearest to* w.

If, in Theorem 3.1 we assume that $P_{T_i} : K \to CBC(K), i = 1,\ldots,N$ are Lipschitz pseudocontractive-type mappings, then, since $P_{T_i}(x)$ is singleton, for every $x \in C$ by Lemma 2.3 and Lemma 2.5 we have $F(T_i) = F(P_{T_i})$, which is closed and convex and $I - P_{T_i}$ is demiclosed at zero for each $i \in \{1,2,...,,N\}$ and hence the following corollary follows.

Corollary 3.6 *Let H be a real Hilbert space and K be a non-empty, closed and convex subset of H. Let $T_i : K \to CBC(K), i = 1,2,\ldots,N$, be a finite family of multivalued mappings. Let $P_{T_i}, i = 1,2,\ldots,N$, be Lipschitz pseudocontractive-type mappings with Lipschitz constants $L_i, i = 1,2,\ldots,N$, respectively. Suppose that $\mathscr{F} = \cap_{i=1}^{N} F(T_i)$ is non-empty. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 = w \in K$ by*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n u_n,\ u_n \in P_{T_n}x_n, \\ z_n = \gamma_n w_n + (1-\gamma_n)x_n,\ w_n \in P_{T_n}y_n, \\ x_{n+1} = \alpha_n w + (1-\alpha_n)z_n,\ n \geq 1 \end{cases} \tag{33}$$

where $T_n := T_n(\mod N)$ *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ *satisfy the following conditions:*

i. $0 \leq \alpha_n \leq c < 1$, $\forall n \geq 1$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

ii. $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \frac{1}{\sqrt{4L^2+1}+1}$, $\forall n \geq 1$ *for* $L := \max\{L_i : 1,2,\ldots,N\}$.

Then, $\{x_n\}$ *converges strongly to some point* p *in* $\mathscr{F}$ *nearest to* w.

In the sequel we shall make use of the following lammas.

Lemma 3.7 *Let K be a closed, convex, nonempty subset of a real Hilbert space H. Let $T : K \to CB(K)$ be a demicontractive-type multivalued mapping with constant $k \in [0,1)$. Assume that $F(T) \neq \emptyset$. If $T(p) = \{p\}, \forall p \in F(T)$, then $F(T)$ is closed and convex.*

Proof. Let $x, y \in K$. By (2) of Lemma 2.2, we have that

$$\|x-y\|^2 = \|x-z\|^2 + \|z-y\|^2 + 2\langle x-z, z-y\rangle.$$

For $p \in F(T), x \in K$, let $u \in Tx$ be such that $\|p-u\| = \inf\{\|p-y\| : y \in Tx\}$. Then, we get that

$$\|p-u\|^2 = \|p-x\|^2 + \|x-u\|^2 + 2\langle p-x, x-u\rangle,$$

which is the same as,

$$d(p,Tx)^2 = \|p-x\|^2 + \|x-u\|^2 + 2\langle p-x, x-u\rangle,$$

which in turn implies that

$$\|p-x\|^2 + \|x-u\|^2 + 2\langle p-x, x-u\rangle \leq D(p,Tx)^2 \leq \|p-x\|^2 + k\|x-u\|^2.$$

Hence,

$$\|x-u\|^2 \leq \left(\frac{2}{1-k}\right)\langle p-x, u-x\rangle. \tag{34}$$

Now, we show that $F(T)$ is closed. Let $\{x_n\} \subseteq F(T)$ be such that $x_n \to z$. Let $u \in Tz$ such that $\|u-z\| = \inf\{\|z-y\| : y \in Tz\}$. Then, from (34) we have that for each $n \geq 1$,

$$\begin{aligned} \|z-u\|^2 &\leq \left(\frac{2}{1-k}\right)\langle x_n - z, u-z\rangle \\ &\to 0, \text{as} \quad n \to \infty. \end{aligned}$$

Hence, $z = u \in Tz$. Therefore, $F(T)$ is closed. Next, let us show that $F(T)$ is convex. Let $p, q \in F(T)$ and $z = \alpha p + (1-\alpha)q$, where $\alpha \in (0,1)$. Then, we want to show that $z \in F(T)$. Let $u \in Tz$ be such that $\|u-z\| = \inf\{\|z-y\| : y \in Tz\}$. But, from (34),

$$\|x-u\|^2 \leq \left(\frac{2}{1-k}\right)\langle p-x, u-x\rangle$$

and

$$\|x-u\|^2 \leq \left(\frac{2}{1-k}\right)\langle q-x, u-x\rangle.$$

Then,

$$
\begin{aligned}
\|z-u\|^2 &= \alpha\|z-u\|^2+(1-\alpha)\|z-u\|^2 \\
&\leq \left(\frac{2}{1-k}\right)\langle \alpha(p-z)+(1-\alpha)(q-z), u-z\rangle \\
&= \left(\frac{2}{1-k}\right)\langle z-z, u-z\rangle = 0.
\end{aligned}
$$

So, $z=u\in Tz$. Therefore, $F(T)$ is convex.

Lemma 3.8 *Let K be a closed, convex, nonempty subset of a real Hilbert space H. Let $T: K\to CB(K)$ be a k-strictly pseudocontractive-type multivalued mapping with constant $k\in[0,1)$. Then, $I-T$ is demiclosed at zero.*

Proof. Let $\{x_n\}\subseteq K$ be such that $x_n \rightharpoonup y$ and suppose $D(x_n, Tx_n)\to 0$. We want to show that $0\in(I-T)y$, i.e., $y\in Ty$.

Let $q\in Ty$ be arbitrary. Then, there exists $y_n\in Tx_n$ such that

$$
\|y_n-q\|\leq D(Tx_n, Ty),\ \forall n\in\mathbb{N}. \tag{35}
$$

Furthermore, since $y_n\in Tx_n$, we have that

$$
\|x_n-y_n\|\leq D(x_n, Tx_n)\to 0. \tag{36}
$$

Now, define $f: H\to[0,\infty)$ by $f(x):=\limsup\limits_{n\to\infty}\|x_n-x\|^2$. Then, by Lemma 2.2 we get that

$$
f(x)=\limsup_{n\to\infty}\|x_n-y\|^2+\|y-x\|^2,\ \forall x\in H,
$$

which implies that

$$
f(x)=f(y)+\|y-x\|^2, \quad \forall x\in H.
$$

Hence,

$$
f(q)=f(y)+\|y-q\|^2. \tag{37}
$$

On the other hand, using (35), (36) and the fact that T is k-strictly pseudocontractive-type we get that

$$\begin{aligned} f(q) &= \limsup_{n\to\infty} \|x_n - q\|^2 \\ &= \limsup_{n\to\infty} \|x_n - y_n + y_n - q\|^2 \\ &\leq \limsup_{n\to\infty} \|y_n - q\|^2 \\ &\leq \limsup_{n\to\infty} D^2(Tx_n, Ty) \\ &\leq \limsup_{n\to\infty} [\|x_n - y\|^2 + k\|x_n - y_n + q - y\|^2] \\ &\leq \limsup_{n\to\infty} [\|x_n - y\|^2 + k\|q - y\|^2], \end{aligned}$$

which gives that

$$f(q) \leq f(y) + k\|q - y\|^2. \tag{38}$$

Thus, from (37) and (38) we get that $\|y-q\|^2 \leq k\|y-q\|^2$ or $(1-k)\|y-q\|^2 \leq 0$. This implies $y = q \in Ty$. Therefore, $I - T$ is demiclosed.

If, in Theorem 3.1, we assume that $T_i, i = 1, \ldots, N$, are k-strictly pseudocontractive-type mappings then by Proposition , T_i are Lipschitz with $L_i = \dfrac{1+\sqrt{k_i}}{1-\sqrt{k_i}}, i = 1, \ldots, N.$ Also by Lemma 3.7 and 3.8, we have that $F(T)$ is closed and convex and $I - T_i$ are demiclosed. Hence, we have the following theorem.

Theorem 3.9 *Let H be a real Hilbert space and K be a non-empty, closed and convex subset of H. Let $T_i : K \to CB(K), i = 1, 2, \ldots, N$, be a finite family of k-strictly pseudocontractive-type mappings. Assume that $\mathscr{F} = \cap_{i=1}^N F(T_i)$ is non-empty with $T_i(p) = \{p\}$, $\forall p \in F(T)$ and for each $i = 1, 2, \ldots, N$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 = w \in K$ by*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n u_n, \; u_n \in T_n x_n, \\ z_n = \gamma_n w_n + (1-\gamma_n)x_n, \; w_n \in T_n y_n, \\ x_{n+1} = \alpha_n w + (1-\alpha_n)z_n, \; n \geq 1 \end{cases} \tag{39}$$

where $T_n := T_n(\mod N)$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ satisfy the following conditions:

i. $0 \leq \alpha_n \leq c < 1$, $\forall n \geq 1$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$, *ii.* $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \dfrac{1}{\sqrt{4L^2+1}+1}$, $\forall n \geq 1$ *for* $L := \max\{\dfrac{1+\sqrt{k_i}}{1-\sqrt{k_i}}, i = 1, \ldots, N\}$.

Then, $\{x_n\}$ *converges strongly to some point* p *in* $\mathscr{F}$ *nearest to* w.

If, in Theorem 3.9, we assume that P_{T_i} are k-strictly pseudocontractive-type mappings, we have that P_{T_i} are Lipschitz, and hence the following corollary follows.

Corollary 3.10 *Let* H *be a real Hilbert space and* K *be a non-empty, closed and convex subset of* H. *Let* $T_i : K \to CBC(K), i = 1, 2, \ldots, N$, *be a finite family of multivalued mappings. Let* $P_{T_i}, i = 1, 2, \ldots, N$, *be k- strictly pseudocontractive-type mappings. Suppose also that* $\mathscr{F} = \cap_{i=1}^{N} F(T_i)$ *is non-empty. Let* $\{x_n\}$ *be the sequence generated from an arbitrary* $x_1 = w \in K$ *by*

$$
\begin{cases}
y_n = (1-\beta_n)x_n + \beta_n u_n, \; u_n \in P_{T_n}x_n, \\
z_n = \gamma_n w_n + (1-\gamma_n)x_n, \; w_n \in P_{T_n}y_n, \\
x_{n+1} = \alpha_n w + (1-\alpha_n)z_n, \; n \geq 1
\end{cases}
\tag{40}
$$

where $T_n := T_n(\mod N)$ *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ *satisfy the following conditions:*

i. $0 \leq \alpha_n \leq c < 1$, $\forall n \geq 1$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

ii. $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \dfrac{1}{\sqrt{4L^2+1}+1}$, $\forall n \geq 1$ *for* $L := \max\{L_i : 1, 2, \ldots, N\}$.

Then, $\{x_n\}$ *converges strongly to some point* p *in* $\mathscr{F}$ *nearest to* w.

If, in Theorem 3.9, we assume that $T_i, i = 1, \ldots, N$, are nonexpansive-type mappings then T_i are Lipschitz with $L = 1$ and k-strictly pseudocontractive-type with $k = 0$. So, we get the following corollary.

Corollary 3.11 *Let* H *be a real Hilbert space and* K *be a non-empty, closed and convex subset of* H. *Let* $T_i : K \to CB(K), i = 1, 2, \ldots, N$, *be a finite family of nonexpansive-type mappings. Assume that* $F = \cap_{i=1}^{N} F(T_i)$ *is non-empty with* $T_i(p) = \{p\}$, $\forall p \in F(T)$ *and for each* $i = 1, 2, \ldots, N$. *Let* $\{x_n\}$ *be the sequence generated from an arbitrary* $x_1 = w \in K$ *by*

$$
\begin{cases}
y_n = (1-\beta_n)x_n + \beta_n u_n, \; u_n \in T_n x_n, \\
z_n = \gamma_n w_n + (1-\gamma_n)x_n, \; w_n \in T_n y_n, \\
x_{n+1} = \alpha_n w + (1-\alpha_n)z_n, \; n \geq 1
\end{cases}
\tag{41}
$$

where $T_n := T_n(\mod N)$ *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ *satisfy the following conditions:*

i. $0 \leq \alpha_n \leq c < 1$, $\forall n \geq 1$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

ii. $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \frac{1}{\sqrt{5}+1}$, $\forall n \geq 1$.

Then, $\{x_n\}$ *converges strongly to some point* p *in* $\mathscr{F}$ *nearest to* w.

Remark 3.12 *The definitions of hemicontractive-type, demicontractive-type, k-strictly pseudo-contractive-type and pseudocontractive-type multivalued mappings used here are those considered by Chidume et al* [3]. *Isiogugu* [4] *defined these mappings somewhat differently (See also* [23]*).*

Remark 3.13 *Theorem 3.1 improves Theorem 1 and Theorem 2 of Sang and Wang* [13] *and Theorem 2.7 of Shahzad and Zegeye* [15] *in the sense that no compactness assumption on either the domain or in the functions* T_i *are assumed. Furthermore, the requirement that* T *satisfies Condition (I) is dispensed with in our more general setting.*

Remark 3.14 *Our work extends the work of Daman and Zegeye* [24] *for the multivalued case.*

Conflict of Interests

The authors declare that there is no conflict of interests.

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