ON CERTAIN INEQUALITIES CONCERNING THE CLASSICAL EULER’S GAMMA FUNCTION

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Abstract. In this paper, some monotonic functions and some inequalities concerning certain ratios of generalized gamma functions are established. The procedure utilizes the series forms of the generalized digamma functions.

Keywords: gamma function; q-gamma function; k-gamma function; (p, q)-gamma function; inequality.

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1. Introduction and Preliminaries

The classical Euler’s Gamma function, \( \Gamma(t) \) and the digamma function, \( \psi(t) \) are well-known in literature as

\[
\Gamma(t) = \int_0^\infty e^{-x} x^{t-1} \, dx \quad \text{and} \quad \psi(t) = \frac{d}{dt} \ln \Gamma(t) = \frac{\Gamma'(t)}{\Gamma(t)}, \quad t > 0.
\]

The \( p \)-Gamma function, \( \Gamma_p(t) \) and the \( p \)-digamma function, \( \psi_p(t) \) are defined for \( p \in \mathbb{N} \) as (see [6])

\[
\Gamma_p(t) = \frac{p! p'}{t(t+1) \cdots (t+p)} \quad \text{and} \quad \psi_p(t) = \frac{d}{dt} \ln \Gamma_p(t) = \frac{\Gamma'_p(t)}{\Gamma_p(t)}, \quad t > 0.
\]

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The \(q\)-Gamma function, \(\Gamma_q(t)\) and the \(q\)-digamma function, \(\psi_q(t)\) are also defined for \(q \in (0, 1)\) as (see [2])

\[
\Gamma_q(t) = (1-q)^{1-t} \prod_{n=1}^{\infty} \frac{1-q^n}{1-q^{n+t}} \quad \text{and} \quad \psi_q(t) = \frac{d}{dt} \ln \Gamma_q(t) = \frac{\Gamma'_q(t)}{\Gamma_q(t)}, \quad t > 0.
\]

Also, the \(k\)-Gamma function, \(\Gamma_k(t)\) and the \(k\)-digamma function, \(\psi_k(t)\) are defined for \(k > 0\) as (see [1])

\[
\Gamma_k(t) = \int_0^\infty e^{-x} x^{t-1} dx \quad \text{and} \quad \psi_k(t) = \frac{d}{dt} \ln \Gamma_k(t) = \frac{\Gamma'_k(t)}{\Gamma_k(t)}, \quad t > 0.
\]

In 2005, Díaz and Teruel [5] defined the \((q,k)\)-Gamma and the \((q,k)\)-digamma functions for \(q \in (0, 1)\) and \(k > 0\) as

\[
\Gamma_{(q,k)}(t) = \frac{(1-q^k)^{\frac{t}{k}-1}}{(1-q)^{\frac{t}{k}-1}} \quad \text{and} \quad \psi_{(q,k)}(t) = \frac{d}{dt} \ln \Gamma_{(q,k)}(t) = \frac{\Gamma'_{(q,k)}(t)}{\Gamma_{(q,k)}(t)}, \quad t > 0,
\]

where \((t)_{n,k} = t(t+k)(t+2k) \ldots (t+(n-1)k) = \prod_{j=0}^{n-1}(t+jk)\) is the \(k\)-generalized Pochhammer symbol.

Also in 2012, Krasniqi and Merovci [4] defined the \((p,q)\)-Gamma and the \((p,q)\)-digamma functions for \(p \in \mathbb{N}\) and \(q \in (0, 1)\) as

\[
\Gamma_{(p,q)}(t) = \frac{[p]^t [p]_q}{[t]_q [t+1]_q \cdots [t+p]_q} \quad \text{and} \quad \psi_{(p,q)}(t) = \frac{d}{dt} \ln \Gamma_{(p,q)}(t) = \frac{\Gamma'_{(p,q)}(t)}{\Gamma_{(p,q)}(t)}, \quad t > 0.
\]

where \([p]_q = \frac{1-q^p}{1-q}\).

The generalized digamma functions \(\psi_q(t)\), \(\psi_k(t)\), \(\psi_{(p,q)}(t)\) and \(\psi_{(q,k)}(t)\) as defined above exhibit the following series representations (see also [8], [9], [10], [11] and [12]).

\[
\psi_q(t) = -\ln(1-q) + \ln q \sum_{n=1}^{\infty} \frac{q^{nt}}{1-q^n}
\]

\[
(2) \quad \psi_k(t) = \frac{\ln k - \gamma}{k} - \frac{1}{t} + \sum_{n=1}^{\infty} \frac{t}{nk(nk+t)}
\]

\[
(3) \quad \psi_{(p,q)}(t) = \ln[p]_q + (\ln q) \sum_{n=1}^{p} \frac{q^{nt}}{1-q^n}
\]

\[
(4) \quad \psi_{(q,k)}(t) = -\ln(1-q) + \ln q \sum_{n=1}^{\infty} \frac{q^{nkt}}{1-q^{nkt}}
\]

where \(\gamma\) is the Euler-Mascheroni’s constant.
In 2010, Krasniqi and Shabani [7] presented the following inequalities:
\[
\frac{p^{-t} e^{-\gamma \Gamma(\alpha)}}{\Gamma_p(\alpha)} < \frac{\Gamma(\alpha + t)}{\Gamma_p(\alpha + t)} < \frac{p^{-t} e^{-\gamma t} \Gamma(\alpha + 1)}{\Gamma_p(\alpha + 1)}
\]
for \( t \in (0, 1) \), where \( \alpha \) is a positive real number such that \( \alpha + t > 1 \).

Also in that same year, Krasniqi, Mansour and Shabani [6] presented the following results.
\[
\frac{(1 - q)^t e^{-\gamma \Gamma(\alpha)}}{\Gamma_q(\alpha)} < \frac{\Gamma(\alpha + t)}{\Gamma_q(\alpha + t)} < \frac{(1 - q)^{t-1} e^{\gamma(1-t)} \Gamma(\alpha + 1)}{\Gamma_q(\alpha + 1)}
\]
for \( t \in (0, 1) \), where \( \alpha \) is a positive real number such that \( \alpha + t > 1 \) and \( q \in (0, 1) \).

Several results of this nature as well as some generalizations have since been established. These can be found in the papers [8], [9], [10], [11] and [12].

In the present paper, our main objective is to present similar results involving the ratios \( \frac{\Gamma_q(t)}{\Gamma_{(p,q)}(t)} \), \( \frac{\Gamma_p(t)}{\Gamma_{(p,q)}(t)} \), \( \frac{\Gamma_k(t)}{\Gamma_{(p,q)}(t)} \) and \( \frac{\Gamma_k(t)}{\Gamma_{(q,k)}(t)} \).

2. Auxiliary Results

**Lemma 2.1.** Let \( \alpha > 0 \), \( t > 0 \), \( p \in \mathbb{N} \) and \( q \in (0, 1) \). Then,
\[
\ln(1-q) + \ln[p]_q + \psi_q(\alpha + t) - \psi_{(p,q)}(\alpha + t) \leq 0.
\]

**Proof.** By the series representations (1) and (3) we have,
\[
\ln(1-q) + \ln[p]_q + \psi_q(t) - \psi_{(p,q)}(t) = (\ln q) \left[ \sum_{n=1}^{\infty} \frac{q^n t}{1-q^n} - \sum_{n=1}^{p} \frac{q^n t}{1-q^n} \right] = (\ln q) \sum_{n=p+1}^{\infty} \frac{q^n t}{1-q^n} \leq 0.
\]
Replacing \( t \) by \( \alpha + t \) completes the proof.

**Lemma 2.2.** Let \( \alpha > 0 \), \( t > 0 \), \( q \in (0, 1) \) and \( k \geq 1 \). Then,
\[
\ln(1-q) - \frac{1}{k} \ln(1-q) + \psi_q(\alpha + t) - \psi_{(q,k)}(\alpha + t) \leq 0.
\]

**Proof.** By the series representations (1) and (4) we have,
\[
\ln(1-q)^{1-\frac{1}{k}} + \psi_q(t) - \psi_{(q,k)}(t) = (\ln q) \left[ \sum_{n=1}^{\infty} \frac{q^n t}{1-q^n} - \frac{q^{nk} t}{1-q^{nk}} \right] \leq 0.
\]
Replacing \( t \) by \( \alpha + t \) completes the proof.
Lemma 2.3. Let $\alpha > 0$, $t > 0$, $k > 0$, $p \in N$ and $q \in (0, 1)$. Then,
\[
\ln[p]q - \frac{\ln k - \gamma}{k} + \frac{1}{\alpha + t} + \psi_k(\alpha + t) - \psi_{(p,q)}(\alpha + t) > 0.
\]

Proof. By the series representations (2) and (3) we have,
\[
\ln[p]q - \frac{\ln k - \gamma}{k} + \frac{1}{\alpha + t} + \psi_k(\alpha + t) - \psi_{(p,q)}(\alpha + t) = \sum_{n=1}^{\infty} \frac{t}{nk(nk+t)} - (\ln q) \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} > 0.
\]
Replacing $t$ by $\alpha + t$ completes the proof.

Lemma 2.4. Let $\alpha > 0$, $t > 0$, $q \in (0, 1)$ and $k > 0$. Then,
\[
-\frac{\ln(k(1-q))}{k} + \frac{\gamma}{k} + \frac{1}{\alpha + t} + \psi_k(\alpha + t) - \psi_{(q,k)}(\alpha + t) > 0.
\]

Proof. By the series representations (2) and (4) we have,
\[
-\frac{\ln(k(1-q))}{k} + \frac{\gamma}{k} + \frac{1}{\alpha + t} + \psi_k(\alpha + t) - \psi_{(q,k)}(\alpha + t) = \sum_{n=1}^{\infty} \frac{t}{nk(nk+t)} - (\ln q) \sum_{n=1}^{\infty} \frac{q^{nk}}{1-q^{nk}} > 0.
\]
Replacing $t$ by $\alpha + t$ completes the proof.

3. Main Results

Theorem 3.1. Define a function $S$ by
\[
S(t) = \frac{(1-q)^t \Gamma_q(\alpha + t)}{[p]_q \Gamma_{(p,q)}(\alpha + t)}, \quad t \in (0, \infty)
\]
for $\alpha > 0$, $p \in N$ and $q \in (0, 1)$. Then $S$ is non-increasing on $t \in (0, \infty)$ and the inequalities
\[
\frac{(1-q)^t \Gamma_q(\alpha)}{[p]_q \Gamma_{(p,q)}(\alpha)} \geq \frac{\Gamma_q(\alpha + t)}{\Gamma_{(p,q)}(\alpha + t)} \geq \frac{(1-q)^{1-t} \Gamma_q(\alpha + 1)}{[p]_q^{\alpha-1} \Gamma_{(p,q)}(\alpha + 1)}
\]
are valid for every $t \in (0, 1)$.

Proof. Let $g(t) = \ln S(t)$ for every $t \in (0, \infty)$. Then,
\[
g(t) = \ln \left( \frac{(1-q)^t \Gamma_q(\alpha + t)}{[p]_q \Gamma_{(p,q)}(\alpha + t)} \right) = t \ln(1-q) + t \ln[p]_q + \ln \Gamma_q(\alpha + t) - \ln \Gamma_{(p,q)}(\alpha + t).
\]
Then,
\[
g'(t) = \ln(1-q) + \ln[p]_q + \psi_q(\alpha + t) - \psi_{(p,q)}(\alpha + t) \leq 0. \quad \text{(by Lemma 2.1)}
\]
That implies \( g \) is non-increasing on \( t \in (0, \infty) \). Hence \( S \) is non-increasing on \( t \in (0, \infty) \) and for every \( t \in (0, 1) \) we have,

\[
S(0) \geq S(t) \geq S(1)
\]

yielding the result.

**Theorem 3.2.** Define a function \( T \) by

\[
T(t) = \frac{(1 - q)^t \Gamma_q(\alpha + t)}{(1 - q)^t \Gamma_{(q,k)}(\alpha + t)}, \quad t \in (0, \infty)
\]

for \( \alpha > 0 \), \( q \in (0, 1) \) and \( k \geq 1 \). Then \( T \) is non-increasing on \( t \in (0, \infty) \) and the inequalities

\[
(1 - q)^{-t} \Gamma_q(\alpha) \geq \frac{\Gamma_q(\alpha + t)}{\Gamma_{(q,k)}(\alpha + t)} \geq \frac{(1 - q)^{-t} \Gamma_q(\alpha + 1)}{(1 - q)^{t(1-t)} \Gamma_{(q,k)}(\alpha + 1)}
\]

are valid for every \( t \in (0, 1) \).

**Proof.** Let \( h(t) = \ln T(t) \) for every \( t \in (0, \infty) \). Then,

\[
h(t) = \ln \frac{(1 - q)^t \Gamma_q(\alpha + t)}{(1 - q)^t \Gamma_{(q,k)}(\alpha + t)} = t \ln(1 - q) - t \ln(1 - q) + \ln \Gamma_q(\alpha + t) - \ln \Gamma_{(q,k)}(\alpha + t). \quad \text{Then},
\]

\[
h'(t) = \ln(1 - q) - \frac{1}{k} \ln(1 - q) + \psi_q(\alpha + t) - \psi_{(q,k)}(\alpha + t) \leq 0. \quad \text{(by Lemma 2.2)}
\]

That implies \( h \) is non-increasing on \( t \in (0, \infty) \). Hence \( T \) is non-increasing on \( t \in (0, \infty) \) and for every \( t \in (0, 1) \) we have,

\[
T(0) \geq T(t) \geq T(1)
\]

establishing the result.

**Theorem 3.3.** Define a function \( U \) by

\[
U(t) = \frac{(\alpha + t)k^{-1} e^{-\alpha} \Gamma_k(\alpha + t)}{[p]_q \Gamma_{(p,q)}(\alpha + t)}, \quad t \in (0, \infty)
\]

for \( \alpha > 0 \), \( p \in \mathbb{N} \), \( q \in (0, 1) \) and \( k > 0 \). Then \( U \) is increasing on \( t \in (0, \infty) \) and the inequalities

\[
\frac{\alpha t e^{-\alpha} \Gamma_k(\alpha)}{\Gamma_{(p,q)}(\alpha)} < \frac{(\alpha + 1)k^{-1} e^{-\alpha} \Gamma_{(p,q)}(\alpha + 1)}{\Gamma_{(p,q)}(\alpha + t)} < \frac{(\alpha + 1)k^{-1} e^{-\alpha} \Gamma_{(p,q)}(\alpha + 1)}{(\alpha + t)[p]_q \Gamma_{(p,q)}(\alpha + 1)}
\]

are valid for every \( t \in (0, 1) \).
Proof. Let \( \mu(t) = \ln U(t) \) for every \( t \in (0, \infty) \). Then,

\[
\mu(t) = \ln \left( \frac{(\alpha + t)k^{-\frac{q}{k}}\Gamma_k(\alpha + t)}{[p]^q\Gamma_{(p,q)}(\alpha + t)} \right) = \ln(\alpha + t) - \frac{t}{k} \ln k + \frac{\gamma}{k} + t \ln[p]_q + \ln \Gamma_k(\alpha + t) - \ln \Gamma_{(p,q)}(\alpha + t).
\]

Then,

\[
\mu'(t) = \ln[p]_q - \frac{\ln k - \gamma}{k} + \frac{1}{\alpha + t} + \psi_k(\alpha + t) - \psi_{(p,q)}(\alpha + t) > 0. \text{ (by Lemma 2.3)}
\]

That implies \( \mu \) is increasing on \( t \in (0, \infty) \). As a result, \( U \) is also increasing on \( t \in (0, \infty) \) and for every \( t \in (0, 1) \) we have,

\[
U(0) < U(t) < U(1)
\]

yielding the result.

**Theorem 3.4.** Define a function \( V \) by

\[
V(t) = \frac{(\alpha + t)\gamma^\frac{p}{k} \Gamma_k(\alpha + t)}{k^\frac{1}{k} (1-q)^\frac{1}{k} \Gamma_{(p,q)}(\alpha + t)}, \quad t \in (0, \infty)
\]

for \( \alpha > 0, q \in (0, 1) \) and \( k > 0 \). Then \( V \) is increasing on \( t \in (0, \infty) \) and the inequalities

\[
\frac{\alpha k^\frac{1}{k} e^\frac{p}{k} \Gamma_k(\alpha)}{(\alpha + t)(1-q)^\frac{1}{k} \Gamma_{(p,q)}(\alpha)} < \frac{\Gamma_k(\alpha + t)}{\Gamma_{(p,q)}(\alpha + t)} < \frac{(\alpha + 1)k^\frac{1}{k} e^\frac{p+1}{k} \Gamma_k(\alpha + 1)}{(\alpha + t)(1-q)^\frac{1}{k} \Gamma_{(p,q)}(\alpha + 1)}
\]

are valid for every \( t \in (0, 1) \).

**Proof.** Let \( \delta(t) = \ln V(t) \) for every \( t \in (0, \infty) \). Then,

\[
\delta(t) = \ln \left( \frac{(\alpha + t)\gamma^\frac{p}{k} \Gamma_k(\alpha + t)}{k^\frac{1}{k} (1-q)^\frac{1}{k} \Gamma_{(p,q)}(\alpha + t)} \right) = \ln(\alpha + t) + \frac{\gamma}{k} - \frac{t}{k} \ln(k(1-q)) + \ln \Gamma_k(\alpha + t) - \ln \Gamma_{(p,q)}(\alpha + t).
\]

Then,

\[
\delta'(t) = -\frac{\ln(k(1-q))}{k} + \frac{\gamma}{k} + \frac{1}{\alpha + t} + \psi_k(\alpha + t) - \psi_{(p,q)}(\alpha + t) > 0. \text{ (by Lemma 2.4)}
\]

That implies \( \delta \) is increasing on \( t \in (0, \infty) \). As a result, \( V \) is also increasing on \( t \in (0, \infty) \) and for every \( t \in (0, 1) \) we have,

\[
V(0) < V(t) < V(1)
\]

establishing the result.
Conflict of Interests.

The author declares that there is no conflict of interests.

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