NOOR ITERATION FOR FIXED POINT AND VARIATIONAL INCLUSION PROBLEMS

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Abstract. In this article, Noor iteration is considered for finding a common element in the set of fixed points of a non-expansive mapping and in the set of solutions of a variational inclusion problem. Strong convergence theorems are established in the framework of Hilbert spaces.

Keywords: monotone operator; nonexpansive mapping; fixed point; Noor iteration.

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1. Introduction-Preliminaries

Variational inclusion problems are being used as mathematical programming models to study a large number of optimization problems arising in finance, economics, network, transportation, and engineering sciences; see [1-21] and the references therein.

Let $H$ be a real Hilbert space $H$ and $A$ a mapping on $H$. Recall that $A$ is said to be monotone if

$$
\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in H;
$$

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A is said to be $\alpha$-strongly monotone if there exists a constant $\alpha > 0$ such that
\[ \langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in H; \]
A is said to be $\alpha$-strongly anti-monotone if there exists a constant $\alpha > 0$ such that
\[ \langle Ax - Ay, x - y \rangle \leq (-\alpha) \|x - y\|^2, \quad \forall x, y \in H; \]
A is said to be $L$-Lipschitz continuous if there exists a constant such that
\[ \|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in H; \]
A is said to be nonexpansive if
\[ \|Ax - Ay\| \leq \|x - y\|, \quad \forall x, y \in H. \]
A is said to be strictly pseudocontractive if
\[ \|Ax - Ay\|^2 \leq \|x - y\|^2 + \kappa \|(I - A)x - (I - A)y\|^2, \quad \forall x, y \in H. \]

Let $C$ be a nonempty, closed and convex subset of $H$. Recall that the classical variational inequality problem is to find $u \in C$ such that
\[ \langle Au, v - u \rangle \geq 0, \quad \forall v \in C. \tag{1.1} \]

One can see that the variational inequality problem (1.1) is equivalent to a fixed point problem. $u \in C$ is a solution of the variational inequality (1.1) if and only if $u \in C$ is a fixed point of the mapping $P_C(I - \lambda A)$, where $I$ is the identity mapping and $\lambda > 0$ is a constant.

Recently, Noor and Huang [15] consider a three-step iterative method for finding a common element in the set of fixed points of a non-expansive mapping and in the set of solutions of the variational inequality problem (1.1) in a real Hilbert space. To be more precise, they introduced the following algorithm:

\[
\begin{cases}
  x_0 \in C, \\
  z_n = (1 - c_n)x_n + c_nSP_C(x_n - \rho T x_n), \\
  y_n = (1 - b_n)x_n + b_nSP_C(y_n - \rho T y_n), \\
  x_{n+1} = (1 - a_n)x_n + a_nSP_C(y_n - \rho T y_n), \quad \forall n \geq 0
\end{cases}
\]
where \( \{a_n\}, \{b_n\} \) and \( \{c_n\} \) are sequences in \([0, 1]\) for all \( n \geq 0 \). \( S \) is a non-expansive mapping and \( T \) is a monotone-type operator. They showed that the sequence \( \{x_n\} \) generated by the above iterative sequence converges strongly to a common element in the set of fixed points of a non-expansive mapping \( S \) and in the set of solutions of the variational inequality problem (1.1); see [15] for details.

In [16], Noor and Huang considered the following variational inclusion problem. Find an \( u \in H \) such that

\[
0 \in Au + Tu,
\]

where \( T \) and \( A \) are monotone operators. They also consider the following three-step iterative algorithm:

\[
\begin{aligned}
& x_0 \in H, \\
& z_n = (1 - c_n)x_n + c_n SJ_A(x_n - \rho Tx_n), \\
& y_n = (1 - b_n)x_n + b_n SJ_A(y_n - \rho Ty_n), \\
& x_{n+1} = (1 - a_n)x_n + a_n SJ_A(y_n - \rho Ty_n), \quad \forall n \geq 0
\end{aligned}
\]

where \( \{a_n\}, \{b_n\} \) and \( \{c_n\} \) are sequences in \([0, 1]\) for all \( n \geq 0 \), \( S \) is a non-expansive mapping, \( J_A = (I + \rho A)^{-1} \). They showed that the sequence \( \{x_n\} \) generated by the above iterative sequence converges strongly to a common element in the set of fixed points of a non-expansive mapping \( S \) and in the set of solutions of the variational inclusion problem (1.2); see [16] for details.

Motivated by the recent research work, we continue to study the problem of finding a solution of the problem by a Noor iteration.

**Lemma 1.1** [22] Suppose that \( \{\delta_n\} \) is a nonnegative sequence satisfying the following inequality

\[
\delta_{n+1} \leq (1 - \lambda_n)\delta_n, \quad \forall n \geq 0,
\]

where \( \{\lambda_n\} \) is a sequence in \([0, 1]\) such that \( \sum_{n=0}^{\infty} \lambda_n = \infty \). Then \( \lim_{n \to \infty} \delta_n = 0 \).

**Lemma 1.2** [21] Let \( H \) be a Hilbert space. An element \( u \in H \) is a solution of the problem (1.3) if and only if \( u \in H \) is a fixed point of the mapping \( J_A(I + \rho T) \), where \( J_A = (I + \rho A)^{-1} \), \( I \) is the identity mapping and \( T \) is a strongly anti-monotone mapping.
Lemma 1.3. Let $H$ be a Hilbert space and $S : H \to H$ a nonexpansive mapping with a fixed point. Assume that $F(S) \cap S(A, T) \neq \emptyset$. If $u \in F(S) \cap S(A, T)$, then $u = SJ_A(I + \rho T)u$.

Proof. Fix $u \in F(S) \cap S(A, T)$. From Lemma 1.2, we see that $u = J_A(I + \rho T)u$. We also have $u = Su$. It follows that $u = J_A(I + \rho T) = Su = SJ_A(I + \rho T)$. This completes the proof.

2. Main results

Theorem 2.1. Let $H$ be a Hilbert space, $A$ a maximal monotone mapping on $H$ and $T$ an $\alpha$-strongly anti-monotone and $\beta$-Lipschitz continuous mapping on $H$. Let $R : H \to H$ be a strictly pseudocontractive mapping with a fixed point and let $\{x_n\}$ be a sequence generated by the following manner:

$$
\begin{cases}
x_0 \in H, \\
 z_n = (1 - c_n)x_n + c_n(\alpha I + (1 - \alpha)R)J_A(x_n + \rho T x_n), \\
y_n = (1 - b_n)x_n + b_n(\alpha I + (1 - \alpha)R)J_A(z_n + \rho T z_n), \\
x_{n+1} = (1 - a_n)x_n + a_n(\alpha I + (1 - \alpha)R)J_A(y_n + \rho T y_n), \quad \forall n \geq 0
\end{cases}
$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $[0, 1]$ for all $n \geq 0$, $J_A = (I + \rho A)^{-1}$ and $\rho$ is a constant satisfying the restriction $0 < \rho < \frac{2\alpha}{\beta^2}$. Assume that $\kappa \in [\alpha, 1)$ $F(R) \cap S(A, T) \neq \emptyset$ and $\sum_{n=0}^{\infty} a_n = \infty$. Then the sequence $\{x_n\}$ converges strongly to a point in $F(R) \cap S(A, T)$.

Proof. Put $S : = \alpha I + (1 - \alpha)R$. From Zhou [23], we see that $S$ is nonexpansive with $F(R) = F(S)$. Let $x^* \in F(S) \cap S(A, T)$. It follows from (2.1) that

$$
\|x_{n+1} - x^*\| = \|(1 - a_n)(x_n - x^*) + a_n(SJ_A(y_n + \rho T y_n) - SJ_A(x^* + \rho T x^*))\|
\leq (1 - a_n)\|x_n - x^*\| + a_n\|J_A(y_n + \rho T y_n) - J_A(x^* + \rho T x^*)\|
\leq (1 - a_n)\|x_n - x^*\| + a_n\|y_n - x^* + \rho(T y_n - T x^*)\|.
$$
Applying Lemma 1.1, we can conclude the desired conclusion immediately. In a similar way, we can obtain that \( \|y_n - x^*\|^2 \leq 2\rho \alpha \|y_n - x^*\|^2 + \rho^2 \beta^2 \|y_n - x^*\|^2 \)

That is, \( \|y_n - x^*\|^2 \leq \theta \|y_n - x^*\|^2 \), where \( \theta = \sqrt{1 - 2\rho \alpha + \rho^2 \beta^2} \). From the assumption \( 0 < \rho < \frac{2\alpha}{\beta^2} \), we see that \( \theta < 1 \).

Next, we estimate \( \|y_n - x^*\| \). It follows that

\[
\|y_n - x^*\| = \|(1 - b_n)(x_n - x^*) + b_n(SJ_A(z_n + \rho Tz_n) - SJ_A(x^* + \rho Tx^*))\|
\leq (1 - b_n)\|x_n - x^*\| + b_n\|J_A(z_n + \rho Tz_n) - J_A(x^* + \rho Tx^*)\|
\leq (1 - b_n)\|x_n - x^*\| + b_n\|z_n - x^* + \rho (Tz_n - Tx^*)\|
\]

From the \( \alpha \)-strongly anti-monotone and \( \beta \)-Lipschitz assumptions on \( T \), we have

\[
\|z_n - x^* + \rho (Tz_n - Tx^*)\|^2 \leq \|z_n - x^*\|^2 - 2\rho \alpha \|z_n - x^*\|^2 + \rho^2 \beta^2 \|z_n - x^*\|^2
\]

That is, \( \|z_n - x^* + \rho (Tz_n - Tx^*)\| \leq \theta \|z_n - x^*\| \).

Finally, we estimate \( \|z_n - x^*\| \). It follows that

\[
\|z_n - x^*\| \leq (1 - c_n)\|x_n - x^*\| + c_n\|J_A(x_n + \rho Tx_n) - J_A(x^* + \rho T x^*)\|
\leq (1 - c_n)\|x_n - x^*\| + c_n\|x_n - x^* + \rho (Tx_n - Tx^*)\|
\]

In a similar way, we can obtain that \( \|x_n - x^* + \rho (Tx_n - Tx^*)\| \leq \theta \|x_n - x^*\| \). Notice that \( \|z_n - x^*\| \leq [1 - c_n(1 - \theta)]\|x_n - x^*\| \), It follows that that

\[
\|y_n - x^*\| \leq (1 - b_n(1 - \theta (1 - c_n(1 - \theta))))\|x_n - x^*\| \leq \|x_n - x^*\|
\]

It follows that

\[
\|x_{n+1} - x^*\| \leq (1 - a_n)\|x_n - x^*\| + a_n \theta \|y_n - x^*\|
\leq [1 - a_n(1 - \theta)]\|x_n - x^*\|
\]

Applying Lemma 1.1, we can conclude the desired conclusion immediately.
Conflict of Interests
The author declares that there is no conflict of interests.

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REFERENCES