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PREFUNCTIONS AND DIFFERENTIAL EQUATIONS VIA SUMUDU TRANSFORM

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Abstract: The purpose of this paper is to study the Sumudu Transform of Prefunctions and obtained the Pretrigonometric,Prehyperbolic and extended Prefunctions are the closed form of solutions of first order, second order and third order nonhomogeneous linear differential equations via Sumudu Transform technique.

Keywords: Initial Value Problems, Prefunctions and Sumudu Transform.

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1. Introduction

It is well known that some basic functions such as polynomial functions, exponential functions, Logarithmic functions and trigonometric functions have played an important role in development of mathematical, physical, biological as well as engineering sciences. Deo and Howell in their paper [1] introduced an alternative method and studied trigonometric and trigonometric like functions. This new approach is simple and useful in the study of

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differential equations. Khandeparkar, Deo and Dhaigude [2], [3] have defined new kind of Exponential, Trigonometric and hyperbolic functions. They are coined as Prefunctions and extended Prefunctions. It is shown that these pre-functions and extended prefunctions satisfy many interesting properties and relations. Some properties and Laplace Transforms of these functions are studied in the papers [4], [5], [6] and solutions of initial value problems for non-homogeneous linear ordinary differential equations,Voltera integral equations, Volltera integro-differential equations and the system of differential equations are obtained. In the classical analysis differential equations play a major role in mathematics, physics and engineering. There are lots of different techniques for solving differential equations. Integral transforms were widely used and thus a lot of work has been done on the theory and applications of integral transforms. Most popular integral transforms are due to Laplace, Fourier, Mellein and Henkel. In 1993, the Sumudu transform was proposed originally by Watugala,[7] and he applied it to the solution of ordinary differential equations in control engineering problems. The Sumudu transform plays a crucious role in the solution of ordinary differential equations and other branches of Mathematics and Physics. Nevertheless, this new transform rivals the Laplace transform in problem solving. Its main advantage is the fact that it may be used to solve problems without resorting to a new frequency domain, because it preserves scale and unit properties.

The aim of this paper is to study the Sumudu Transform of Prefunctions and obtained the Pretrigonometric, Prehyperbolic and extended Prefunctions are the solutions of first order, second order as well as third order nonhomogeneous linear differential equations via Sumudu transform technique.

2. Definitions of Prefunctions

In this section, we take overview of pre-function and extended prefunctions.

The pre-exponential functions is denoted by pexp (t, α) and is defined as,

$$pexp(t,\alpha) = 1 + \sum_{n=1}^{\infty} \frac{t^{n+\alpha}}{\Gamma(n+1+\alpha)},\ t \in R \text{ and } \alpha \geq 0 \tag{2.1}$$

The preexponential functions is denoted by pexp (t, α) and is defined as,

$$pexp(-t,\alpha) = 1 + (-1)^{\alpha} \sum_{n=1}^{\infty} (-1)^{n} \frac{t^{n+\alpha}}{\Gamma(n+1+\alpha)},\ t\, \varepsilon\ R \text{ and } \alpha \geq 0 \tag{2.2}$$

The pre-cosine function is denoted by pcos (t, α) and is defined as,

$$pcos(t,\alpha) = 1 + \sum_{n=1}^{\infty}(-1)^n \frac{t^{2n+\alpha}}{\Gamma(2n+1+\alpha)},\ t\,\varepsilon\ R \text{ and } \alpha \geq 0 \quad (2.3)$$

The pre-sine function is denoted by $psin(t,\alpha)$ and is defined as,

$$psin(t,\alpha) = \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n+\alpha+1}}{\Gamma(2n+2+\alpha)},\ t \in R \text{ and } \alpha \geq 0 \quad (2.4)$$

The prehyperbolic cosine function is denoted by pcosh (t, α) and is defined as,

$$pcosh(t,\alpha) = 1 + \sum_{n=1}^{\infty} \frac{t^{2n+\alpha}}{\Gamma(2n+1+\alpha)},\ t \in R \text{ and } \alpha \geq 0 \quad (2.5)$$

The prehyperbolic sine function is denoted by $psinh(t,\alpha)$ and is defined,

$$psinh(t,\alpha) = \sum_{n=0}^{\infty} \frac{t^{2n+\alpha+1}}{\Gamma(2n+2+\alpha)},\ t \in R \text{ and } \alpha \geq 0 \quad (2.6)$$

Extended Prefunctions: The extended pretrigonometric functions are defined by,

$$pM_{3,0}(t,\alpha) = 1 + \sum_{n=1}^{\infty}(-1)^n \frac{t^{3n+\alpha}}{\Gamma(3n+1+\alpha)},\ t \in R \text{ and } \alpha \geq 0 \quad (2.7)$$

$$pM_{3,1}(t,\alpha) = \sum_{n=0}^{\infty}(-1)^n \frac{t^{3n+1+\alpha}}{\Gamma(3n+2+\alpha)},\ t \in R \text{ and } \alpha \geq 0 \quad (2.8)$$

$$pM_{3,2}(t,\alpha) = \sum_{n=0}^{\infty}(-1)^n \frac{t^{3n+2+\alpha}}{\Gamma(3n+3+\alpha)},\ t \in R \text{ and } \alpha \geq 0 \quad (2.9)$$

$$pN_{3,0}(t,\alpha) = 1 + \sum_{n=1}^{\infty} \frac{t^{3n+\alpha}}{\Gamma(3n+1+\alpha)},\ t \in R \text{ and } \alpha \geq 0 \quad (3.0)$$

$$pN_{3,1}(t,\alpha) = \sum_{n=0}^{\infty} \frac{t^{3n+1+\alpha}}{\Gamma(3n+2+\alpha)},\ t \in R \text{ and } \alpha \geq 0 \quad (3.1)$$

$$pN_{3,2}(t,\alpha) = \sum_{n=0}^{\infty} \frac{t^{3n+2+\alpha}}{\Gamma(3n+3+\alpha)},\ t \in R \text{ and } \alpha \geq 0 \quad (3.2)$$

3. The Sumudu Transform of Prefunctions and extended Prefunctions

Definition 3.1: Watugala, [7] introduced the Sumudu transform of a function f(t) denoted by F(u) and defined as,

$$F(u) = \int_0^{\infty}(1/u)\, exp(-t/u) f(t) dt. \quad (3.3)$$

Recently, Khandeparkar, Deo and Dhaigude [2] have defined some new and interesting functions such as, pcos (t, α), psin (t, α), pcosh (t, α), psinh (t,α) for α ≥ 0.It is possible that the Sumudu transform of all such functions can be obtained and can be employed in solving differential equations involving preexponential, pretrigonometric and prehyperbolic functions. . We obtain the Sumudu transforms of these prefunctions by using the above definition (3.1) of Sumudu transform.

1. The Sumudu transform of $\boldsymbol{pexp(t,\alpha)}$:

We know that,

$pexp(t,\alpha) = 1 + \sum_{n=1}^{\infty} \frac{(t)^{n+\alpha}}{\Gamma(n+1+\alpha)}$, $t \in R$ and $\alpha \geq 0$.

Now, we calculate the Sumudu transform of this prefunction as follows,

$$S[pexp(t,\alpha)] = \int_0^{\infty} (1/u)\, exp(-t/u) pexp(t,\alpha) dt$$

$$= \int_0^{\infty} (1/u)\, e^{\frac{-t}{u}} [1 + \sum_{n=1}^{\infty} \frac{(t)^{n+\alpha}}{\Gamma(n+1+\alpha)}] dt.$$

$$= \int_0^{\infty} (1/u)\, e^{\frac{-t}{u}} 1dt + \sum_{n=1}^{\infty} \frac{1}{\Gamma(n+1+\alpha)} \int_0^{\infty} (1/u)\, e^{\frac{-t}{u}} (t)^{n+\alpha} dt$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{\Gamma(n+1+\alpha)} \Gamma(n+1+\alpha) u^{n+\alpha}$$

$$= 1 + \sum_{1}^{\infty} u^{n+\alpha}$$

$$= 1 + \frac{(u)^{\alpha+1}}{1-u}$$

$$S[pexp(t,\alpha)] = 1 + \frac{(u)^{\alpha+1}}{1-u}$$

It is interesting to note that in particular α = 0,

$$S[pexp(t,0)] \quad = S[pexp(t)] = \frac{1}{(1-u)}$$

Clearly, $S^{-1}\left[1 + \frac{(u)^{\alpha+1}}{1-u}\right] = pexp(t,\alpha)$

2. The Sumudu transform of $pexp(-t,\alpha)$:

We know that,

$$pexp(-t,\alpha) = 1 + (-1)^{\alpha} \sum_{n=1}^{\infty} (-1)^n \frac{(t)^{n+\alpha}}{\Gamma(n+1+\alpha)}, \quad t \in R \quad \text{and } \alpha \geq 0$$

Now, we calculate the Sumudu Transforms of this prefunction as follows,

$$S[pexp(-t,\alpha)] = \int_0^{\infty} (1/u)\, exp(-t/u) pexp(-t,\alpha) dt$$

$$= \int_0^{\infty} (1/u)\, e^{\frac{-t}{u}} [1 + (-1)^{\alpha} \sum_{n=1}^{\infty} (-1)^n \frac{(t)^{n+\alpha}}{\Gamma(n+1+\alpha)}] dt$$

$$= \int_0^{\infty} (1/u)\, e^{\frac{-t}{u}} 1dt + (-1)^{\alpha} \sum_{n=1}^{\infty} \frac{(-1)^n}{\Gamma(n+1+\alpha)} \int_0^{\infty} (1/u)\, e^{\frac{-t}{u}} (t)^{n+\alpha} dt$$

$$= 1 + (-1)^{\alpha} \sum_{n=1}^{\infty} \frac{(-1)^n}{\Gamma(n+1+\alpha)} (n+\alpha)!\, u^{n+\alpha}$$

$$= 1 + (-1)^{\alpha} \sum_{n=1}^{\infty} (-1)^n u^{n+\alpha}$$

$$= 1 - \frac{(-1)^{\alpha} (u)^{\alpha+1}}{1+u}$$

$$S[pexp(-t,\alpha)] = 1 - \frac{(-1)^{\alpha} (u)^{\alpha+1}}{1+u}$$

In particular, α = 0,

$$\mathrm{S}[pexp(-t,0)] = \mathrm{S}[pexp(-t)] = \frac{1}{(1+u)}$$

Clearly, $S^{-1}\left[1-\frac{(-1)^{\alpha}(\mathrm{u})^{\alpha+1}}{1+\mathrm{u}}\right] = pexp(-t,\alpha)$

On similar lines we obtain the Sumudu transform of other Prefunctions

We omit the details. We list below the Sumudu transform of prefunctions.

3. $\mathrm{S}[psin(t,\alpha)] = \frac{\mathrm{u}^{1+\alpha}}{1+\mathrm{u}^2}$

4. $\mathrm{S}[pcos(t,\alpha)] = 1-\frac{\mathrm{u}^{\alpha+2}}{1+\mathrm{u}^2}$

5. $\mathrm{S}[psinh(t,\alpha)] = \frac{\mathrm{u}^{1+\alpha}}{1-\mathrm{u}^2}$

6. $S[pcosh(t,\alpha)] = 1+\frac{\mathrm{u}^{\alpha+2}}{1-\mathrm{u}^2}$

7. $S[pM_{30}(t,\alpha)] = 1-\frac{\mathrm{u}^{\alpha+3}}{1+\mathrm{u}^3}$

8. $S[pM_{31}(t,\alpha)] = \frac{\mathrm{u}^{1+\alpha}}{1+\mathrm{u}^3}$

9. $S[pM_{32}(t,\alpha)] = \frac{\mathrm{u}^{2+\alpha}}{1+\mathrm{u}^3}$

10. $S[pN_{30}(t,\alpha)] = 1+\frac{\mathrm{u}^{\alpha+3}}{1-\mathrm{u}^3}$

11. $S[pN_{31}(t,\alpha)] = \frac{\mathrm{u}^{1+\alpha}}{1-\mathrm{u}^3}$

12. $S[pN_{32}(t,\alpha)] = \frac{\mathrm{u}^{2+\alpha}}{1-\mathrm{u}^3}$

13. $S\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right] = u^{\alpha}$

4. Prefunctions as a solution of Initial Value Problems

The infinite series expansions of preexponential, pretrigonometric and prehyperbolic functions are given in[2],[3], we have obtained the Sumudu transform of these functions. Below we consider a sequence of general form nonhomogeneous linear differential equations with constant coefficients of the type,

$X^{n}(t,\alpha)+X(t,\alpha)=f(t,\alpha)$, $n = 1,2,$ ...and α ≥ 0. With suitable initial conditions. It is interesting to note that the Solution of the IVPs is prefunctions.

IVP.1. Consider the following IVP for first order nonhomogeneous differential equation,

$$x'(t,\alpha) - x(t,\alpha) = \frac{t^\alpha}{\Gamma(\alpha+1)} - 1 \tag{4.1}$$

with initial condition,

$$x(0,\alpha) = 1 \tag{4.2}$$

Taking the Sumudu transform of both the sides of equation (4.1), we have,

$$S\{x'(t,\alpha)\} - S\{x(t,\alpha)\} = S\left\{\left(\frac{t^\alpha}{\Gamma(\alpha+1)} - 1\right)\right\}$$

$$\frac{1}{u}[X(u,\alpha) - x(0,\alpha)] - X(u,\alpha) = u^\alpha - 1$$

Using equation (4.2) and on simplification, we get,

$$X(u,\alpha) = 1 + \frac{\mathrm{u}^{\alpha+1}}{1-\mathrm{u}}$$

Taking Inverse Sumudu transform on both the sides we have,

$$S^{-1}[X(u,\alpha)] = S^{-1}[1 + \frac{\mathrm{u}^{\alpha+1}}{1-\mathrm{u}}]$$

$$x(t,\alpha) = pexp(t,\alpha)$$

Hence, the preexponential function $x(t,\alpha) = pexp(t,\alpha)$ is the solution of IVPs (4.1)-(4.2)

IVP.2. Consider the following IVP for first order nonhomogeneous differential equation,

$$x'(t,\alpha) + x(t,\alpha) = 1 - (-1)^\alpha \frac{t^\alpha}{\Gamma(\alpha+1)} \tag{4.3}$$

With initial condition,

$$x(0,\alpha) = 1 \tag{4.4}$$

Taking the Sumudu transform of both the sides of equation (4.1), we have,

$$S\{x'(t,\alpha)\} + S\{x(t,\alpha)\} = S\left\{\left[1 - (-1)^\alpha \frac{t^\alpha}{\Gamma(\alpha+1)}\right]\right\}$$

$$\frac{1}{u}[X(u,\alpha) - x(0,\alpha)] - X(u,\alpha) = 1 - (-1)^\alpha u^\alpha$$

Using equation (4.4) and on simplification, we get,

$$X(u,\alpha) = 1 - (-1)^\alpha \frac{\mathrm{u}^{\alpha+1}}{1+\mathrm{u}}$$

Taking Inverse Sumudu transform on both the sides we have,

$$S^{-1}[X(u,\alpha)] = S^{-1}[1 - (-1)^\alpha \frac{\mathrm{u}^{\alpha+1}}{1+\mathrm{u}}]$$

$$x(t,\alpha) = pexp(-t,\alpha)$$

Hence the preexponential function $x(t,\alpha) = pexp(-t,\alpha)$ is the solution of IVPs (4.3)-(4.4).

IVP.3. Consider the following IVP for second order nonhomogeneous differential equation,

$$x''(t,\alpha) + x(t,\alpha) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \tag{4.5}$$

with initial condition,

$$x(0,\alpha) = x'(0,\alpha) = 0 \tag{4.6}$$

Taking the Sumudu transform of both the sides of equation (4.5), we have,

$$S\{x''(t,\alpha)\} + S\{x(t,\alpha)\} = S\left\{\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)\right\}$$

$$\frac{1}{u^2}[X(u,\alpha) - x(0,\alpha) - ux'(0,\alpha)] + X(u,\alpha) = u^{\alpha-1}$$

Using equation (4.6) and on simplification, we get,

$$X(u,\alpha) = \frac{\mathrm{u}^{\alpha+1}}{1+\mathrm{u}^2}$$

Taking Inverse Sumudu transform on both the sides we have,

$$S^{-1}[X(u,\alpha)] = S^{-1}\left[\frac{\mathrm{u}^{\alpha+1}}{1+\mathrm{u}^2}\right]$$

$$x(t,\alpha) = psin(t,\alpha)$$

Hence the preexponential function $x(t,\alpha) = psin(t,\alpha)$ is the solution of IVPs (4.5)-(4.6)

IVP.4. Consider the following IVP for second order nonhomogeneous differential equation,

$$x''(t,\alpha) + x(t,\alpha) = 1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)} \tag{4.7}$$

with initial condition,

$$x(0,\alpha) = 1, x'(0,\alpha) = 0 \tag{4.8}$$

Taking the Sumudu transform of both the sides of equation (4.7), we have,

$$S\{x''(t,\alpha)\} + S\{x(t,\alpha)\} = S\left\{\left(1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)\right\}$$

$$\frac{1}{u^2}[X(u,\alpha) - x(0,\alpha) - ux'(0,\alpha)] + X(u,\alpha) = 1 - u^{\alpha}$$

Using equation (4.8) and on simplification, we get,

$$X(u,\alpha) = 1 - \frac{\mathrm{u}^{\alpha+2}}{1+\mathrm{u}^2}$$

Taking Inverse Sumudu transform on both the sides we have,

$$S^{-1}[X(u,\alpha)] = S^{-1}\left[1 - \frac{\mathrm{u}^{\alpha+2}}{1+\mathrm{u}^2}\right]$$

$$x(t,\alpha) = pcos(t,\alpha)$$

Hence the preexponential function $x(t,\alpha) = pcos(t,\alpha)$ is the solution of IVPs (4.7)-(4.8)

IVP.5. Consider the following IVP for third order nonhomogeneous differential equation,

$$x'''(t,\alpha) + x(t,\alpha) = 1 - \frac{t^\alpha}{\Gamma(\alpha+1)} \tag{4.9}$$

together with initial condition,

$$x(0,\alpha) = 1, x'(0,\alpha) = x''(0,\alpha) = 0 \tag{5.0}$$

Taking the Sumudu transform of both the sides of equation (4.9), we have,

$$S\{x'''(t,\alpha)\} + S\{x(t,\alpha)\} = S\{ (1 - \frac{t^\alpha}{\Gamma(\alpha+1)}) \}$$

$$\frac{1}{u^3}[X(u,\alpha) - x(0,\alpha) - ux'(0,\alpha) - u^2x''(0,\alpha)] + X(u,\alpha) = 1 - u^\alpha$$

Using equation (5.0) and on simplification, we get,

$$X(u,\alpha) = 1 - \frac{\mathrm{u}^{\alpha+3}}{1+\mathrm{u}^3}$$

Taking Inverse Sumudu transform on both the sides we have,

$$S^{-1}[X(u,\alpha)] = S^{-1}[1 - \frac{\mathrm{u}^{\alpha+3}}{1+\mathrm{u}^3}\ ]$$

$$x(t,\alpha) = pM_{3,0}(t,\alpha)$$

Hence the preexponential function $x(t,\alpha) = pM_{3,0}\ (t,\alpha)$ is the solution of IVPs (4.9)-(5.0).

IVP.6. Consider the following IVP for third order nonhomogeneous differential equation,

$$x'''(t,\alpha) - x(t,\alpha) = \frac{t^\alpha}{\Gamma(\alpha+1)} - 1 \tag{5.1}$$

together with initial condition,

$$x(0,\alpha) = 1, x'(0,\alpha) = x''(0,\alpha) = 0 \tag{5.2}$$

Taking the Sumudu transform of both the sides of equation (5.1), we have,

$$S\{x'''(t,\alpha)\} - S\{x(t,\alpha)\} = S\{(\frac{t^\alpha}{\Gamma(\alpha+1)} - 1)\}$$

$$\frac{1}{u^3}[X(u,\alpha) - x(0,\alpha) - ux'(0,\alpha) - u^2x''(0,\alpha)] - X(u,\alpha) = u^\alpha - 1$$

Using equation (5.2) and on simplification, we get,

$$X(u,\alpha) = 1 + \frac{\mathrm{u}^{\alpha+3}}{1-\mathrm{u}^3}$$

Taking Inverse Sumudu transform on both the sides we have,

$$S^{-1}[X(u,\alpha)] = S^{-1}[1 + \frac{\mathrm{u}^{\alpha+3}}{1-\mathrm{u}^3}\ ]$$

$$x(t,\alpha) = pN_{3,0}(t,\alpha)$$

Hence the preexponential function $x(t,\alpha) = pN_{3,0}\ (t,\alpha)$ is the solution of IVPs (5.1)-(5.2).

Conclusion: In this paper we have studied the Sumudu Transform of Prefunctions and obtained the Pre-trigonometric, Pre-hyperbolic and extended Prefunctions are the closed form of solutions of first order, second order as well as third order nonhomogeneous linear differential equations via Sumudu Transform technique.

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Conflict of Interests

The author declares that there is no conflict of interests.

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