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OPERATOR INEQUALITIES ASSOCIATED WITH RELATIVE OPERATOR ENTROPIES

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Abstract. In this paper, we introduce the notions of operator (α,β,γ)-mean, relative operator (α,β,γ)-entropy and Tsallis relative operator (α,β,γ)-entropy. We give upper and lower bounds of relative operator (α,β,γ)-entropy, relative operator $(0,\beta,\gamma)$-entropy and Tsallis relative operator (α,β,γ)-entropy. Our results are refinements and generalizations of some existing inequalities due to Furuichi, Zou and Nikoufar.

Keywords: Tsallis relative operator (α,β,γ)-entropy; Relative operator (α,β,γ)-entropy; Operator (α,β,γ)-mean; Operator inequality; Positive invertible operator.

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1. Introduction

Let A and B be two invertible positive operators on a finite dimensional Hilbert space. Relative operator entropy is defined by (see [3])

$$S(A|B) = A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}},$$

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which is an extension of the operator entropy introduced in [13] and [16]. More generally, the generalized relative operator entropy

$$S_q(A|B) = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^q(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}},$$

for positive operators A and B and $q \in \mathbb{R}$ was defined in [11]. We notice that when $q = 0$, we have $S_0(A|B) = S(A|B)$. Furuta [9] (see also [10]) proved the following inequality for $a > 0$

$$(1-\log a)A - \frac{1}{a}AB^{-1}A \leqslant S(A|B) \leqslant (\log a - 1)A + \frac{1}{a}B,$$

as a generalization of the upper and lower bounds of

$$A - AB^{-1}A \leqslant S(A|B) \leqslant B - A,$$

which was obtained in [4].

For positive operators A, B and $0 < \lambda \leqslant 1$, Tsallis relative operator entropy is defined as follows (see [17])

$$T_\lambda(A|B) = \frac{A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\lambda A^{\frac{1}{2}} - A}{\lambda} = \frac{A\sharp_\lambda B - A}{\lambda},$$

where $A\sharp_\lambda B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\lambda A^{\frac{1}{2}}$ is the λ-geometric mean [12]. When $\lambda = \frac{1}{2}$, $A\sharp_\lambda B$ is denoted by $A\sharp B$ and called geometric mean. In particular $A\sharp_0 B = A$, $A\sharp_1 B = B$ and $A\sharp_{-1} B = AB^{-1}A$. Tsallis relative operator entropy can be rewritten as

$$T_\lambda(A|B) = A^{\frac{1}{2}}\ln_\lambda(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}},$$

where one-parameter extended logarithmic function $\ln_\lambda t$ is defined by $\ln_\lambda t = \frac{t^\lambda - 1}{\lambda}$ for $t > 0$. $\ln_\lambda t$ uniformly converges to the usual logarithmic function $\log t$ when $\lambda \to 0$. So Tsallis relative operator entropy $T_\lambda(A|B)$ is a one-parameter extension of relative operator entropy $S(A|B)$ in the sense that $\lim_{\lambda\to 0} T_\lambda(A|B) = S(A|B)$. For more information on the Tsallis relative operator entropy the reader is referred to [6], [7] and [17].

The relation between $S(A|B)$, $T_\lambda(A|B)$ and $T_{-\lambda}(A|B)$ was considered in [5] and the following inequalities was proved

$$T_{-\lambda}(A|B) \leqslant S(A|B) \leqslant T_\lambda(A|B) \tag{1.1}$$

$$A - AB^{-1}A \leqslant T_\lambda(A|B) \leqslant B - A, \tag{1.2}$$

and for $a>0$

$$A\sharp_\lambda B-\frac{1}{a}A\sharp_{\lambda-1}B+(\ln_\lambda \frac{1}{a})A\leqslant T_\lambda(A|B)\leqslant \frac{1}{a}B-A-(\ln_\lambda \frac{1}{a})A\sharp_\lambda B. \tag{1.3}$$

In [15], operator (α,β)-geometric mean for real numbers α, β was introduced as a generalization of the operator α-geometric mean $A\sharp_\alpha B$ as follows

$$A\sharp_{(\alpha,\beta)}B=A^{\frac{\beta}{2}}\left(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}}\right)^{\alpha}A^{\frac{\beta}{2}}.$$

In particular we have

$$A\sharp_{(\alpha,1)}B=A\sharp_\alpha B,\quad A\sharp_{(-1,\beta)}B=A^\beta B^{-1}A^\beta,\quad A\sharp_{(0,\beta)}B=A^\beta,\quad A\sharp_{(1,\beta)}B=B.$$

The notion of relative operator (α,β)-entropy and Tsallis relative operator (α,β)-entropy was defined in [14] as a parameter extensions of relative operator entropy and Tsallis relative operator entropy. For invertible positive operators A, B and real numbers α, β, relative operator (α,β)-entropy is

$$S_{\alpha,\beta}(A|B)=A^{\frac{\beta}{2}}\left(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}}\right)^{\alpha}\left(\log A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}}\right)A^{\frac{\beta}{2}},$$

and Tsallis relative operator (α,β)-entropy for $\alpha\neq 0$ is

$$T_{\alpha,\beta}(A|B)=A^{\frac{\beta}{2}}\ln_\alpha\left(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}}\right)A^{\frac{\beta}{2}}.$$

In particular $S_{q,1}(A|B)=S_q(A|B)$, $S_{0,1}(A|B)=S(A|B)$, $T_{\lambda,1}(A|B)=T_\lambda(A|B)$ and $\lim_{\alpha\to 0}T_{\alpha,\beta}(A|B)=S_{0,\beta}(A|B)$.

According to the definitions of $T_{\lambda,\beta}(A|B)$ and $S_{0,\beta}(A|B)$, the following inequalities were proved in [14] for $\lambda\in(0,1]$ and $\beta>0$

$$T_{-\lambda,\beta}(A|B)\leqslant S_{0,\beta}(A|B)\leqslant T_{\lambda,\beta}(A|B), \tag{1.4}$$

$$A^\beta-A^\beta B^{-1}A^\beta\leqslant T_{\lambda,\beta}(A|B)\leqslant B-A^\beta, \tag{1.5}$$

and for $a>0$

$$A\sharp_{(\lambda,\beta)}B-\frac{1}{a}A\sharp_{(\lambda-1,\beta)}B+(\ln_\lambda \frac{1}{a})A^\beta\leqslant T_{\lambda,\beta}(A|B)\leqslant \frac{1}{a}B-A^\beta-(\ln_\lambda \frac{1}{a})A\sharp_{(\lambda,\beta)}B. \tag{1.6}$$

We notice that these three inequalities are a generalization and refinement of the inequalities (1.1), (1.2) and (1.3).

In this paper, we introduce three parameter extensions of operator mean, relative operator entropy and Tsallis relative operator entropy and present some new operator inequalities. Our results will recover and generalize some existing inequalities in [8], [14] and [18]. A generalization of inequalities (1.4) and (1.6) is obtained in Theorem 2.7 and Theorem 2.9. In Theorem 2.11 we will give a refinement of Theorem 2.7. Some theorems of [18] will be extracted according to Theorem 2.11. New upper and lower bounds of three parameter Tsallis relative operator entropy will be introduced in Theorem 2.16. We will obtained tight bounds of Theorem 2.9 when $a \geqslant 1$. The main result of [8] and [18] will be concluded according to Theorem 2.16. Theorem 2.20 will give precise upper and lower bounds of Theorem 2.7 which recover some main operator inequalities of [8]. Indeed, we will extend corollary 2.5, propostion 2.3 and propostion 2.4 in [8] by Theorem 2.20. We will get a generalization of the two last results of [8] in Theorem 2.23 and Theorem 2.25. At the end, in Theorem 2.27 we obtain upper and lower bounds of three parameter relative operator entropy.

2. Main Results

We introduce three parameter extension of operator mean, relative operator entropy and T-sallis relative operator entropy in the following:

Definition 2.1. Operator (α,β,γ)-mean

$$A\sharp_{(\alpha,\beta,\gamma)}B \equiv A^{\frac{\gamma}{2}}\left(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}}\right)^{\alpha}A^{\frac{\gamma}{2}}$$

for positive invertible operators A,B and real numbers α,β,γ. We would remark that $A\sharp_{(1,\beta,\gamma)}B \equiv A^{\frac{\gamma}{2}}\left(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}}\right)A^{\frac{\gamma}{2}}$, $A\sharp_{(-1,\beta,\gamma)}B \equiv A^{\frac{\gamma}{2}}\left(A^{\frac{\beta}{2}}B^{-1}A^{\frac{\beta}{2}}\right)A^{\frac{\gamma}{2}}$, $A\sharp_{(\alpha,\beta,\beta)}B \equiv A\sharp_{(\alpha,\beta)}B$, $A\sharp_{(0,\beta,\gamma)}B \equiv A^{\gamma}$ and $A\sharp_{(\alpha,1,1)}B \equiv A\sharp_{\alpha}B$.

Definition 2.2. Relative operator (α,β,γ)-entropy

$$S_{\alpha,\beta,\gamma}(A|B) \equiv A^{\frac{\gamma}{2}}\left(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}}\right)^{\alpha}\left(\log A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}}\right)A^{\frac{\gamma}{2}}$$

for positive invertible operators A,B and real numbers α,β,γ. In particular we have $S_{0,\beta,\gamma}(A|B) \equiv A^{\frac{\gamma}{2}}\left(\log A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}}\right)A^{\frac{\gamma}{2}}$, $S_{\alpha,\beta,\beta}(A|B) \equiv S_{\alpha,\beta}(A|B)$ and $S_{0,1,1}(A|B) \equiv S(A|B)$.

Definition 2.3. Tsallis relative operator (α,β,γ)-entropy

$$T_{\alpha,\beta,\gamma}(A|B) \equiv A^{\frac{\gamma}{2}} \ln_\alpha \left(A^{-\frac{\beta}{2}} B A^{-\frac{\beta}{2}}\right) A^{\frac{\gamma}{2}}$$

for positive invertible operators A, B and real numbers $\alpha \neq 0, \beta, \gamma$. We note that $T_{1,\beta,\gamma}(A|B) \equiv A\sharp_{(1,\beta,\gamma)}B - A^\gamma$, $T_{-1,\beta,\gamma}(A|B) \equiv A^\gamma - A\sharp_{(-1,\beta,\gamma)}B$, $T_{\alpha,\beta,\beta}(A|B) \equiv T_{\alpha,\beta}(A|B)$ and $T_{\alpha,1,1}(A|B) \equiv T_\alpha(A|B)$.

Definition 2.4. Let f, g, h be real valued continuous functions on the closed interval I such that $g, h > 0$. For invertible positive operators A and B, we define

$$Q_{f,g,h}(B|A) = h(A)^{\frac{1}{2}} f\left(g(A)^{-\frac{1}{2}} B g(A)^{-\frac{1}{2}}\right) h(A)^{\frac{1}{2}}.$$

We would remark that if $g = h$, $Q_{f,g,h}(B|A)$ is equal to generalized perspective function $P_{f\triangle h}(B|A)$ introduced in [1](see also [2]).

Theorem 2.5. *Let r, q, k, g and h be real valued continuous functions on the closed interval I such that $g, h > 0$. If for $t \in I$ we have $r(t) \leqslant q(t) \leqslant k(t)$, then for invertible positive operators A and B*

$$Q_{r,g,h}(B|A) \leqslant Q_{q,g,h}(B|A) \leqslant Q_{k,g,h}(B|A)$$

Proof. According to the assumption, we have

$$r\left(g(A)^{-\frac{1}{2}} B g(A)^{-\frac{1}{2}}\right) \leqslant q\left(g(A)^{-\frac{1}{2}} B g(A)^{-\frac{1}{2}}\right) \leqslant k\left(g(A)^{-\frac{1}{2}} B g(A)^{-\frac{1}{2}}\right),$$

by multiplying $h(A)^{\frac{1}{2}}$ from both sides, the desired inequality is obtained. □

Corollary 2.6. *Let $f_1, f_2, \ldots, f_n, g$ and h be real valued continuous functions on the closed interval I such that $g, h > 0$. If for $t \in I$ we have $f_1(t) \leqslant f_2(t) \leqslant f_3(t) \leqslant \ldots \leqslant f_n(t)$, then for invertible positive operators A and B*

$$Q_{f_1,g,h}(B|A) \leqslant Q_{f_2,g,h}(B|A) \leqslant Q_{f_3,g,h}(B|A) \leqslant \ldots \leqslant Q_{f_n,g,h}(B|A).$$

Theorem 2.7. *Let $\beta, \gamma > 0$ and $\lambda \in (0,1]$. For any invertible positive operators A and B, we have*

$$T_{-\lambda,\beta,\gamma}(A|B) \leqslant S_{0,\beta,\gamma}(A|B) \leqslant T_{\lambda,\beta,\gamma}(A|B).$$

Proof. Since for $t > 0$ we have

$$\ln_{-\lambda} t \leqslant \log t \leqslant \ln_{\lambda} t,$$

by putting $r(t) = \ln_{-\lambda} t$, $q(t) = \log t$, $k(t) = \ln_{\lambda} t$, $g(t) = t^{\beta}$, $h(t) = t^{\gamma}$ and using Theorem 2.5, we obtain the result. $\square$

Remark 2.8. According to Theorem 2.7, we get the inequality (1.4) by replacing γ with β.

Theorem 2.9. *Let $a, \beta, \gamma > 0$ and $\lambda \in (0,1]$. For any invertible positive operators A and B, we have*

$$A\natural_{(\lambda,\beta,\gamma)}B - \frac{1}{a}A\natural_{(\lambda-1,\beta,\gamma)}B + (\ln_{\lambda}\frac{1}{a})A^{\gamma} \leqslant T_{\lambda,\beta,\gamma}(A|B) \leqslant \frac{1}{a}A\natural_{(1,\beta,\gamma)}B - A^{\gamma} - (\ln_{\lambda}\frac{1}{a})A\natural_{(\lambda,\beta,\gamma)}B.$$

Proof. The following inequality holds for $a, t > 0$ and $\lambda \in (0,1]$ (see [5])

$$t^{\lambda} - (\frac{1}{a})t^{\lambda-1} + (\ln_{\lambda}\frac{1}{a}) \leqslant \ln_{\lambda} t \leqslant (\frac{1}{a})t - (\ln_{\lambda}\frac{1}{a})t^{\lambda} - 1.$$

Applying Theorem 2.5 for $r(t) = t^{\lambda} - (\frac{1}{a})t^{\lambda-1} + (\ln_{\lambda}\frac{1}{a})$, $q(t) = \ln_{\lambda} t$, $k(t) = (\frac{1}{a})t - (\ln_{\lambda}\frac{1}{a})t^{\lambda} - 1$, $g(t) = t^{\beta}$ and $h(t) = t^{\gamma}$, give the desired result. $\square$

Remark 2.10. We notice that Theorem 2.9 recover the inequalities (1.6), if we replace γ with β.

Theorem 2.11. *Let $a, \beta, \gamma > 0$ and $\lambda \in (0,1]$. For any invertible positive operators A and B, we have*

$$\begin{aligned}(1-\log a)A^{\gamma} - \frac{1}{a}A\natural_{(-1,\beta,\gamma)}B &\leqslant a^{-\lambda}T_{-\lambda,\beta,\gamma}(A|B) - \left(\log a + \ln_{\lambda}\frac{1}{a}\right)A^{\gamma} \\ &\leqslant S_{0,\beta,\gamma}(A|B) \\ &\leqslant (\log a)A^{\gamma} + T_{\lambda,\beta,\gamma}(A|B) + (\ln_{\lambda}\frac{1}{a})A\natural_{(\lambda,\beta,\gamma)}B \\ &\leqslant (\log a - 1)A^{\gamma} + \frac{1}{a}A\natural_{(1,\beta,\gamma)}B.\end{aligned} \tag{2.1}$$

Proof. For $t > 0$ we have the following inequality

$$\ln_{-\lambda}(at) \leqslant \log(at) \leqslant \ln_{\lambda}(at),$$

which is equivalent to the following inequality

$$-\log a+\frac{(at)^{-\lambda}-1}{-\lambda}\leqslant \log(t)\leqslant \frac{(at)^{\lambda}-1}{\lambda}-\log a,$$

by easy calculation we get

$$-\log a-\ln_{\lambda}\frac{1}{a}+a^{-\lambda}\ln_{-\lambda}t\leqslant \log t\leqslant \ln_{\lambda}t+t^{\lambda}\ln_{\lambda}a-\log a,$$

by putting $r(t)=-\log a-\ln_{\lambda}\frac{1}{a}+a^{-\lambda}\ln_{-\lambda}t$, $q(t)=\log t$, $k(t)=\ln_{\lambda}t+t^{\lambda}\ln_{\lambda}a-\log a$, $g(t)=t^{\beta}$, $h(t)=t^{\gamma}$ and using Theorem 2.5, we obtain the following inequality

$$\begin{aligned}&a^{-\lambda}T_{-\lambda,\beta,\gamma}(A|B)-\left(\log a+\ln_{\lambda}\frac{1}{a}\right)A^{\gamma}\\&\leqslant S_{0,\beta,\gamma}(A|B)\\&\leqslant -(\log a)A^{\gamma}+T_{\lambda,\beta,\gamma}(A|B)+(\ln_{\lambda}a)A\sharp_{(\lambda,\beta,\gamma)}B,\end{aligned}\tag{2.2}$$

if we replace a with $\frac{1}{a}$ in the right side of the previous inequality, we obtain

$$\begin{aligned}a^{-\lambda}T_{-\lambda,\beta,\gamma}(A|B)-\left(\log a+\ln_{\lambda}\frac{1}{a}\right)A^{\gamma}&\leqslant S_{0,\beta,\gamma}(A|B)\\&\leqslant(\log a)A^{\gamma}+T_{\lambda,\beta,\gamma}(A|B)+(\ln_{\lambda}\frac{1}{a})A\sharp_{(\lambda,\beta,\gamma)}B,\end{aligned}\tag{2.3}$$

which is the second and third part of (2.1).

According to Theorem 2.9, we have $T_{\lambda,\beta,\gamma}(A|B)\leqslant \frac{1}{a}A\sharp_{(1,\beta,\gamma)}B-A^{\gamma}-(\ln_{\lambda}\frac{1}{a})A\sharp_{(\lambda,\beta,\gamma)}B$, so

$$(\log a)A^{\gamma}+T_{\lambda,\beta,\gamma}(A|B)+(\ln_{\lambda}\frac{1}{a})A\sharp_{(\lambda,\beta,\gamma)}B\leqslant(\log a-1)A^{\gamma}+\frac{1}{a}A\sharp_{(1,\beta,\gamma)}B,\tag{2.4}$$

which is the last inequality of (2.1).

Let $f(\lambda)=\lambda(t-1)-(t^{\lambda}-1)$ defined for $t>0$ and $\lambda\in(0,1]$. Since $\frac{d^2}{d\lambda}f(\lambda)<0$ and $f(0)=f(1)=0$, then $f(\lambda)\geqslant 0$. It means that

$$\frac{t^{\lambda}-1}{\lambda}\leqslant t-1.\tag{2.5}$$

So for $\frac{1}{at}>0$, we have the following inequality

$$\frac{(\frac{1}{at})^{\lambda}-1}{\lambda}\leqslant(\frac{1}{at})-1,$$

which is equivalent to

$$\ln_{\lambda}\frac{1}{a}+a^{-\lambda}\ln_{\lambda}(\frac{1}{t})\leqslant(\frac{1}{at})-1,$$

if we multiple both sides by (-1), we obtain

$$a^{-\lambda}\ln_{-\lambda} t \geqslant 1-\frac{1}{at}+\ln_{\lambda}\frac{1}{a}.$$

By putting $r(t)=1-\frac{1}{at}+\ln_{\lambda}\frac{1}{a}$, $q(t)=a^{-\lambda}\ln_{-\lambda} t$, $g(t)=t^{\beta}$, $h(t)=t^{\gamma}$ and using Theorem 2.5, we get the following inequality

$$a^{-\lambda}T_{-\lambda,\beta,\gamma}(A|B) \geqslant (1+\ln_{\lambda}\frac{1}{a})A^{\gamma}-\frac{1}{a}A\sharp_{(-1,\beta,\gamma)}B. \tag{2.6}$$

According to (2.6), we have the following inequality

$$a^{-\lambda}T_{-\lambda,\beta,\gamma}(A|B)-\left(\log a+\ln_{\lambda}\frac{1}{a}\right)A^{\gamma} \geqslant (1-\log a)A^{\gamma}-\frac{1}{a}A\sharp_{(-1,\beta,\gamma)}B, \tag{2.7}$$

which is the first inequality of (2.1). Now our proof is completed by inequalities (2.3), (2.4) and (2.7). □

The following corollaries are an immediate conclusions of Theorem 2.11:

Corollary 2.12. ([18], Theorem 2.3) *If* $a>0$ *and* $\lambda\in(0,1]$, *then for any invertible positive operators A and B*

$$\begin{aligned}(1-\log a)A-\frac{1}{a}AB^{-1}A &\leqslant a^{-\lambda}T_{-\lambda}(A|B)-\left(\log a+(\ln_{\lambda}\frac{1}{a})\right)A\\ &\leqslant S(A|B)\\ &\leqslant (\log a)A+T_{\lambda}(A|B)+(\ln_{\lambda}\frac{1}{a})A\sharp_{\lambda}B\\ &\leqslant (\log a-1)A+\frac{1}{a}B.\end{aligned}$$

Proof. We notice that Theorem 2.11 generalize the above inequalities. The result now follows from Theorem 2.11 by putting $\gamma=\beta=1$. □

Corollary 2.13. ([18], Theorem 2.2) *Let* $a>0$ *and* $\lambda\in(0,1]$. *For any invertible positive operators A and B, we have*

$$-\left(\log a+(\ln_{\lambda}\frac{1}{a})\right)A+a^{-\lambda}T_{-\lambda}(A|B) \leqslant S(A|B) \leqslant T_{\lambda}(A|B)+(\ln_{\lambda}a)A\sharp_{\lambda}B-(\log a)A,$$

Proof. By putting $\gamma=\beta=1$ in (2.2), we obtain the result. □

Corollary 2.14. ([18], Theorem 2.1) *Let $a > 0$ and $\lambda \in (0,1]$. For any invertible positive operators A and B, we have*

$$a^{-\lambda} T_{-\lambda}(A|B) \geqslant A - \frac{1}{a} AB^{-1}A + (\ln_{\lambda} \frac{1}{a})A.$$

Proof. The desired inequality is concluded from inequality (2.6) as $\gamma = \beta = 1$. □

Remark 2.15. Theorem 2.11 recover the inequalities of Theorem 2.7, if we put $a = 1$.

Theorem 2.16. *Let $a, \beta, \gamma > 0$, $\lambda \in (0,1]$ and $t \in [0,1]$. For any invertible positive operators A and B, the following inequalities (i) and (ii) hold.*

(i) If $0 < a \leqslant 1$, then

$$\begin{aligned} A\sharp_{(\lambda,\beta,\gamma)}B - \frac{1}{a}A\sharp_{(\lambda-1,\beta,\gamma)}B + (\ln_{\lambda}\frac{1}{a})A^{\gamma} &\leqslant l_3 A\sharp_{(\lambda,\beta,\gamma)}B - l_1 A\sharp_{(\lambda-1,\beta,\gamma)}B - l_2 A^{\gamma} \\ &\leqslant T_{\lambda,\beta,\gamma}(A|B) \\ &\leqslant l_1 A\sharp_{(1,\beta,\gamma)}B + l_2 A\sharp_{(\lambda,\beta,\gamma)}B - l_3 A^{\gamma} \\ &\leqslant \frac{1}{a}A\sharp_{(1,\beta,\gamma)}B - (\ln_{\lambda}\frac{1}{a})A\sharp_{(\lambda,\beta,\gamma)}B - A^{\gamma}. \end{aligned} \tag{2.8}$$

(ii) If $a \geqslant 1$, then

$$\begin{aligned} l_3 A\sharp_{(\lambda,\beta,\gamma)}B - l_1 A\sharp_{(\lambda-1,\beta,\gamma)}B - l_2 A^{\gamma} &\leqslant A\sharp_{(\lambda,\beta,\gamma)}B - \frac{1}{a}A\sharp_{(\lambda-1,\beta,\gamma)}B + (\ln_{\lambda}\frac{1}{a})A^{\gamma} \\ &\leqslant T_{\lambda,\beta,\gamma}(A|B) \\ &\leqslant \frac{1}{a}A\sharp_{(1,\beta,\gamma)}B - (\ln_{\lambda}\frac{1}{a})A\sharp_{(\lambda,\beta,\gamma)}B - A^{\gamma} \\ &\leqslant l_1 A\sharp_{(1,\beta,\gamma)}B + l_2 A\sharp_{(\lambda,\beta,\gamma)}B - l_3 A^{\gamma}, \end{aligned} \tag{2.9}$$

where

$$l_1 = \frac{\lambda a^{\lambda-1}}{\lambda\left[ta^{\lambda} + (1-t)\right]}, \quad l_2 = \frac{t(a^{\lambda}-1)}{\lambda\left[ta^{\lambda} + (1-t)\right]}, \quad l_3 = \frac{\lambda a^{\lambda} + (t-1)(a^{\lambda}-1)}{\lambda\left[ta^{\lambda} + (1-t)\right]}.$$

Proof. *(i)* The inequalities which is obtained in ([18], Theorem 2.4), can be rewritten as follows for $t > 0$

$$l_3 t^{\lambda} - l_1 t^{\lambda-1} - l_2 \leqslant \ln_{\lambda} t \leqslant l_1 t + l_2 t^{\lambda} - l_3.$$

by putting $r(t) = l_3 t^{\lambda} - l_1 t^{\lambda-1} - l_2$, $q(t) = \ln_{\lambda} t$, $k(t) = l_1 t + l_2 t^{\lambda} - l_3$, $g(t) = t^{\beta}$, $h(t) = t^{\gamma}$ and using Theorem 2.5, we obtain the following inequality

$$\begin{aligned} l_3 A\sharp_{(\lambda,\beta,\gamma)}B - l_1 A\sharp_{(\lambda-1,\beta,\gamma)}B - l_2 A^{\gamma} &\leqslant T_{\lambda,\beta,\gamma}(A|B) \\ &\leqslant l_1 A\sharp_{(1,\beta,\gamma)}B + l_2 A\sharp_{(\lambda,\beta,\gamma)}B - l_3 A^{\gamma}, \end{aligned} \tag{2.10}$$

which is the second and third part of inequalities (2.8).

According to the proof of Furuichi's main theorem [8], we have the following inequalities for $t > 0$ and $0 < a \leqslant 1$

$$t^{\lambda} - \frac{1}{a}t^{\lambda-1} + (\ln_{\lambda}\frac{1}{a}) \leqslant l_3 t^{\lambda} - l_1 t^{\lambda-1} - l_2$$

$$l_1 t + l_2 t^{\lambda} - l_3 \leqslant \frac{1}{a}t - (\ln_{\lambda}\frac{1}{a})t^{\lambda} - 1$$

We put $r_1(t) = t^{\lambda} - \frac{1}{a}t^{\lambda-1} + (\ln_{\lambda}\frac{1}{a})$, $q_1(t) = l_3 t^{\lambda} - l_1 t^{\lambda-1} - l_2$, $r_2(t) = l_1 t + l_2 t^{\lambda} - l_3$ and $q_2(t) = \frac{1}{a}t - (\ln_{\lambda}\frac{1}{a})t^{\lambda} - 1$. Since $r_1(t) \leqslant q_1(t)$ and $r_2(t) \leqslant q_2(t)$, by putting $g(t) = t^{\beta}$, $h(t) = t^{\gamma}$ and using Theorem 2.5, we get the following inequalities

$$A\sharp_{(\lambda,\beta,\gamma)}B - \frac{1}{a}A\sharp_{(\lambda-1,\beta,\gamma)}B + (\ln_{\lambda}\frac{1}{a})A^{\gamma} \leqslant l_3 A\sharp_{(\lambda,\beta,\gamma)}B - l_1 A\sharp_{(\lambda-1,\beta,\gamma)}B - l_2 A^{\gamma}, \tag{2.11}$$

$$l_1 A\sharp_{(1,\beta,\gamma)}B + l_2 A\sharp_{(\lambda,\beta,\gamma)}B - l_3 A^{\gamma} \leqslant \frac{1}{a}A\sharp_{(1,\beta,\gamma)}B - (\ln_{\lambda}\frac{1}{a})A\sharp_{(\lambda,\beta,\gamma)}B - A^{\gamma}. \tag{2.12}$$

According to the inequalities (2.10), (2.11) and (2.12) the proof of *(i)* is completed.

(ii) According to the proof of Furuichi's main theorem [8], the following inequalities are obtained for $a \geqslant 1$ and $t > 0$

$$t^{\lambda} - \frac{1}{a}t^{\lambda-1} + (\ln_{\lambda}\frac{1}{a}) \geqslant l_3 t^{\lambda} - l_1 t^{\lambda-1} - l_2,$$

$$l_1 t + l_2 t^{\lambda} - l_3 \geqslant \frac{1}{a}t - (\ln_{\lambda}\frac{1}{a})t^{\lambda} - 1.$$

We define $r_1(t) = l_3 t^{\lambda} - l_1 t^{\lambda-1} - l_2$, $q_1(t) = t^{\lambda} - \frac{1}{a}t^{\lambda-1} + (\ln_{\lambda}\frac{1}{a})$, $r_2(t) = \frac{1}{a}t - (\ln_{\lambda}\frac{1}{a})t^{\lambda} - 1$ and $q_2(t) = l_1 t + l_2 t^{\lambda} - l_3$. Since $r_1(t) \leqslant q_1(t)$ and $r_2(t) \leqslant q_2(t)$, by putting $g(t) = t^{\beta}$, $h(t) = t^{\gamma}$ and using Theorem 2.5, we get the following inequalities

$$l_3 A\sharp_{(\lambda,\beta,\gamma)}B - l_1 A\sharp_{(\lambda-1,\beta,\gamma)}B - l_2 A^{\gamma} \leqslant A\sharp_{(\lambda,\beta,\gamma)}B - \frac{1}{a}A\sharp_{(\lambda-1,\beta,\gamma)}B + (\ln_{\lambda}\frac{1}{a})A^{\gamma}, \tag{2.13}$$

$$\frac{1}{a}A\sharp_{(1,\beta,\gamma)}B - (\ln_{\lambda}\frac{1}{a})A\sharp_{(\lambda,\beta,\gamma)}B - A^{\gamma} \leqslant l_1 A\sharp_{(1,\beta,\gamma)}B + l_2 A\sharp_{(\lambda,\beta,\gamma)}B - l_3 A^{\gamma}. \tag{2.14}$$

Now inequalities (2.9) of *(ii)* are obtained by Theorem 2.9, inequalities (2.13) and (2.14). □

Corollary 2.17. ([8], Theorem 2.1) *Let* $a>0$ *and* $\lambda\in(0,1]$. *For any invertible positive operators A and B, the following inequalities (i) and (ii) hold.*

(i) If $0<a\leqslant 1$, *then*

$$\begin{aligned} A\sharp_\lambda B-\frac{1}{a}A\sharp_{\lambda-1}B+(\ln_\lambda\frac{1}{a})A &\leqslant l_3A\sharp_\lambda B-l_1A\sharp_{\lambda-1}B-l_2A\\ &\leqslant T_\lambda(A|B)\\ &\leqslant l_1B+l_2A\sharp_\lambda B-l_3A\\ &\leqslant \frac{1}{a}B-(\ln_\lambda\frac{1}{a})A\sharp_\lambda B-A. \end{aligned}$$

(ii) If $a\geqslant 1$, *then*

$$\begin{aligned} l_3A\sharp_\lambda B-l_1A\sharp_{\lambda-1}B-l_2A &\leqslant A\sharp_\lambda B-\frac{1}{a}A\sharp_{\lambda-1}B+(\ln_\lambda\frac{1}{a})A\\ &\leqslant T_\lambda(A|B)\\ &\leqslant \frac{1}{a}B-(\ln_\lambda\frac{1}{a})A\sharp_\lambda B-A\\ &\leqslant l_1B+l_2A\sharp_\lambda B-l_3A, \end{aligned}$$

where

$$l_1=\frac{\lambda a^{\lambda-1}}{\lambda\left[ta^\lambda+(1-t)\right]},\quad l_2=\frac{t(a^\lambda-1)}{\lambda\left[ta^\lambda+(1-t)\right]},\quad l_3=\frac{\lambda a^\lambda+(t-1)(a^\lambda-1)}{\lambda\left[ta^\lambda+(1-t)\right]}.$$

Proof. It follows from Theorem 2.16 by putting $\gamma=\beta=1$. □

Corollary 2.18. ([18], Theorem 2.4) *Let* $a>0$ *and* $\lambda\in(0,1]$ *and* $t\in[0,1]$. *For any invertible positive operators A and B, we have*

$$l_3A\sharp_\lambda B-l_1A\sharp_{\lambda-1}B-l_2A\leqslant T_\lambda(A|B)\leqslant l_1B+l_2A\sharp_\lambda B-l_3A,$$

where

$$l_1=\frac{\lambda a^{\lambda-1}}{\lambda\left[ta^\lambda+(1-t)\right]},\quad l_2=\frac{t(a^\lambda-1)}{\lambda\left[ta^\lambda+(1-t)\right]},\quad l_3=\frac{\lambda a^\lambda+(t-1)(a^\lambda-1)}{\lambda\left[ta^\lambda+(1-t)\right]}.$$

Proof. We obtain the result by using inequalities (2.10) with $\gamma=\beta=1$. □

Remark 2.19.

(i) Inequalities (2.8) give precise bounds of inequalities (2.10) when $0 < a \leqslant 1$.

(ii) The inequalities (2.9) are a refinement of Theorem 2.9. Indeed, the inequalities (2.9) shows that we can obtain tight bounds of Theorem 2.9 when $a \geqslant 1$.

Theorem 2.20. *Let $a, \beta, \gamma > 0$ and $\lambda \in (0,1]$. For any invertible positive operators A and B we have*

(i) If $0 < a \leqslant 1$, then

$$\begin{aligned} & A^\gamma - (\frac{1}{a})A\sharp_{(-1,\beta,\gamma)}B + (\ln_\lambda \frac{1}{a})A\sharp_{(-\lambda,\beta,\gamma)}B \\ & \leqslant (a^\lambda - \ln_\lambda a)A^\gamma - a^{\lambda-1}A\sharp_{(-1,\beta,\gamma)}B \\ & \leqslant T_{-\lambda,\beta,\gamma}(A|B) \leqslant S_{0,\beta,\gamma}(A|B) \leqslant T_{\lambda,\beta,\gamma}(A|B) \\ & \leqslant (\ln_\lambda a - a^\lambda)A^\gamma + a^{\lambda-1}A\sharp_{(1,\beta,\gamma)}B \\ & \leqslant \frac{1}{a}A\sharp_{(1,\beta,\gamma)}B - A^\gamma - (\ln_\lambda \frac{1}{a})A\sharp_{(\lambda,\beta,\gamma)}B. \end{aligned} \tag{2.15}$$

(ii) If $a \geqslant 1$, then

$$\begin{aligned} & (a^\lambda - \ln_\lambda a)A^\gamma - a^{\lambda-1}A\sharp_{(-1,\beta,\gamma)}B \\ & \leqslant A^\gamma - (\frac{1}{a})A\sharp_{(-1,\beta,\gamma)}B + (\ln_\lambda \frac{1}{a})A\sharp_{(-\lambda,\beta,\gamma)}B \\ & \leqslant T_{-\lambda,\beta,\gamma}(A|B) \leqslant S_{0,\beta,\gamma}(A|B) \leqslant T_{\lambda,\beta,\gamma}(A|B) \\ & \leqslant \frac{1}{a}A\sharp_{(1,\beta,\gamma)}B - A^\gamma - (\ln_\lambda \frac{1}{a})A\sharp_{(\lambda,\beta,\gamma)}B \\ & \leqslant (\ln_\lambda a - a^\lambda)A^\gamma + a^{\lambda-1}A\sharp_{(1,\beta,\gamma)}B. \end{aligned} \tag{2.16}$$

Proof. *(i)* According to (2.5), the following inequality holds for $t, a > 0$

$$\frac{(\frac{t}{a})^\lambda - 1}{\lambda} \leqslant \frac{t}{a} - 1,$$

which is equivalent to

$$\ln_\lambda t \leqslant (\ln_\lambda a - a^\lambda) + a^{\lambda-1}t.$$

by putting $r(t) = \ln_\lambda t$, $q(t) = (\ln_\lambda a - a^\lambda) + a^{\lambda-1}t$, $g(t) = t^\beta$, $h(t) = t^\gamma$ and using Theorem 2.5, we get the following inequality

$$T_{\lambda,\beta,\gamma}(A|B) \leqslant (\ln_\lambda a - a^\lambda)A^\gamma + a^{\lambda-1}A\sharp_{(1,\beta,\gamma)}B. \tag{2.17}$$

According to Theorem 2.7, inequalities (2.6) and (2.17) we get

$$\begin{aligned}(1+\ln_\lambda \frac{1}{a})a^\lambda A^\gamma - a^{\lambda-1}A\natural_{(-1,\beta,\gamma)}B &\leqslant T_{-\lambda,\beta,\gamma}(A|B)\\ &\leqslant S_{0,\beta,\gamma}(A|B) \\ &\leqslant T_{\lambda,\beta,\gamma}(A|B) \leqslant (\ln_\lambda a - a^\lambda)A^\gamma + a^{\lambda-1}A\natural_{(1,\beta,\gamma)}B.\end{aligned} \tag{2.18}$$

Now it remains to prove the first and last inequalities of (2.15). For $0<a\leqslant 1$, the following inequality holds by Theorem 2.16

$$l_1 A\natural_{(1,\beta,\gamma)}B + l_2 A\natural_{(\lambda,\beta,\gamma)}B - l_3 A^\gamma \leqslant \frac{1}{a}A\natural_{(1,\beta,\gamma)}B - (\ln_\lambda \frac{1}{a})A\natural_{(\lambda,\beta,\gamma)}B - A^\gamma.$$

If we put $t=0$, we have $l_1 = a^{\lambda-1}$, $l_2 = 0$ and $l_3 = a^\lambda - \ln_\lambda a$. So

$$a^{\lambda-1}A\natural_{(1,\beta,\gamma)}B - (a^\lambda - \ln_\lambda a)A^\gamma \leqslant \frac{1}{a}A\natural_{(1,\beta,\gamma)}B - (\ln_\lambda \frac{1}{a})A\natural_{(\lambda,\beta,\gamma)}B - A^\gamma, \tag{2.19}$$

which is the last inequality of (2.15). The inequality (2.19) is equivalent to

$$a^{\lambda-1}t - (a^\lambda - \ln_\lambda a) \leqslant \frac{1}{a}t - (\ln_\lambda \frac{1}{a})t^\lambda - 1, \tag{2.20}$$

for $t>0$. Replacing t with $\frac{1}{t}$ in (2.20) and then multiplying both sides by (-1), we get

$$-a^{\lambda-1}(\frac{1}{t}) + (a^\lambda - \ln_\lambda a) \geqslant 1 - (\frac{1}{a})(\frac{1}{t}) + (\ln_\lambda \frac{1}{a})t^{-\lambda}.$$

by putting $r(t) = 1-(\frac{1}{a})(\frac{1}{t}) + (\ln_\lambda \frac{1}{a})t^{-\lambda}$, $q(t) = -a^{\lambda-1}(\frac{1}{t}) + (a^\lambda - \ln_\lambda a)$, $g(t)=t^\beta$, $h(t)=t^\gamma$ and using Theorem 2.5, we obtain the following inequality

$$-a^{\lambda-1}A\natural_{(-1,\beta,\gamma)}B + (a^\lambda - \ln_\lambda a)A^\gamma \geqslant A^\gamma - (\frac{1}{a})A\natural_{(-1,\beta,\gamma)}B + (\ln_\lambda \frac{1}{a})A\natural_{(-\lambda,\beta,\gamma)}B. \tag{2.21}$$

So according to (2.18), (2.19) and (2.21), the proof of *(i)* is completed.

(ii) For $t>0$, the following inequality [5] holds for $a>0$

$$\ln_\lambda \frac{1}{t} \leqslant \frac{1}{a}(\frac{1}{t}) - (\ln_\lambda \frac{1}{a})t^{-\lambda} - 1,$$

which can be rewritten as follows

$$\ln_{-\lambda} t \geqslant 1 - \frac{1}{a}(\frac{1}{t}) + (\ln_\lambda \frac{1}{a})t^{-\lambda}.$$

by putting $r(t)=1-\frac{1}{a}(\frac{1}{t})+(\ln_\lambda \frac{1}{a})t^{-\lambda}$, $q(t)=\ln_{-\lambda} t$, $g(t)=t^\beta$, $h(t)=t^\gamma$ and using Theorem 2.5, we get the following inequality

$$T_{-\lambda,\beta}(A|B,\gamma) \geqslant A^\gamma-(\frac{1}{a})A\sharp_{(-1,\beta,\gamma)}B+(\ln_\lambda \frac{1}{a})A\sharp_{(-\lambda,\beta,\gamma)}B. \tag{2.22}$$

So according to Theorem 2.7, Theorem 2.9 and inequality (2.22) we deduce

$$\begin{aligned} & A^\gamma-(\frac{1}{a})A\sharp_{(-1,\beta,\gamma)}B+(\ln_\lambda \frac{1}{a})A\sharp_{(-\lambda,\beta,\gamma)}B \\ & \leqslant T_{-\lambda,\beta,\gamma}(A|B) \\ & \leqslant S_{0,\beta,\gamma}(A|B) \\ & \leqslant T_{\lambda,\beta,\gamma}(A|B) \leqslant \frac{1}{a}A\sharp_{(1,\beta,\gamma)}B-A^\gamma-(\ln_\lambda \frac{1}{a})A\sharp_{(\lambda,\beta,\gamma)}B. \end{aligned} \tag{2.23}$$

To complete the proof, we must obtain the first and last inequalities of (2.16). For $a \geqslant 1$, according to Theorem 2.16 we have

$$\frac{1}{a}A\sharp_{(1,\beta,\gamma)}B-(\ln_\lambda \frac{1}{a})A\sharp_{(\lambda,\beta,\gamma)}B-A^\gamma \leqslant l_1A\sharp_{(1,\beta,\gamma)}B+l_2A\sharp_{(\lambda,\beta,\gamma)}B-l_3A^\gamma,$$

since for $t=0$ we have $l_1=a^{\lambda-1}$, $l_2=0$ and $l_3=a^\lambda-\ln_\lambda a$, then we deduce the following inequality

$$\frac{1}{a}A\sharp_{(1,\beta,\gamma)}B-(\ln_\lambda \frac{1}{a})A\sharp_{(\lambda,\beta,\gamma)}B-A^\gamma \leqslant a^{\lambda-1}A\sharp_{(1,\beta,\gamma)}B-(a^\lambda-\ln_\lambda a)A^\gamma, \tag{2.24}$$

which is equivalent to the following scalar inequality

$$\frac{1}{a}t-(\ln_\lambda \frac{1}{a})t^\lambda-1 \leqslant a^{\lambda-1}t-(a^\lambda-\ln_\lambda a),$$

for $t>0$. Now by replacing t with $\frac{1}{t}$ and multiplying both sides by (-1), we get

$$1+(\ln_\lambda \frac{1}{a})t^{-\lambda}-\frac{1}{a}(\frac{1}{t}) \geqslant -a^{\lambda-1}(\frac{1}{t})+(a^\lambda-\ln_\lambda a).$$

by putting $r(t)=-a^{\lambda-1}(\frac{1}{t})+(a^\lambda-\ln_\lambda a)$, $q(t)=1+(\ln_\lambda \frac{1}{a})t^{-\lambda}-\frac{1}{a}(\frac{1}{t})$, $g(t)=t^\beta$, $h(t)=t^\gamma$ and using Theorem 2.5, we obtain the following inequality

$$A^\gamma+(\ln_\lambda \frac{1}{a})A\sharp_{(-\lambda,\beta,\gamma)}B-(\frac{1}{a})A\sharp_{(-1,\beta,\gamma)}B \geqslant (a^\lambda-\ln_\lambda a)A^\gamma-a^{\lambda-1}A\sharp_{(-1,\beta,\gamma)}B. \tag{2.25}$$

Now according to inequalities (2.23), (2.24) and (2.25), we get the inequalities (2.16) and this completes the proof of *(ii)*. □

Corollary 2.21. ([8], Corollary 2.5) *Let $a > 0$ and $\lambda \in (0,1]$. For any invertible positive operators A and B we have*

(i) If $0 < a \leqslant 1$, then

$$\begin{aligned} A - (\frac{1}{a})AB^{-1}A + (\ln_\lambda \frac{1}{a})A\sharp_{-\lambda}B &\leqslant (a^\lambda - \ln_\lambda a)A - a^{\lambda-1}(AB^{-1}A) \\ &\leqslant T_{-\lambda}(A|B) \leqslant S(A|B) \leqslant T_\lambda(A|B) \\ &\leqslant (\ln_\lambda a - a^\lambda)A + a^{\lambda-1}B \\ &\leqslant \frac{1}{a}B - A - (\ln_\lambda \frac{1}{a})A\sharp_\lambda B. \end{aligned}$$

(ii) If $a \geqslant 1$, then

$$\begin{aligned} (a^\lambda - \ln_\lambda a)A - a^{\lambda-1}AB^{-1}A &\leqslant A - (\frac{1}{a})AB^{-1}A + (\ln_\lambda \frac{1}{a})A\sharp_{-\lambda}B \\ &\leqslant T_{-\lambda}(A|B) \leqslant S(A|B) \leqslant T_\lambda(A|B) \\ &\leqslant \frac{1}{a}B - A - (\ln_\lambda \frac{1}{a})A\sharp_\lambda B \\ &\leqslant (\ln_\lambda a - a^\lambda)A + a^{\lambda-1}B. \end{aligned}$$

Proof. By putting $\gamma = \beta = 1$ in Theorem 2.20, we get the result. □

Remark 2.22.

(i) Putting $\gamma = \beta = 1$ in (2.18), we get the inequalities shown in ([8], Proposition 2.3). Moreover, the inequalities (2.23) are a generalization of the inequalities obtained in ([8], Proposition 2.4).

(ii) Theorem 2.20 give precise upper and lower bounds of Theorem 2.7 when $0 < a \leqslant 1$ and $a \geqslant 1$.

Theorem 2.23.

(i) If $\lambda \in [\frac{1}{2}, 1]$, $\alpha \in [-1,0) \cup (0,1]$, $\beta, \gamma > 0$ and $0 < A^\beta \leqslant B$, then

$$\begin{aligned} 0 &\leqslant T_{-1,\beta,\gamma}(A|B) \leqslant T_{-\lambda,\beta,\gamma}(A|B) \leqslant 2A^{\frac{\gamma}{2}}\left[I - A^{\frac{\gamma}{2}}\left(\frac{A\sharp_{(1,\beta,\gamma)}B + A^\gamma}{2}\right)^{-1}A^{\frac{\gamma}{2}}\right]A^{\frac{\gamma}{2}} \\ &\leqslant \frac{2}{\alpha}A^{\frac{\gamma}{2}}\left[I - A^{\frac{\gamma}{2}}\left(\frac{A^\gamma + A\sharp_{(\alpha,\beta,\gamma)}B}{2}\right)^{-1}A^{\frac{\gamma}{2}}\right]A^{\frac{\gamma}{2}} \leqslant S_{0,\beta,\gamma}(A|B) \\ &\leqslant \frac{A\sharp_{(\frac{\alpha}{2},\beta,\gamma)}B - A\sharp_{(-\frac{\alpha}{2},\beta,\gamma)}B}{\alpha} \leqslant A\sharp_{(\frac{1}{2},\beta,\gamma)}B - A^\gamma\left(A^{-1}\sharp_{(\frac{1}{2},\beta,\gamma)}B^{-1}\right)A^\gamma \leqslant T_{\lambda,\beta,\gamma}(A|B) \leqslant T_{1,\beta,\gamma}(A|B). \end{aligned}$$

(ii) If $\lambda \in [\frac{1}{2}, 1]$, $\alpha \in [-1,0) \cup (0,1]$, $\beta, \gamma > 0$ and $0 < B \leqslant A^{\beta}$, then

$$\begin{aligned} T_{-1,\beta,\gamma}(A|B) &\leqslant T_{-\lambda,\beta,\gamma}(A|B) \leqslant A\natural_{(\frac{1}{2},\beta,\gamma)}B - A^{\gamma}\left(A^{-1}\natural_{(\frac{1}{2},\beta,\gamma)}B^{-1}\right)A^{\gamma} \leqslant \frac{A\natural_{(\frac{\alpha}{2},\beta,\gamma)}B - A\natural_{(-\frac{\alpha}{2},\beta,\gamma)}B}{\alpha} \\ &\leqslant S_{0,\beta,\gamma}(A|B) \leqslant \frac{2}{\alpha}A^{\frac{\gamma}{2}}\left[I - A^{\frac{\gamma}{2}}\left(\frac{A^{\gamma} + A\natural_{(\alpha,\beta,\gamma)}B}{2}\right)^{-1}A^{\frac{\gamma}{2}}\right]A^{\frac{\gamma}{2}} \\ &\leqslant 2A^{\frac{\gamma}{2}}\left[I - A^{\frac{\gamma}{2}}\left(\frac{A\natural_{(1,\beta,\gamma)}B + A^{\gamma}}{2}\right)^{-1}A^{\frac{\gamma}{2}}\right]A^{\frac{\gamma}{2}} \leqslant T_{\lambda,\beta,\gamma}(A|B) \leqslant T_{1,\beta,\gamma}(A|B) \leqslant 0. \end{aligned}$$

Proof. *(i)* According to ([8], Lemma 2.6), for $\lambda \in [\dfrac{1}{2}, 1]$, $\alpha \in [-1,0) \cup (0,1]$ and $t \geqslant 1$, we have

$$\begin{aligned} 0 \leqslant 1 - \frac{1}{t} \leqslant \ln_{-\lambda} t \leqslant \frac{2(t-1)}{t+1} &\leqslant \frac{2\ln_{\alpha} t}{t^{\alpha}+1} \\ &\leqslant \log t \qquad (2.26) \\ &\leqslant t^{-\frac{\alpha}{2}}\ln_{\alpha} t \leqslant \frac{t-1}{\sqrt{t}} \leqslant \ln_{\lambda} t \leqslant t-1. \end{aligned}$$

Define $f_1(t) = 0$, $f_2(t) = 1 - \frac{1}{t}$, $f_3(t) = \ln_{-\lambda} t$, $f_4(t) = \frac{2(t-1)}{t+1}$, $f_5(t) = \frac{2\ln_{\alpha} t}{t^{\alpha}+1}$, $f_6(t) = \log t$, $f_7(t) = t^{-\frac{\alpha}{2}}\ln_{\alpha} t$, $f_8(t) = \frac{t-1}{\sqrt{t}}$, $f_9(t) = \ln_{\lambda} t$ and $f_{10}(t) = t-1$. Apply Corollary 2.6 with $g(t) = t^{\beta}$ and $h(t) = t^{\gamma}$ to conclude the following inequality

$$\begin{aligned} 0 \leqslant A^{\gamma} - A\natural_{(-1,\beta,\gamma)}B \leqslant T_{-\lambda,\beta,\gamma}(A|B) &\leqslant 2A^{\frac{\gamma}{2}}\left[\frac{(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}} - 1)}{(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}} + 1)}\right]A^{\frac{\gamma}{2}} \\ &\leqslant A^{\frac{\gamma}{2}}\left[\frac{2\ln_{\alpha}(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})}{(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})^{\alpha} + 1}\right]A^{\frac{\gamma}{2}} \leqslant S_{0,\beta,\gamma}(A|B) \\ &\leqslant A^{\frac{\gamma}{2}}\left[(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})^{-\frac{\alpha}{2}}\ln_{\alpha}(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})\right]A^{\frac{\gamma}{2}} \qquad (2.27) \\ &\leqslant A^{\frac{\gamma}{2}}\left[(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})^{\frac{1}{2}} - (A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})^{-\frac{1}{2}}\right]A^{\frac{\gamma}{2}} \\ &\leqslant T_{\lambda,\beta,\gamma}(A|B) \leqslant A\natural_{(1,\beta,\gamma)}B - A^{\gamma}. \end{aligned}$$

We notice that $t \geqslant 1$ implies $0 < A^{\beta} \leqslant B$. By some mathematical calculation we get

(a)

$$\frac{A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}}-1}{A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}}+1}=\frac{A^{\frac{\gamma}{2}}(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}}-1)A^{\frac{\gamma}{2}}}{A^{\frac{\gamma}{2}}(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}}+1)A^{\frac{\gamma}{2}}}=\frac{A\natural_{(1,\beta,\gamma)}B-A^{\gamma}}{A\natural_{(1,\beta,\gamma)}B+A^{\gamma}}$$

$$=I-\left(\frac{2A^{\gamma}}{A\natural_{(1,\beta,\gamma)}B+A^{\gamma}}\right)$$

$$=I-A^{\frac{\gamma}{2}}\left(\frac{A\natural_{(1,\beta,\gamma)}B+A^{\gamma}}{2}\right)^{-1}A^{\frac{\gamma}{2}}.$$

(b)

$$A^{\frac{\gamma}{2}}\left[\frac{2\ln_{\alpha}(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})}{(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})^{\alpha}+1}\right]A^{\frac{\gamma}{2}}=\frac{2}{\alpha}A^{\frac{\gamma}{2}}\left[\frac{(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})^{\alpha}-1}{(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})^{\alpha}+1}\right]A^{\frac{\gamma}{2}}$$

$$=\frac{2}{\alpha}A^{\frac{\gamma}{2}}\left[\frac{A\natural_{(\alpha,\beta,\gamma)}B-A^{\gamma}}{A\natural_{(\alpha,\beta,\gamma)}B+A^{\gamma}}\right]A^{\frac{\gamma}{2}}$$

$$=\frac{2}{\alpha}A^{\frac{\gamma}{2}}\left[I-\frac{2A^{\gamma}}{A\natural_{(\alpha,\beta,\gamma)}B+A^{\gamma}}\right]A^{\frac{\gamma}{2}}$$

$$=\frac{2}{\alpha}A^{\frac{\gamma}{2}}\left[I-A^{\frac{\gamma}{2}}\left(\frac{A\natural_{(\alpha,\beta,\gamma)}B+A^{\gamma}}{2}\right)^{-1}A^{\frac{\gamma}{2}}\right]A^{\frac{\gamma}{2}}.$$

(c)

$$A^{\frac{\gamma}{2}}\left[(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})^{-\frac{\alpha}{2}}\ln_{\alpha}(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})\right]A^{\frac{\gamma}{2}}=A^{\frac{\gamma}{2}}\left[\frac{(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})^{\frac{\alpha}{2}}-(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})^{-\frac{\alpha}{2}}}{\alpha}\right]A^{\frac{\gamma}{2}}$$

$$=\frac{A\natural_{(\frac{\alpha}{2},\beta,\gamma)}B-A\natural_{(-\frac{\alpha}{2},\beta,\gamma)}B}{\alpha}.$$

(d)

$$A^{\frac{\gamma}{2}}\left[(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})^{\frac{1}{2}}-(A^{-\frac{\beta}{2}}BA^{-\frac{\beta}{2}})^{-\frac{1}{2}}\right]A^{\frac{\gamma}{2}}=A\natural_{(\frac{1}{2},\beta,\gamma)}B-A^{\frac{\gamma}{2}}\left[\left(A^{\frac{\beta}{2}}B^{-1}A^{\frac{\beta}{2}}\right)^{\frac{1}{2}}\right]A^{\frac{\gamma}{2}}$$

$$=A\natural_{(\frac{1}{2},\beta,\gamma)}B-A^{\gamma}\left[A^{-1}\natural_{(\frac{1}{2},\beta,\gamma)}B^{-1}\right]A^{\gamma}.$$

So according to inequalities (2.27) and the previous calculations, we obtain the inequalities of *(i)* and this completes the proof.

(ii) For $\lambda \in [\frac{1}{2}, 1]$, $\alpha \in [-1,0) \cup (0,1]$ and $0 < t \leqslant 1$, we have ([8], Lemma 2.6)

$$\begin{aligned} 1-\frac{1}{t} \leqslant \ln_{-\lambda} t \leqslant \frac{t-1}{\sqrt{t}} &\leqslant t^{-\frac{\alpha}{2}} \ln_\alpha t \\ &\leqslant \log t \\ &\leqslant \frac{2\ln_\alpha t}{t^\alpha+1} \leqslant \frac{2(t-1)}{t+1} \leqslant \ln_\lambda t \leqslant t-1 \leqslant 0. \end{aligned} \tag{2.28}$$

The proof is completed similarly as in *(i)* according to Corollary 2.6. We notice that $0 < t \leqslant 1$ is equivalent to $0 < B \leqslant A^\beta$. □

Corollary 2.24. ([8], Theorem 2.8)

(i) If $\lambda \in [\frac{1}{2}, 1]$, $\alpha \in [-1,0) \cup (0,1]$ and $0 < A \leqslant B$, then

$$\begin{aligned} 0 &\leqslant T_{-1}(A|B) \leqslant T_{-\lambda}(A|B) \leqslant 2A^{\frac{1}{2}} \left[I - A^{\frac{1}{2}} \left(\frac{A+B}{2} \right)^{-1} A^{\frac{1}{2}} \right] A^{\frac{1}{2}} \\ &\leqslant \frac{2}{\alpha} A^{\frac{1}{2}} \left[I - A^{\frac{1}{2}} \left(\frac{A + A\sharp_\alpha B}{2} \right)^{-1} A^{\frac{1}{2}} \right] A^{\frac{1}{2}} \leqslant S(A|B) \\ &\leqslant \frac{A\sharp_{\frac{\alpha}{2}} B - A\sharp_{-\frac{\alpha}{2}} B}{\alpha} \leqslant A\sharp_{\frac{1}{2}} B - A\left(A^{-1} \sharp_{\frac{1}{2}} B^{-1} \right) A \leqslant T_\lambda(A|B) \leqslant T_1(A|B). \end{aligned}$$

(ii) If $\lambda \in [\frac{1}{2}, 1]$, $\alpha \in [-1,0) \cup (0,1]$ and $0 < B \leqslant A$, then

$$\begin{aligned} T_{-1}(A|B) \leqslant T_{-\lambda}(A|B) \leqslant A\sharp_{\frac{1}{2}} B - A\left(A^{-1} \sharp_{\frac{1}{2}} B^{-1} \right) A &\leqslant \frac{A\sharp_{\frac{\alpha}{2}} B - A\sharp_{-\frac{\alpha}{2}} B}{\alpha} \\ \leqslant S(A|B) &\leqslant \frac{2}{\alpha} A^{\frac{1}{2}} \left[I - A^{\frac{1}{2}} \left(\frac{A + A\sharp_\alpha B}{2} \right)^{-1} A^{\frac{1}{2}} \right] A^{\frac{1}{2}} \\ \leqslant 2A^{\frac{1}{2}} \left[I - A^{\frac{1}{2}} \left(\frac{A+B}{2} \right)^{-1} A^{\frac{1}{2}} \right] A^{\frac{1}{2}} &\leqslant T_\lambda(A|B) \leqslant T_1(A|B) \leqslant 0. \end{aligned}$$

Proof. We note that Theorem 2.23 is a generalization of the above inequalities. Indeed, the result follows from Theorem 2.23 by putting $\gamma = \beta = 1$. □

Theorem 2.25. *We have*

(i) If $\lambda \in (0,1]$, $\beta,\gamma > 0$ *and* $0 < A^{\beta} \leqslant B$, *then*

$$\begin{aligned}0 \leqslant T_{-1,\beta,\gamma}(A|B) \leqslant T_{-\lambda,\beta,\gamma}(A|B) &\leqslant \frac{2}{\lambda}A^{\frac{\gamma}{2}}\left[I - A^{\frac{\gamma}{2}}\left(\frac{A^{\gamma}+A\sharp_{(\lambda,\beta,\gamma)}B}{2}\right)^{-1}A^{\frac{\gamma}{2}}\right]A^{\frac{\gamma}{2}} \\ &\leqslant S_{0,\beta,\gamma}(A|B) \leqslant \frac{A\sharp_{(\frac{\lambda}{2},\beta,\gamma)}B - A\sharp_{(-\frac{\lambda}{2},\beta,\gamma)}B}{\lambda} \\ &\leqslant T_{\lambda,\beta,\gamma}(A|B) \leqslant T_{1,\beta,\gamma}(A|B).\end{aligned}$$

(ii) If $\lambda \in (0,1]$, $\beta,\gamma > 0$ *and* $0 < B \leqslant A^{\beta}$, *then*

$$\begin{aligned}T_{-1,\beta,\gamma}(A|B) \leqslant T_{-\lambda,\beta,\gamma}(A|B) &\leqslant \frac{A\sharp_{(\frac{\lambda}{2},\beta,\gamma)}B - A\sharp_{(-\frac{\lambda}{2},\beta,\gamma)}B}{\lambda} \\ &\leqslant S_{0,\beta,\gamma}(A|B) \leqslant \frac{2}{\lambda}A^{\frac{\gamma}{2}}\left[I - A^{\frac{\gamma}{2}}\left(\frac{A^{\gamma}+A\sharp_{(\lambda,\beta,\gamma)}B}{2}\right)^{-1}A^{\frac{\gamma}{2}}\right]A^{\frac{\gamma}{2}} \\ &\leqslant T_{\lambda,\beta,\gamma}(A|B) \leqslant T_{1,\beta,\gamma}(A|B) \leqslant 0.\end{aligned}$$

Proof. According to the inequalities (2.26) and (2.28), we have

(i) If $\lambda \in (0,1]$ and $t \geqslant 1$, then

$$\ln_{-\lambda} t \leqslant \frac{2\ln_{\lambda} t}{t^{\lambda}+1} \leqslant \log t \leqslant t^{-\frac{\lambda}{2}} \ln_{\lambda} t \leqslant \ln_{\lambda} t. \tag{2.29}$$

(ii) If $\lambda \in (0,1]$ and $0 < t \leqslant 1$, then

$$\ln_{-\lambda} t \leqslant t^{-\frac{\lambda}{2}} \ln_{\lambda} t \leqslant \log t \leqslant \frac{2\ln_{\lambda} t}{t^{\lambda}+1} \leqslant \ln_{\lambda} t. \tag{2.30}$$

Now the proof of the Theorem can be completed by using Corollary 2.6, Theorem 2.23 and inequalities (2.29) and (2.30). □

Corollary 2.26. ([8], Corollary 2.9)

(i) If $\lambda \in (0,1]$ *and* $0 < A \leqslant B$, *then*

$$\begin{aligned}0 \leqslant T_{-1}(A|B) \leqslant T_{-\lambda}(A|B) &\leqslant \frac{2}{\lambda}A^{\frac{1}{2}}\left[I - A^{\frac{1}{2}}\left(\frac{A+A\sharp_{\lambda}B}{2}\right)^{-1}A^{\frac{1}{2}}\right]A^{\frac{1}{2}} \\ &\leqslant S(A|B) \leqslant \frac{A\sharp_{\frac{\lambda}{2}}B - A\sharp_{-\frac{\lambda}{2}}B}{\lambda} \\ &\leqslant T_{\lambda}(A|B) \leqslant T_{1}(A|B).\end{aligned}$$

(ii) If $\lambda \in (0,1]$ *and* $0 < B \leqslant A$, *then*

$$\begin{aligned} T_{-1}(A|B) \leqslant T_{-\lambda}(A|B) &\leqslant \frac{A\sharp_{\frac{\lambda}{2}}B - A\sharp_{-\frac{\lambda}{2}}B}{\lambda} \\ &\leqslant S(A|B) \leqslant \frac{2}{\lambda}A^{\frac{1}{2}}\left[I - A^{\frac{1}{2}}\left(\frac{A + A\sharp_{\lambda}B}{2}\right)^{-1}A^{\frac{1}{2}}\right]A^{\frac{1}{2}} \\ &\leqslant T_{\lambda}(A|B) \leqslant T_{1}(A|B) \leqslant 0. \end{aligned}$$

Proof. The desired inequalities is obtained by putting $\gamma = \beta = 1$ in Theorem 2.25. □

Theorem 2.27. *Let* $a, \beta, \gamma > 0$ *and* $\lambda \in (0,1]$. *For any invertible positive operators A and B we have*

$$\begin{aligned} A\sharp_{(\lambda,\beta,\gamma)}B - \frac{1}{a}A\sharp_{(\lambda-1,\beta,\gamma)}B + (\ln_{\lambda}\frac{1}{a})A^{\gamma} &\leqslant S_{\lambda,\beta,\gamma}(A|B) \\ &\leqslant \frac{1}{a}A\sharp_{(\lambda+1,\beta,\gamma)}B - (\ln_{\lambda}\frac{1}{a})A\sharp_{(2\lambda,\beta,\gamma)}B - A\sharp_{(\lambda,\beta,\gamma)}B. \end{aligned}$$

Proof. We have the following inequality for $t > 0$ (see [14])

$$t^{\lambda} - \frac{1}{a}t^{\lambda-1} + \ln_{\lambda}\frac{1}{a} \leqslant t^{\lambda}\log t \leqslant t^{\lambda}\left(\frac{1}{a}t - (\ln_{\lambda}\frac{1}{a})t^{\lambda} - 1\right).$$

Applying Theorem 2.5 for $r(t) = t^{\lambda} - \frac{1}{a}t^{\lambda-1} + \ln_{\lambda}\frac{1}{a}$, $q(t) = t^{\lambda}\log t$, $k(t) = t^{\lambda}\left(\frac{1}{a}t - (\ln_{\lambda}\frac{1}{a})t^{\lambda} - 1\right)$, $g(t) = t^{\beta}$, $h(t) = t^{\gamma}$, we obtain the desired result.

Corollary 2.28. ([14], Corollary 3.4) *Let* $a, \beta > 0$ *and* $\lambda \in (0,1]$. *For any invertible positive operators A and B we have*

$$\begin{aligned} A\sharp_{(\lambda,\beta)}B - \frac{1}{a}A\sharp_{(\lambda-1,\beta)}B + (\ln_{\lambda}\frac{1}{a})A^{\beta} &\leqslant S_{\lambda,\beta}(A|B) \\ &\leqslant \frac{1}{a}A\sharp_{(\lambda+1,\beta)}B - (\ln_{\lambda}\frac{1}{a})A\sharp_{(2\lambda,\beta)}B - A\sharp_{(\lambda,\beta)}B. \end{aligned}$$

Proof. It follows from Theorem 2.27 by replacing γ with β.

Conflict of Interests

The authors declare that there is no conflict of interests.

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