GENERALIZED HERMITE-HADAMARD INEQUALITY FOR LIPSCHITZ FUNCTIONS

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Abstract. In this paper, we establish some Hermite-Hadamard type inequalities for Lipschitz functions defined on invex subsets of real line.

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1. Introduction

Let \( I = [c,d] \) be an interval on the real line \( \mathbb{R} \), \( f : I \to \mathbb{R} \) be a convex function and \( a, b \in [c,d], a < b \). We consider the well-known Hermite-Hadamard inequality

\[
\frac{f(a+b)}{2} \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]

Several refinements and generalizations of Hermite-Hadamard have been found in [1-5, 8-12, 16] and references therein. In recent years several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of preinvex functions introduced by Ben-Israel and Mond in [7] (see [6, 14] for more property and generalizations).

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Now, we recall some notions in invexity analysis which will be used throughout the paper (see [2, 13, 14, 17] and references therein). A set \( S \subseteq \mathbb{R} \) is said to be invex with respect to the map \( \eta : S \times S \rightarrow S \), if for every \( x, y \in S \) and every \( t \in [0, 1] \), \( y + t\eta(x, y) \in S \). Recall that an \( \eta \)-path for \( x, y \in S \) is a subset of \( S \) defined by

\[
P_{xy} := \{x + t\eta(y, x) \mid 0 \leq t \leq 1\}.
\]

It is obvious that every convex set is invex with respect to the map \( \eta(x, y) = x - y \), but there exist invex sets which are not convex. The mapping \( \eta : S \times S \rightarrow S \) is said to satisfies the condition \( C \) if for every \( x, y \in S \) and \( t \in [0, 1] \),

\[
\eta(y, y + t\eta(x, y)) = -t\eta(x, y),
\]

\[
\eta(x, y + t\eta(x, y)) = (1 - t)\eta(x, y).
\]

For every \( x, y \in S \) and every \( t_1, t_2 \in [0, 1] \) from condition \( C \) we have

\[
\eta(y + t_2\eta(x, y), y + t_1\eta(x, y)) = (t_2 - t_1)\eta(x, y).
\]  \( \tag{1.2} \)

Let \( S \subseteq \mathbb{R} \) be an invex set with respect to \( \eta : S \times S \rightarrow S \). Then, the function \( f : S \rightarrow \mathbb{R} \) is said to be preinvex with respect to \( \eta \), if for every \( x, y \in S \) and \( t \in [0, 1] \),

\[
f(y + t\eta(x, y)) \leq tf(x) + (1 - t)f(y).
\]

Every convex function is a preinvex with respect to the map \( \eta(x, y) = x - y \) but the converse does not holds. The following Hermite-Hadamard inequality for preinvex functions is introduced in [15],

\[
f(a + \frac{1}{2}\eta(b, a)) \leq \frac{1}{\eta(b, a)} \int_{a}^{a + \eta(b, a)} f(x) \, dx \leq \frac{f(a) + f(b)}{2}, \]  \( \tag{1.3} \)

where \( a, b \in S \), (see also [5]).

On the other hand, Dragomir in [9] defined two mapping \( H, F : [0, 1] \rightarrow \mathbb{R} \), as follows and established several important results in connection to Hermite-Hadamard inequality.

\[
H(t) := \frac{1}{b - a} \int_{a}^{b} f(tx + (1 - t)\frac{a + b}{2}) \, dx,
\]

\[
F(t) := \frac{1}{(b - a)^2} \int_{a}^{b} \int_{a}^{b} f(tx + (1 - t)y) \, dx \, dy. \]  \( \tag{1.4} \)
Since then numerous articles have appeared in the literature reflecting further applications and properties of mappings $H, F$, (see [3, 8, 10, 11, 16]) and references therein. In [10] Dragomir by relaxing convexity and utilizing two above mapping, introduced some Hermite-Hadamard inequalities for Lipschitz functions defined on intervals as follows;

**Theorem 1.1.** Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a $M$–Lipschitz function and $a, b \in I$ with $a < b$. Then, we have the following inequalities

$$
\left| f\left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{M}{4} (b-a),
$$

(1.5)

and

$$
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{M}{3} (b-a).
$$

(1.6)

**Theorem 1.2.** Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a $M$–Lipschitz function and $a, b \in I$ with $a < b$. Then,

(i) The mapping $H$ is $\frac{M}{4} (b-a)$–Lipschitz on $[0, 1]$.

(ii) For every $t \in [0, 1]$ we have the following inequalities

$$
\left| H(t) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{M(1-t)}{4} (b-a),
$$

(1.7)

$$
\left| f\left( \frac{a+b}{2} \right) - H(t) \right| \leq \frac{Mt}{4} (b-a),
$$

(1.8)

and

$$
\left| H(t) - t \frac{1}{b-a} \int_a^b f(x)dx - (1-t)f\left( \frac{a+b}{2} \right) \right| \leq \frac{t(1-t)M}{2} (b-a).
$$

(1.9)

**Theorem 1.3.** Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a $M$–Lipschitz function and $a, b \in I$ with $a < b$. Then,

(i) $F(t) = F(1-t)$, for all $t \in [0, 1]$

(ii) The mapping $F$ is a $\frac{M(b-a)}{3}$–Lipsschitz function on $[0, 1]$.

(iii) We have the following inequalities

$$
\left| F(t) - \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left( \frac{x+y}{2} \right) dxdy \right| \leq \frac{M|2t-1|}{6} (b-a),
$$

(1.10)

$$
\left| F(t) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{Mt}{3} (b-a),
$$

(1.11)
The main purpose of this paper is to establishing some new inequalities involving generalizations of two above mappings for Lipschitz functions on invex subsets of \( \mathbb{R} \).

2. Main results

At first we start with the following theorem connecting two inequalities of Hermite-Hadamard type for Lipschitz functions defined on invex sets.

**Theorem 2.1.** Let \( S \subseteq \mathbb{R} \) be an invex set with respect to \( \eta : S \times S \to S \). Suppose that \( \eta \) satisfies condition C. Assume that \( f : S \to \mathbb{R} \) is a \( M \)-Lipschitz function and \( a, b \in S \) with \( \eta(a, b) \neq 0 \).

Then,

(i) \[
\left| f(a + \frac{1}{2} \eta(a, b)) - \frac{1}{\eta(a, b)} \int_b^c f(x) dx \right| \leq \frac{1}{4} M |\eta(a, b)|,
\]

where \( c := b + \eta(a, b) \).

(ii) If \( a > b \) and \( \eta(a, b) \geq a - b \), then,

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{\eta(a, b)} \int_b^c f(x) dx \right| \leq \frac{M(a - b)^3}{3 \eta(a, b)^2} + \frac{M \eta(a, b)}{2} + \frac{M(b - a)}{2},
\]

where \( c := b + \eta(a, b) \).

**Proof.** (i) Let \( a, b \in S \) and \( t \in [0, 1] \). Then,

\[
\left| tf(a) + (1-t)f(b) - f(b + t \eta(a, b)) \right|
= \left| t(f(a) - f(b + t \eta(a, b))) + (1-t)(f(b) - f(b + t \eta(a, b))) \right|
\leq t \left| f(a) - f(b + t \eta(a, b)) \right| + (1-t) \left| f(b) - f(b + t \eta(a, b)) \right|
\leq tM |a - b - t \eta(a, b)| + (1-t)M |\eta(a, b)|.
\]
For $t = \frac{1}{2}$, we get
\[
\left| \frac{f(a) + f(b)}{2} - f(b + \frac{1}{2} \eta(a, b)) \right| \leq \frac{1}{2} M \left| a - b - \frac{1}{2} \eta(a, b) \right| + \frac{1}{2} M |\eta(a, b)|. \tag{2.4}
\]

If in (2.4) we put $b + t \eta(a, b)$ and $b + (1 - t) \eta(a, b)$ instate of $a$ and $b$, respectively then we obtain
\[
\left| \frac{f(b + t \eta(a, b)) + f(b + (1 - t) \eta(a, b))}{2} - f(b + \frac{1}{2} \eta(a, b)) \right| \leq \frac{1}{2} M |2t - 1||\eta(a, b)|. \tag{2.5}
\]

Integrating on $[0, 1]$ implies that
\[
\left| \int_{0}^{1} f(b + t \eta(a, b)) dt + \int_{0}^{1} f(b + (1 - t) \eta(a, b)) dt - f(b + \frac{1}{2} \eta(a, b)) \right| \leq \frac{1}{4} M |\eta(a, b)|. \tag{2.6}
\]

(ii) For every $t \in [0, 1]$, from (2.3) we have
\[
\left| tf(a) + (1-t)f(b) - f(b + t \eta(a, b)) \right| \leq t M |a - b - t \eta(a, b)| + t(1-t)M \eta(a, b). \tag{2.7}
\]

By integrating on $[0, 1]$ we get
\[
\left| f(a) \int_{0}^{1} t dt + f(b) \int_{0}^{1} (1-t) dt - \int_{0}^{1} f(a + t \eta(a, b)) dt \right| \leq \left| \frac{f(a) + f(b)}{2} - \frac{1}{\eta(a, b)} \int_{a}^{c} f(x) dx \right| \leq \left[ \int_{0}^{1} t M |a - b - t \eta(a, b)| dt \right] + \int_{0}^{1} t(1-t)M \eta(a, b) dt \tag{2.8}
\]
\[
= \left[ \frac{M(a - b)^3}{3\eta(a, b)^2} + \frac{M \eta(a, b)}{3} + \frac{M(b - a)}{2} \right] + \frac{M \eta(a, b)}{6} \leq \frac{M(a - b)^3}{3\eta(a, b)^2} + \frac{M \eta(a, b)}{2} + \frac{M(b - a)}{2}.
\]
Note that, by simple computation we have
\[
\int_0^1 t|a-b-t\eta(a,b)|\,dt = \int_0^{\lambda} t(a-b-t\eta(a,b))\,dt + \int_{\lambda}^1 t(b-a+t\eta(a,b))\,dt = \frac{(a-b)^3}{3\eta(a,b)^2} + \frac{\eta(a,b)}{3} + \frac{b-a}{2},
\]
where \(\lambda := \frac{a-b}{\eta(a,b)}\).

**Remark 2.1.** If in Theorem 2.1, \(\eta(x,y) = x-y\), for every \(x,y \in S\), then we have the results in Theorem 1.1, as a special case.

Motivated by [9] for a real valued function \(f\) defined on an invex set \(S \subseteq \mathbb{R}\) with respect to \(\eta: S \times S \to S\), we consider two mappings \(H, F: [0,1] \to \mathbb{R}\), as follows;

\[
H(t) := \frac{1}{\eta(b,a)} \int_a^c f(a + \frac{1}{2}\eta(b,a) + t\eta(y,a + \frac{1}{2}\eta(b,a)))\,dy,
\]
and

\[
F(t) := \frac{1}{\eta(b,a)^2} \int_a^c \int_a^c f(x + t\eta(y,x))\,dxdy,
\]
where \(a,b \in S\) and \(c := a + \eta(b,a)\). Note that in the special case, if \(\eta(x,y) = x-y\) for every \(x,y \in S\) then, the mappings \(H\) and \(F\) reduce to mappings \(H\) and \(F\) defined in (1.4), respectively. The following theorem is a generalization of theorem 1.2 in invexity setting.

**Theorem 2.2.** Let \(S \subseteq \mathbb{R}\) be an invex set with respect to \(\eta: S \times S \to S\). Suppose that \(\eta\) satisfies condition \(C\) and for every \(x \neq y \in S\), \(\eta(y,x) \neq 0\). Assume that \(f: S \to \mathbb{R}\) is a \(M\)–Lipschitz function. Then, for every \(a,b \in S\) one has

(i) The mapping \(H\) is \(\frac{M}{4}||\eta(b,a)||\)–Lipschitz on \([0,1]\).

(ii) For every \(t \in [0,1]\) we have the following inequalities

\[
\left|H(t) - \frac{1}{\eta(b,a)} \int_a^c f(x)\,dx\right| \leq \frac{M(1-t)}{4} \left|\eta(b,a)\right|, \quad (2.9)
\]

\[
\left|f(a + \frac{1}{2}\eta(b,a)) - H(t)\right| \leq \frac{Mt}{4} \left|\eta(b,a)\right|, \quad (2.10)
\]
and

\[
\left| H(t) - t \frac{1}{\eta(b,a)} \int_a^c f(x) dx - (1-t)f(a + \frac{1}{2} \eta(b,a)) \right| \\
\leq \frac{t(1-t)M}{2} |\eta(b,a)|,
\]

where \( c := a + \eta(b,a) \).

**Proof.** Let \( t_1, t_2 \in [0,1] \). Then,

\[
\left| H(t_2) - H(t_1) \right| \\
= \frac{1}{|\eta(b,a)|} \left| \int_a^c f(a + \frac{1}{2} \eta(b,a) + t_2 \eta(x,a + \frac{1}{2} \eta(b,a))) - f(a + \frac{1}{2} \eta(b,a) + t_1 \eta(x,a + \frac{1}{2} \eta(b,a)) \right| dx \\
\leq \frac{M}{|\eta(b,a)|} \left| \int_a^c t_2 \eta(x,a + \frac{1}{2} \eta(b,a)) - t_1 \eta(x,a + \frac{1}{2} \eta(b,a)) \right| dx \\
= \frac{M|t_2 - t_1|}{|\eta(b,a)|} \left| \int_a^c \eta(x,a + \frac{1}{2} \eta(b,a)) \right| dx \\
= \frac{M|\eta(b,a)|}{4} |t_2 - t_1|.
\]

Indeed, if we choose the change of variable \( x := a + s \eta(b,a), s \in [0,1] \), and using (1.2) then we have

\[
\int_a^c |\eta(x,a + \frac{1}{2} \eta(b,a))| dx \\
\int_0^1 |\eta(a + s \eta(b,a), a + \frac{1}{2} \eta(b,a))| |\eta(b,a)| ds \\
= |\eta(b,a)|^2 \int_0^1 |s - \frac{1}{2}| ds = \frac{1}{4} |\eta(b,a)|^2,
\]

this completes the proof of (i).

(ii) It is easy to see that

\[ H(0) = f(a + \frac{1}{2} \eta(b,a)) \],

and

\[ H(1) = \frac{1}{\eta(b,a)} \int_a^c f(x) dx. \]
Now, the inequalities (2.9) and (2.10) follow from (2.12) by choosing \( t_1 = t, t_2 = 1 \) and \( t_1 = 0, t_2 = t \), respectively. Inequality (2.11) follow by adding \( t \) times (2.9) and \( (1 - t) \) times (2.10).

This completes the proof.

**Theorem 2.2.** Let \( S \subseteq \mathbb{R} \) be an invex set with respect to \( \eta : S \times S \to \mathbb{R} \). Suppose that for every \( x \neq y \in S \), \( \eta(y, x) \neq 0 \). If \( f : S \to \mathbb{R} \) is a \( M \)-Lipschitz function then; for every \( a, b \in S \),

(i) \( F(t) = F(1 - t), \) for all \( t \in [0, 1] \)

(ii) If \( \eta \) satisfies condition C then, \( F \) is a \( \frac{M|\eta(b,a)|}{3} \)-Lipschitz function on \([0, 1]\).

(iii) If \( \eta \) satisfies condition C then, for every \( t \in [0, 1] \) we have the following inequalities

\[
\left| F(t) - \frac{1}{\eta(b,a)^2} \int_a^c \int_a^c f(x + \frac{1}{2} \eta(y,x)) \, dx \, dy \right| \leq \frac{M|2t - 1|}{6} |\eta(b,a)|, \tag{2.14}
\]

\[
\left| F(t) - \frac{1}{\eta(b,a)} \int_a^c f(x) \, dx \right| \leq \frac{Mt}{3} |\eta(b,a)|, \tag{2.15}
\]

and

\[
|F(t) - H(t)| \leq \frac{M(1 - t)}{4} |\eta(b,a)|. \tag{2.16}
\]

**Proof.** (i) It is obvious.

(ii) Let \( t_1, t_2 \in [0, 1] \). Then,

\[
|F(t_2) - F(t_1)|
= \frac{1}{\eta(b,a)^2} \left| \int_a^c \int_a^c \left[ f(x + t_2 \eta(y,x)) - f(x + t_1 \eta(y,x)) \right] \, dx \, dy \right|
\leq \frac{1}{\eta(b,a)^2} \int_a^c \int_a^c \left| f(x + t_2 \eta(y,x)) - f(x + t_1 \eta(y,x)) \right| \, dx \, dy
\leq \frac{M|t_2 - t_1|}{\eta(b,a)^2} \int_a^c \int_a^c |\eta(y,x)| \, dx \, dy.
\tag{2.17}
\]

On the other hand, if we use the change of variables \( x := a + r \eta(b,a), y := a + s \eta(b,a), r, s \in [0, 1] \) then by simple computation we get

\[
\partial(x, y) = \begin{pmatrix}
\eta(y,x) & 0 \\
0 & \eta(y,x)
\end{pmatrix},
\]
hence \( \det \frac{\partial (x,y)}{\partial (r,s)} = \eta(b,a)^2 \). Now, by using (1.2) we obtain

\[
\frac{1}{\eta(b,a)^2} \int_a^c \int_a^c |\eta(y,x)| dx dy = \frac{1}{\eta(b,a)^2} \int_0^1 \int_0^1 |\eta(a + s\eta(b,a), a + r\eta(b,a))| \eta(b,a)^2 dr ds \tag{2.18}
\]

\[
= |\eta(b,a)| \int_0^1 \int_0^1 |s - r| dr ds = \frac{|\eta(b,a)|}{3}.
\]

By combining (2.17) and (2.18) it follows that

\[
|F(t_2) - F(t_1)| \leq M |\eta(b,a)| |t_2 - t_1|. \tag{2.19}
\]

(iii) The inequalities (2.14) and (2.15) follows from (2.19) if we choose \( t_1 = \frac{1}{2}, t_2 = t \) and \( t_1 = 0, t_2 = t \), respectively.

Now, we prove the inequality (2.16). Since \( f \) is \( M \)-Lipschitz if we set

\[
y := a + s\eta(b,a), \quad x := a + r\eta(b,a), \quad r, s \in [0,1]
\]

then, by using (1.2) we have

\[
\left| f(y + t\eta(x,y)) - f\left(a + \frac{1}{2}\eta(b,a) + t\eta(x,a + \frac{1}{2}\eta(b,a))\right) \right| \\
\leq M \left| y + t\eta(x,y) - a - \frac{1}{2}\eta(b,a) - t\eta(x,a + \frac{1}{2}\eta(b,a)) \right| \\
= M \left| s\eta(b,a) + t(r-s)\eta(b,a) - \frac{1}{2}\eta(b,a)) - t(r - \frac{1}{2})\eta(b,a)) \right| \\
= M \left| s\eta(b,a) - ts\eta(b,a) - \frac{1}{2}\eta(b,a)) + \frac{1}{2}t\eta(b,a)) \right| \\
= M \left| (1-t)s\eta(b,a) - (1-t)\frac{1}{2}\eta(b,a)) \right| \\
= (1-t)M \left| y - a - \frac{1}{2}\eta(b,a) \right| , \text{ for all } t \in [0,1]. \tag{2.20}
\]
By integrating the inequality (2.20) on $P_{ab} \times P_{ab}$ we have
\[
\left| \frac{1}{\eta(b,a)^2} \int_a^c \int_a^c f\left(x + \frac{1}{2} \eta(y,x)\right) dxdy - \frac{1}{\eta(b,a)} \int_a^c f\left(a + \frac{1}{2} \eta(b,a) + t \eta(x,a + \frac{1}{2} \eta(b,a))\right) dx \right| \\
\leq (1-t)M \frac{1}{\eta(b,a)} \int_a^c \left| y - a - \frac{1}{2} \eta(b,a) \right| dy \\
= \frac{(1-t)M}{4} |\eta(b,a)|.
\]

This completes the proof.

**Corollary 2.2.** If in Theorem 2.2, $\eta(x,y) = x - y$, for every $x, y \in S$, then we have the results in Theorem 1.3, as a special case.

**Conflict of Interests**

The author declares that there is no conflict of interests.

**References**


