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ON FIXED POINTS OF ϕ-TYPE ĆIRIĆ OPERATORS

S. PAL[1], P. DAS[2], A. P. BAISNAB[1,*]

[1]Department of Mathematics, Lady Brabourne College, Kolkata, West Bengal, India

[2]Department of Mathematics, Jadavpur University, Kolkata-700032, West Bengal, India

Abstract. Over a complete metric space endowed with an integral metric, some fixed point theorems for operators including Ćirić type operators satisfying non-expansive type condition have been established.

Keywords: Fixed point; Complete metric space: Ćirić operator; Integral metric.

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1. Introduction and preliminaries

In 1968, Kannan [1] first considered discontinuous operators, and successfully proved some fixed point theorems for such operators.

In a metric space (X,d), an operator $T:(X,d)\to(X,d)$ is called a Kannan operator if T satisfies the condition:

$$d(T(x),T(y))\leq\beta[d(x,T(x))+d(y,T(y))],$$

*Corresponding author

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where $0 \leq \beta < \frac{1}{2}$ for all $x, y \in X$. See [1] and the references therein. Immediately thereafter Kannan operator T has been extended to a form like satisfying the condition below:

$$d(T(x), T(y)) \leq \alpha[d(x, T(x)) + d(y, T(y))] + \beta d(x, y) + \gamma \max[d(x, T(y)), d(y, T(x))],$$

where $0 \leq \alpha, \beta, \gamma$ with $\max(\alpha, \beta) + \gamma < 1$ for all $x, y \in X$.

This extended form of Kannan operator has been studied by researchers when (X,d) is complete.

In 1975 , Ćirić considered another type of operator $T : (X, d) \to (X, d)$ satisfying

$$d(T^n(x), T^n(y)) \leq q^n(x, y)\delta(x, y),\ n = 1, 2, \ldots$$

for all $x, y \in X$, where q and δ are two non-negative real-valued functions over $X \times X$ satisfying $q(x, y) < 1$ for all $d(x, y) \in X \times X$ with $\sup\limits_{x,y \in X} q(x, y) = 1$.

These operators T are designated as Ćirić operators. Let R^+ denote the set of all non-negative reals with usual topology and $\phi : R^+ \to R^+$ be non-negative, finitely Lebesgue summable over each compact set of R^+ satisfying following properties:

(i) $\int_0^{u+v} \phi(t)dt \leq \int_0^u \phi(t)dt + \int_0^v \phi(t)dt$ if $u, v \in R^+$ and

(ii) for every $\varepsilon > 0$, $\int_0^{\varepsilon} \phi(t)dt > 0$.

Ćirić operators include a contraction operator over a metric space but converse is not true. It is known that a Ćirić operator may not possess a fixed point, and that is why, attempts have been in progress to frame appropriate conditions to be satisfied by Ćirić operator to attract a fixed point. Relevant references have been cited at the end in this connection.

In this paper, we define for the first time a ϕ-type Ćirić operator over (X,d) via integral metric, and search for appropriate conditions that ensure fixed point of ϕ-type Ćirić operator over a complete metric space; allied matters like continuity of fixed points have also been investigated into. Examples are cited either in support of relevant Theorem or to examine strength of hypothesis made in Theorem.

2. Main results

Definition 2.1. In a metric space (X,d), an operator $T:(X,d)\to(X,d)$ is said to be a ϕ-type Ćirić operator if

$$\int_0^{d(T^n(x),T^n(y))}\phi(t)dt \le q^n(x,y)\delta(x,y), n=1,2,...,$$

where ϕ, q and δ are as above.

Here 1-type Ćirić operator is a Ćirić operator.

Theorem 2.2. *Let (X,d) be a complete metric space and $T:(X,d)\to(X,d)$ be a ϕ-type Ćirić operator satisfying the following condition:*

$$\begin{aligned}\int_0^{d(T(x),T(y))}\phi(t)dt &\le \alpha\int_0^{d(x,T(x))+d(y,T(y))}\phi(t)dt+\beta\int_0^{d(x,y)}\phi(t)dt\\ &\quad +\gamma\int_0^{\max\{d(x,T(y)),d(y,T(x))\}}\phi(t)dt\end{aligned}$$

for all $x,y\in X$ such that $0\le\alpha,\beta,\gamma$ with $\max(\alpha,\beta)+\gamma<1$.
Then T has a unique fixed point u in X, and for each $x\in X$, $\lim_{n\to\infty}T^n(x)=u$.

Proof. Take $x=x_0\in X$, then for positive m and n,

$$\begin{aligned}&\int_0^{d(T^m(x_0),T^n(x_0)}\phi(t)dt\\ =&\int_0^{d(T(T^{m-1}(x_0)),T(T^{n-1}(x_0))}\phi(t)dt\\ \le&\ \alpha\int_0^{d(T^{m-1}(x_0)),T^m(x_0))+d(T^{n-1}(x_0),T^n(x_0))}\phi(t)dt+\beta\int_0^{d(T^{m-1}(x_0),T^{n-1}(x_0))}\phi(t)dt\\ &+\gamma\int_0^{\max\{d(T^{m-1}(x_0),T^n(x_0)),d(T^{n-1}(x_0),T^m(x_0))\}}\phi(t)dt\\ \le&\ \alpha\int_0^{d(T^{m-1}(x_0),T^m(x_0))+d(T^{n-1}(x_0),T^n(x_0))}\phi(t)dt\\ &+\beta\int_0^{d(T^{m-1}(x_0),T^m(x_0))+d(T^m(x_0),T^n(x_0))+d(T^n(x_0),T^{n-1}(x_0))}\phi(t)dt\\ &+\gamma\int_0^{\max\{d(T^{m-1}(x_0),T^m(x_0)+d(T^m(x_0),T^n(x_0)),d(T^{n-1}(x_0),T^n(x_0))+d(T^n(x_0),T^m(x_0))\}}\phi(t)dt\end{aligned}$$

$$\begin{aligned}
&\le \alpha \int_0^{d(T^{m-1}(x_0),T^m(x_0))+d(T^{n-1}(x_0),T^n(x_0))} \phi(t)dt \\
&\quad +\beta \int_0^{d(T^{m-1}(x_0),T^m(x_0))+d(T^m(x_0),T^n(x_0))+d(T^n(x_0),T^{n-1}(x_0))} \phi(t)dt \\
&\quad +\gamma \int_0^{d(T^{m-1}(x_0),T^m(x_0)+d(T^m(x_0),T^n(x_0))+d(T^{n-1}(x_0),T^n(x_0))} \phi(t)dt \\
&\le (2\max\{\alpha,\beta\}+\gamma) \int_0^{d(T^{m-1}(x_0),T^m(x_0))+d(T^{n-1}(x_0),T^n(x_0))} \phi(t)dt \\
&\quad +(\beta+\gamma) \int_0^{d(T^m(x_0),T^n(x_0))} \phi(t)dt.
\end{aligned}$$

This gives,

$$\int_0^{d(T^m(x_0),T^n(x_0)} \phi(t)dt \le \frac{2\max(\alpha,\beta)+\gamma}{1-\beta-\gamma} \int_0^{d(T^{m-1}(x_0),T^m(x_0))+d(T^{n-1}(x_0),T^n(x_0))} \phi(t)dt,$$

using property (i) of ϕ. And therefore,

$$\begin{aligned}
&\int_0^{d(T^m(x_0),T^n(x_0)} \phi(t)dt \\
&\le \frac{2\max\{\alpha,\beta\}+\gamma}{1-\beta-\gamma}[q^{m-1}(x_0,T(x_0))+q^{n-1}(x_0,T(x_0))]\delta(x_0,T(x_0)) \\
&\to 0 \text{ as } m,n\to\infty.
\end{aligned}$$

So $\{T^n(x_0)\}$ is Cauchy in (X,d) which is complete.

Let $\lim_{n\to\infty} T^n(x_0) = u$ for some $u \in X$. Now

$$\begin{aligned}
&\int_0^{d(T^n(x_0),T(u))} \phi(t)dt \\
&\le \alpha \int_0^{d(T^{n-1}(x_0),T^n(x_0))+d(u,T(u))} \phi(t)dt + \beta \int_0^{d(T^{n-1}(x_0),u)} \phi(t)dt \\
&\quad +\gamma \int_0^{\max\{d(T^{n-1}(x_0),T(u)),d(u,T^n(x_0))\}} \phi(t)dt.
\end{aligned}$$

Passing on limit as $n\to\infty$, we have

$$\int_0^{d(u,T(u))} \phi(t)dt \le (\alpha+\gamma) \int_0^{d(u,T(u))} \phi(t)dt.$$

That is,

$$\int_0^{d(u,T(u))} \phi(t)dt = 0.$$

Hence using property (ii) of ϕ, we have $u = T(u)$. For uniqueness of fixed point u of T, let $v \in X$ with $v = T(v)$. Then

$$\begin{aligned}\int_0^{d(u,v)} \phi(t)dt &= \int_0^{d(T(u),T(v))} \phi(t)dt = \int_0^{d(T^n(u),T^n(v))} \phi(t)dt \\ &\leq q^n(u,v)\delta(u,v) \to 0 \text{ as } n \to \infty.\end{aligned}$$

Hence $u = v$.

Theorem 2.3. *Let (X,d) be a metric space, and $T : (X,d) \to (X,d)$ be a ϕ-type Ćirić operator satisfying*

$$\begin{aligned}\int_0^{d(T(x),T(y))} \phi(t)dt \quad \leq \quad & \alpha \int_0^{d(x,T(x))+d(y,T(y))} \phi(t)dt + \beta \int_0^{d(x,y)} \phi(t)dt \\ & +\gamma \int_0^{\max\{d(x,T(y)),d(y,T(x))\}} \phi(t)dt\end{aligned}$$

for all $x, y \in X$ such that $0 \leq \alpha, \beta, \gamma$ with $\max\{\alpha, \beta\} + \gamma < 1$. If for some point $x_0 \in X$, $\{T^n(x_0)\}$ has a convergent subsequence converging to $u \in X$, the u is the unique fixed point of T, and $\lim_{n\to\infty} T^n(x_0) = u$.

Proof. Let $\lim_{k\to\infty} T^{n_k}(x_0) = u \in X$. Then

$$\begin{aligned}& \int_0^{d(u,T(u))} \phi(t)dt \\ & \leq \int_0^{d(u,T^{n_k+1}(x_0))+d(T^{n_k+1}(x_0),T(u))} \phi(t)dt \\ & \leq \int_0^{d(u,T^{n_k+1}(x_0))} \phi(t)dt + \alpha \int_0^{d(T^{n_k}(x_0),T^{n_k+1}(x_0))+d(u,T(u))} \phi(t)dt \\ & +\beta \int_0^{d(T^{n_k}(x_0),u)} \phi(t)dt + \gamma \int_0^{\max\{d(T^{n_k}(x_0),T(u)),d(u,T^{n_k+1}(x_0))\}} \phi(t)dt \\ & \leq \int_0^{d(u,T^{n_k}(x_0))+d(T^{n_k}(x_0),T^{n_k+1}(x_0))} \phi(t)dt \\ & +\alpha \int_0^{d(T^{n_k}(x_0),T^{n_k+1}(x_0))+d(u,T(u))} \phi(t)dt + \beta \int_0^{d(T^{n_k}(x_0),u)} \phi(t)dt \\ & +\gamma \int_0^{\max\{d(T^{n_k}(x_0),T(u)),d(u,T^{n_k}(x_0))+d(T^{n_k}(x_0),T^{n_k+1}(x_0))\}} \phi(t)dt\end{aligned}$$

$$\leq \int_0^{d(u,T^{n_k}(x_0))} \phi(t)dt + q^{n_k}(x_0,T(x_0))\delta(x_0,T(x_0))$$

$$+\alpha q^{n_k}(x_0,T(x_0))\delta(x_0,T(x_0)) + \alpha \int_0^{d(u,T(u))} \phi(t)dt + \beta \int_0^{d(T^{n_k}(x_0),u)} \phi(t)dt$$

$$+\gamma\{\int_0^{\max\{d(T^{n_k}(x_0),T(u)),d(u,T^{n_k}(x_0))\}} \phi(t)dt + q^{n_k}(x_0,T(x_0))\delta(x_0,T(x_0))\}$$

$$\to (\alpha+\gamma)\int_0^{d(u,T(u))} \phi(t)dt \text{ as } k \to \infty.$$

This gives $u = T(u)$, and uniqueness of u is clear.

If (X,d) is a complete metric space it is known that a Kannan operator $T : (X,d) \to (X,d)$ satisfying

$$d(T(x),T(y)) \leq \beta[d(x,T(x)) + d(y,T(y))],$$

where $0 \leq \beta < \frac{1}{2}$ for all $x,y \in X$ has a unique fixed point in X. See [1]. If contractive parameter β exceeds $\frac{1}{2}$, T may fail to possess a fixed point in X. Theorem 2.2 above permits β to exceed $\frac{1}{2}$ to the extent of $\beta < 1$, and ensures a fixed point of T provided that T is a ϕ-type Ćirić operator. Here is an Example to support this statement.

Example 2.4. Take $X = \{0,1\}$ with usual metric d of reals, and $T : X \to X$ where $T(0) = 1$ and $T(1) = 0$. One finds that

$$d(T(x),T(y)) \leq \beta\,[d(x,T(x)) + d(y,T(y))]$$

for all $x,y \in X$ by taking $\beta = \frac{3}{4} > \frac{1}{2}$. Thus conditions of Theorem 2.2 are partially met by T. Now,

$$d(T^0(0),T^n(1)) = d(0,1) = 1 > q^n(x,y)\delta(x,y)$$

for large N where $q(x,y)$ and $\delta(x,y)$ are as in Ćirić operator. Thus T is not 1-type Ćirić operator(taking $\phi \equiv 1$) over X; and T has no fixed point in X.

Example 2.5. Let $X = [0,1]$, and $\phi : R^+ \to R^+$ ($R^+ =$ set of non-negative reals) with usual metric of reals, where $\phi(t) = \frac{1}{2}t^2$ as $t \in R^+$. Take $T : X \to X$ as $T(x) = \frac{x}{2}$ when $x \in [0,1]$.

Now $x,y\in X$ gives

$$\begin{aligned}\int_0^{d(T^n(x),T^n(y))}\phi(t)dt &= \int_0^{d(\frac{x}{2^n},\frac{y}{2^n})}\frac{1}{2}t^2dt\\ &= \frac{1}{6}\left|\frac{x}{2^n}-\frac{y}{2^n}\right|^3=\frac{1}{2^{3n}}\frac{|x-y|^3}{6}.\end{aligned}$$

Let $\delta,q:[0,1]\to$ Reals be taken as

$$q(x,y)=\begin{cases}1-|x-y|\,,if\,|x-y|<\frac{7}{8}\\ \frac{1}{2},\ if\,|x-y|\geq\frac{7}{8}\end{cases}$$

and $\delta(x,y)=\frac{|x-y|^3}{6}$ for all $x,y\in[0,1]$.

Now take $\alpha=\gamma=\frac{1}{16}$ and $\beta=\frac{1}{8}$; then T is a ϕ-type Ćirić operator that fits in Theorem 2.2; it supports Theorem 2.2. 0 is, of course, the unique fixed point of T.

3. Continuity

Theorem 3.1. *Let (X,d) be a complete metric space, and each $T_j:X\to X$ satisfies conditions like*

$$(i)\int_0^{d(T_j^n(x),T_j^n(y))}\phi(t)dt\leq q^n(x,y)\delta(x,y),n=1,2,...$$

and

$$\begin{aligned}(ii)\int_0^{d(T_j(x),T_j(y))}\phi(t)dt &\leq \alpha\int_0^{d(x,T_j(x))+d(y,T_j(y))}\phi(t)dt+\beta\int_0^{d(x,y)}\phi(t)dt\\ &+\gamma\int_0^{\max\{d(x,T_j(y)),d(y,T_j(x))\}}\phi(t)dt\end{aligned}$$

for all $x,y\in X$ where $0\leq\alpha,\beta,\gamma$ with $\max\{\alpha,\beta\}+\gamma<1$. If $u_j=T_j(u_j)\in X$, and $T:X\to X$ is an operator such that $T^n(x)=\lim_{j\to\infty}T_j^n(x)$ for all $x\in X,n=1,2,\ldots..$ Then T has a unique fixed point u in X if and only if $u=\lim_{j\to\infty}u_j$.

Proof. Here as $x\in X$, $T^n(x)=\lim_{j\to\infty}T_j^n(x)$. In (ii) of Theorem 3.1 make $j\to\infty$ to obtain

$$\begin{aligned}\int_0^{d(T(x),T(y))}\phi(t)dt &\leq \alpha\int_0^{d(x,T(x))+d(y,T(y))}\phi(t)dt+\beta\int_0^{d(x,y)}\phi(t)dt\\ &+\gamma\int_0^{\max\{d(x,T(y)),d(y,T(x))\}}\phi(t)dt,\end{aligned}$$

where $\alpha,\beta,\gamma \geq 0$ with $\max(\alpha,\beta)+\gamma < 1$. Now

$$\begin{aligned}\int_0^{d(T^n(x),T^n(y))}\phi(t)dt &\leq \int_0^{d(T^n(x),T_j^n(x))}\phi(t)dt \\ &\quad+\int_0^{d(T_j^n(x),T_j^n(y))}\phi(t)dt+\int_0^{d(T_j^n(y),T^n(y))}\phi(t)dt \\ &\leq \int_0^{d(T^n(x),T_j^n(x))}\phi(t)dt+q^n(x,y)\delta(x,y)+\int_0^{d(T_j^n(y),T^n(y))}\phi(t)dt.\end{aligned}$$

Passing on limit $j\to\infty$, we have

$$\int_0^{d(T^n(x),T^n(y))}\phi(t)dt \leq q^n(x,y)\delta(x,y)$$

for all $x,y\in X$. Now Theorem 2.2 applies to conclude that T has a unique fixed point $u\in X$. So $T(u)=u$. If $T_j(u_j)=u_j, i=1,2,...,$ we have

$$\begin{aligned}\int_0^{d(u,u_j)}\phi(t)dt &= \int_0^{d(T(u),T_j(u_j))}\phi(t)dt \\ &\leq \int_0^{d(T(u),T_j(u))}\phi(t)dt+\int_0^{d(T_j(u),T_j(u_j))}\phi(t)dt \\ &\leq \int_0^{d(T(u),T_j(u))}\phi(t)dt+\alpha\int_0^{d(u,T_j(u))+d(u_j,T_j(u_j))}\phi(t)dt \\ &\quad+\beta\int_0^{d(u,u_j)}\phi(t)dt+\gamma\int_0^{\max\{d(u,T_j(u_j)),d(u_j,T_j(u))\}}\phi(t)dt\end{aligned}$$

or,

$$\begin{aligned}\int_0^{d(u,u_j)}\phi(t)dt &\leq \int_0^{d(u,T_j(u))}\phi(t)dt+\alpha\int_0^{d(u,T_j(u))}\phi(t)dt+\beta\int_0^{d(u,u_j)}\phi(t)dt \\ &\quad+\gamma\int_0^{\max\{d(u,u_j),d(u_j,T_j(u))\}}\phi(t)dt.\end{aligned}$$

Case I. Suppose $\max\{d(u,u_j),d(u_j,T_j(u))\}=d(u,u_j)$. Then from above

$$(1-\beta-\gamma)\int_0^{d(u,u_j)}\phi(t)dt \leq (1+\alpha)\int_0^{d(u,T_j(u))}\phi(t)dt.$$

Therefore

$$\int_0^{d(u,u_j)}\phi(t)dt \leq \frac{1+\alpha}{1-\beta-\gamma}\int_0^{d(u,T_j(u))}\phi(t)dt \to 0 \text{ as } j\to\infty$$

and we have $\lim_{j\to\infty} u_j = u$.

Case II. Suppose $\max\{d(u,u_j), d(u_j,T_j(u))\} = d(u_j,T_j(u))$. Then

$$\begin{aligned}\int_0^{d(u,u_j)} \phi(t)dt &\leq (1+\alpha)\int_0^{d(u,T_j(u))} \phi(t)dt + \beta\int_0^{d(u,u_j)} \phi(t)dt \\ &\quad +\gamma\int_0^{d(u_j,T_j(u))} \phi(t)dt\end{aligned}$$

or

$$\int_0^{d(u,u_j)} \phi(t)dt \leq \frac{1+\alpha}{1-\beta}\int_0^{d(u,T_j(u))} \phi(t)dt + \frac{\gamma}{1-\beta}[\int_0^{d(u_j,u)} \phi(t)dt + \int_0^{d(u,T_j(u))} \phi(t)dt]$$

or $(1-\frac{\gamma}{1-\beta})\int_0^{d(u,u_j)} \phi(t)dt \to 0$ as $j\to\infty$, where $\lim_{j\to\infty} T_j(u) = T(u)$ and we have $\lim_{j\to\infty} u_j = u$.

Conversely, let $\lim_{j\to\infty} u_j = u$. We have

$$\begin{aligned}\int_0^{d(u_j,T_j(u))} \phi(t)dt &= \int_0^{d(T_j(u_j),T_j(u))} \phi(t)dt \\ &\leq \alpha\int_0^{d(u_j,T_j(u_j))+d(u,T_j(u))} \phi(t)dt + \beta\int_0^{d(u_j,u)} \phi(t)dt \\ &\quad +\gamma\int_0^{\max\{d(u_j,T_j(u)),d(u,u_j)\}} \phi(t)dt.\end{aligned}$$

As $j\to\infty$, we have

$$\int_0^{d(u,T(u))} \phi(t)dt \leq (\alpha+\gamma)\int_0^{d(u,T(u))} \phi(t)dt.$$

So, $(1-\alpha-\gamma)\int_0^{d(u,T(u))} \phi(t)dt = 0$. Hence $T(u) = u$.

4. Simultaneous fixed points

Theorem 4.1. *Let (X,d) be a complete metric space, and $T_1,T_2 : (X,d) \to (X,d)$ be two commuting operators satisfying*

$$(i)\int_0^{d((T_2T_1)^n(x),(T_2T_1)^n(y))} \phi(t)dt \leq q^n(x,y)\delta(x,y), n = 1,2,...$$

and

$$\begin{aligned}(ii)\int_0^{d(T_1(x),T_2(y))} \phi(t)dt &\leq \alpha\int_0^{d(x,T_1(x))+d(y,T_2(y))} \phi(t)dt + \beta\int_0^{d(x,y)} \phi(t)dt \\ &\quad +\gamma\int_0^{\max\{d(x,T_2(y)),d(y,T_1(x))\}} \phi(t)dt\end{aligned}$$

for all $x,y \in X$ where $\alpha,\beta,\gamma \geq 0$ with $\max\{\alpha,\beta\}+\gamma < 1$, and where ϕ,q,δ have the properties as in Theorem 2.2.

Then T_1 and T_2 have a unique common fixed point in X.

Proof. Let x_0 be any point in X, and define $x_{2n+1} = T_1(x_{2n}), x_{2n+2} = T_2(x_{2n+1})$, for $n \geq 0$. Then $x_{2n} = T_2(x_{2n-1}) = T_2T_1(x_{2n-2}) = ... = (T_2T_1)^n(x_0)$ and $x_{2n+1} = (T_2T_1)^n(T_1(x_0))$. So

$$\int_0^{d(x_{2n},x_{2n+1})}\phi(t)dt = \int_0^{d((T_2T_1)^n(x_0),(T_2T_1)^n(T_1(x_0)))}\phi(t)dt \leq q^n(x_0,T_1(x_0))\delta(x_0,T_1(x_0)) \quad ...(4.1)$$

and

$$\begin{aligned}\int_0^{d(x_{2n+1},x_{2n+2})}\phi(t)dt &= \int_0^{d(T_1(x_{2n}),T_2(x_{2n+1}))}\phi(t)dt \\ &\leq \alpha\int_0^{d(x_{2n},T_1(x_{2n}))+d(x_{2n+1},T_2(x_{2n+1}))}\phi(t)dt + \beta\int_0^{d(x_{2n},x_{2n+1})}\phi(t)dt \\ &\quad +\gamma\int_0^{\max\{d(x_{2n},T_2(x_{2n+1})),d(x_{2n+1},T_1(x_{2n}))\}}\phi(t)dt \\ &= \alpha\int_0^{d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})}\phi(t)dt + \beta\int_0^{d(x_{2n},x_{2n+1})}\phi(t)dt \\ &\quad +\gamma\int_0^{\max\{d(x_{2n},x_{2n+2}),d(x_{2n+1},x_{2n+1})\}}\phi(t)dt \\ &\leq (\alpha+\beta)\int_0^{d(x_{2n},x_{2n+1})}\phi(t)dt + \alpha\int_0^{d(x_{2n+1},x_{2n+2})}\phi(t)dt \\ &\quad +\gamma\int_0^{d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})}\phi(t)dt.\end{aligned}$$

This gives

$$\begin{aligned}\int_0^{d(x_{2n+1},x_{2n+2})}\phi(t)dt &\leq (\frac{\alpha+\beta+\gamma}{1-\alpha-\gamma})\int_0^{d(x_{2n},x_{2n+1})}\phi(t)dt \\ &\leq (\frac{\alpha+\beta+\gamma}{1-\alpha-\gamma})q^n(x_0,T_1(x_0))\delta(x_0,T_1(x_0)). \quad (4.2)\end{aligned}$$

From (4.1) and (4.2) we have for $n \geq 0$

$$\int_0^{d(x_{n+1},x_n)}\phi(t)dt \leq \max(1,\frac{\alpha+\beta+\gamma}{1-\alpha-\gamma})q^n(x_0,T_1(x_0))\delta(x_0,T_1(x_0)). \quad (4.3)$$

Since $\lim_{n\to\infty} q^n(x_0,T_1(x_0)) = 0$, we have $\int_0^{d(x_{n+1},x_n)}\phi(t)dt \to 0$ as $n\to\infty$ and $\lim_{n\to\infty} d(x_{n+1},x_n) = 0$. Suppose m,n are positive integers with $n > m$. Then taking $l = \max(1,\frac{\alpha+\beta+\gamma}{1-\alpha-\gamma})\delta(x_0,T_1(x_0))$, where $lq^n(x_0, T_1(x_0)) < 1$ for large n. Then we have from (4.3),

$$\begin{aligned}\int_0^{d(x_m,x_n)} \phi(t)dt &\leq \int_0^{d(x_m,x_{m+1})} \phi(t)dt + \int_0^{d(x_{m+1},x_{m+2})} \phi(t)dt + \ldots \\ &+ \int_0^{d(x_{n-1},x_n)} \phi(t)dt \\ &\leq l(q^m(x_0,T_1(x_0)) + q^{m+1}(x_0,T_1(x_0)) + \ldots..) \\ &= l\frac{q^m(x_0,T_1(x_0))}{1-q(x_0,T_1(x_0))},\end{aligned}$$

which $\to 0$ as $m \to \infty$. Thus $\{x_n\}$ is shown to be Cauchy in X; and let $\lim_{n\to\infty} x_n = u \in X$.

Now

$$\begin{aligned}\int_0^{d(T_1(u),x_{2n+2})} \phi(t)dt &= \int_0^{d(T_1(u),T_2(x_{2n+1}))} \phi(t)dt \\ &\leq \alpha \int_0^{d(u,T_1(u))+d(x_{2n+1},T_2(x_{2n+1}))} \phi(t)dt + \beta \int_0^{d(u,x_{2n+1})} \phi(t)dt \\ &\quad +\gamma \int_0^{\max\{d(u,T_2(x_{2n+1})),d(x_{2n+1},T_1(u))\}} \phi(t)dt \\ &= \alpha \int_0^{d(u,T_1(u))+d(x_{2n+1},x_{2n+2})} \phi(t)dt + \beta \int_0^{d(u,x_{2n+1})} \phi(t)dt \\ &\quad +\gamma \int_0^{\max\{d(u,x_{2n+2}),d(x_{2n+1},T_1(u))\}} \phi(t)dt.\end{aligned}$$

Passing on limit as $n \to \infty$ we get

$$\int_0^{d(u,T_1(u))} \phi(t)dt \leq (\alpha+\gamma) \int_0^{d(u,T_1(u))} \phi(t)dt.$$

This gives $d(u, T_1(u)) = 0$; so u is a fixed point of T_1. Similarly, it is shown that u is a fixed point of T_2. For uniqueness of u, let $w \in X$ such that $T_1(w) = w = T_2(w)$. Then

$$\begin{aligned}\int_0^{d(u,w)} \phi(t)dt &= \int_0^{d(T_1(u),T_2(w))} \phi(t)dt \\ &\leq \alpha \int_0^{d(u,T_1(u))+d(w,T_2(w))} \phi(t)dt + \beta \int_0^{d(u,w)} \phi(t)dt \\ &\quad +\gamma \int_0^{\max\{d(u,T_2(w)),d(w,T_1(u))\}} \phi(t)dt \\ &= (\beta+\gamma) \int_0^{d(u,w)} \phi(t)dt.\end{aligned}$$

Since $\max\{\alpha,\beta\} + \gamma < 1$, we have $\int_0^{d(u,w)} \phi(t)dt = 0$ which implies $u = w$ and so the fixed point is unique.

Conflict of Interests

The authors declare that there is no conflict of interests.

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