JENSEN’S INEQUALITY FOR HH-CONVEX FUNCTIONS AND RELATED RESULTS

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Abstract. In this paper we obtain Jensen’s inequality for HH-convex functions. Also we get inequalities alike to Hermite-Hadamard inequality for HH-convex functions. Some examples are given.

Keywords: Jensen’s inequality; HH-convex; Integral inequality.

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1. INTRODUCTION

Let \( \mu \) be a positive measure on \( X \) such that \( \mu(X) = 1 \). If \( f \) is a real-valued function in \( L^1(\mu) \), \( a < f(x) < b \) for all \( x \in X \) and \( \phi \) is convex on \((a,b)\), then

\[
\phi\left(\int_X f \, d\mu\right) \leq \int_X (\phi \cdot f) \, d\mu
\]

The inequality (1) is known as Jensen’s inequality [3],[4].

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**Definition 1.1.** A function \( \varphi : (a, b) \rightarrow (0, \infty) \), where \( 0 < a < b \leq \infty \), is called HH-convex (according to the harmonic mean) if the inequality

\[
\varphi \left( \frac{x}{\lambda} + \frac{1}{1-\lambda} y \right) \leq \frac{1}{\lambda \varphi(x) + (1-\lambda) \varphi(y)}
\]

or

\[
\varphi \left( \frac{xy}{\lambda y + (1-\lambda)x} \right) \leq \frac{\varphi(x)\varphi(y)}{\lambda \varphi(y) + (1-\lambda)\varphi(x)}
\]

holds, where \( a < x < b, a < y < b \), and \( 0 \leq \lambda \leq 1 \).

In this paper we prove Jensen’s inequality and alike to Hermite-Hadamard inequality for HH-convex functions. First we need the following theorem.

**Theorem 1.2.** A function \( \varphi \) is HH-convex on \((a, b)\) if for \( 0 < a < s < t < u < b \) the following inequality holds

\[
\frac{1}{s} - \frac{1}{t} \leq \frac{1}{\frac{s}{t} - \frac{1}{u}}.
\]

**Proof.** Let \( \varphi \) be HH-convex and \( \lambda = \frac{s(u-t)}{t(u-s)} \), then

\[
t = \frac{1}{\lambda s + \frac{1-\lambda}{u}}.
\]

Hence

\[
\varphi(t) \leq \frac{1}{\frac{s}{t} \varphi(s) + \frac{u}{t} \varphi(u)}
\]

It follows that

\[
\frac{1}{s} - \frac{1}{t} \leq \frac{1}{\frac{s}{t} \varphi(s) + \frac{u}{t} \varphi(u)} \leq \frac{1}{s(u-t)} \frac{1}{\varphi(s)} + \frac{u(t-s)}{t(u-s)} \frac{1}{\varphi(u)}
\]

\[
\Rightarrow \frac{s(u-t)}{t(u-s)} \left( \frac{1}{\varphi(s)} - \frac{1}{\varphi(t)} \right) \leq \frac{u(t-s)}{t(u-s)} \left( \frac{1}{\varphi(t)} - \frac{1}{\varphi(u)} \right)
\]

since \( 0 < s < t < u \), we obtain

\[
\frac{1}{s} - \frac{1}{t} \leq \frac{1}{\frac{s}{t} - \frac{1}{u}}.
\]
Conversely let the inequality (3) holds, and \( \lambda \in [0, 1], 0 < a < x < y < b, \) then \( x \leq \frac{1}{\lambda} \frac{1}{x} + \frac{1}{y} \leq y. \)

By inequality (3) we have

\[
\frac{1}{\phi(x)} - \frac{1}{\phi(\frac{xy}{\lambda y + (1 - \lambda)x})} \leq \frac{1}{\lambda y + (1 - \lambda)x} - \frac{1}{\phi(y)}
\]

\[
\Rightarrow \frac{1}{\phi(x)} - \frac{1}{\phi(\frac{xy}{\lambda y + (1 - \lambda)x})} \leq \frac{1}{\lambda y + (1 - \lambda)x} - \frac{1}{\phi(y)}
\]

\[
\Rightarrow \frac{\lambda}{\phi(x)} - \frac{\lambda}{\phi(\frac{xy}{\lambda y + (1 - \lambda)x})} \leq \frac{1 - \lambda}{\phi(y)} - \frac{1}{\phi(y)}
\]

\[
\Rightarrow \frac{1}{\phi(\frac{xy}{\lambda y + (1 - \lambda)x})} \geq \frac{1 - \lambda}{\phi(y)} + \frac{\lambda}{\phi(x)}
\]

\[
\Rightarrow \phi(\frac{xy}{\lambda y + (1 - \lambda)x}) \leq \frac{1}{\frac{1}{\phi(x)} + \frac{1}{\phi(y)}}
\]

Thus \( \phi \) is HH-convex. \( \square \)

By similar way to the convex functions we can prove if \( \phi \) is HH-convex on \((a, b)\), then \( \phi \) is continuous on \((a, b)\).

2. Main Results

**Theorem 2.1.** Let \( \mu \) be a positive measure on a \( \sigma \)-algebra \( \mathfrak{m} \) in a set \( X \), so that \( \mu(X) = 1 \). If \( f \) is a real function in \( L^1(\mu) \), \( 0 < a < f(x) < b \) for all \( x \in X \), and if \( \phi \) is HH-convex on \((a, b)\), then

\[
\phi \left( \frac{1}{\int_X \frac{d\mu}{f}} \right) \leq \frac{1}{\int_X \frac{d\mu}{\phi \circ f}}
\]

**Proof.** Put \( t = \frac{1}{\int_X \frac{d\mu}{f}} \). Then \( a < t < b \). Let

\[
M = \sup_{a < x < t} \frac{1}{\phi(x)} - \frac{1}{\phi(t)}
\]

\[
\frac{1}{s} - \frac{1}{t}
\]

\[
M \leq \frac{1}{\phi(t)} - \frac{1}{\phi(t)}
\]

Therefore, we have

\[
\phi(\frac{1}{\int_X \frac{d\mu}{f}}) \leq \frac{1}{\int_X \frac{d\mu}{\phi \circ f}}
\]
Then $M$ is no larger than any of the quotients on the right side of (3), for any $u \in (t, b)$. It follows that

$$
\frac{1}{\phi(s)} - \frac{1}{\phi(t)} \leq M \quad \text{or} \quad \frac{1}{\phi(s)} - \frac{1}{\phi(t)} \leq M \left( \frac{1}{s} - \frac{1}{t} \right)
$$

Hence, for any $x \in X$, we have

$$
\frac{1}{\phi(f(x))} - \frac{1}{\phi(t)} \leq M \left( \frac{1}{f(x)} - \frac{1}{t} \right)
$$

since $\phi$ is continuous, $\phi \circ f$ is measureable, and since $f \in L^1(\mu)$, $f(x) > a > 0$, so $\frac{1}{f} \in L^1(\mu)$. By integrating both sides with respect to measure $\mu$, we obtain

$$
\int_X \frac{d\mu}{\phi \circ f} - \frac{1}{\phi(t)} \leq M \left( \int_X \frac{d\mu}{f} - \frac{1}{t} \right) \quad (\mu(X) = 1)
$$

Now set $t = \frac{1}{\int_X \frac{d\mu}{f}}$. It follows that

$$
\int_X \frac{d\mu}{\phi \circ f} - \frac{1}{\phi \left( \frac{1}{\int_X \frac{d\mu}{f}} \right)} \leq 0
$$

or

$$
\phi \left( \frac{1}{\int_X \frac{d\mu}{f}} \right) \geq \frac{1}{\int_X \frac{d\mu}{\phi \circ f}}
$$

\[\square\]

Corollary 2.2. Let $f : [a, b] \rightarrow (0, \infty)$ ($b > a > 0$) be a continuous function and $\phi : J \rightarrow (0, \infty)$ be a HH-convex function on an interval $J$ which includes the image of $f$. Then

$$
\phi \left( \frac{ab}{b-a} \int_a^b \frac{dx}{x^2(f(x))} \right) \leq \frac{1}{ab} \int_a^b \frac{dx}{x^2(\phi \circ f)(x)}
$$

Proof. In theorem 2.1, put $X = [a, b]$ and $d\mu = \frac{dx}{x^2}$. \[\square\]

In the following theorem we prove a version for the inverse of Corollary 2.2

Theorem 2.3. Let $\phi : (0, \infty) \rightarrow (0, \infty)$ be a function such that the inequality (5) holds, for every positive real bounded measurable function $f$. Then $\phi$ is HH-convex.
Proof. Let $\lambda \in [0, 1]$, $c, d \in (0, \infty)$. Define

$$f(x) = \begin{cases} 
  c & a \leq x < \frac{ab}{\lambda a+(1-\lambda)b} \\
  d & \frac{ab}{\lambda a+(1-\lambda)b} \leq x \leq b 
\end{cases}$$

we have

$$\frac{ab}{b-a} \int_a^b \frac{dx}{x^2 f(x)} = \frac{ab}{b-a} \left[ \frac{\lambda a+(1-\lambda)b}{cx^2} + \frac{\lambda a+(1-\lambda)b}{dx^2} \right]$$

$$= \frac{ab}{b-a} \left[ \frac{1}{c} \left( \frac{\lambda a+(1-\lambda)b}{ab} + \frac{1}{a} \right) + \frac{1}{d} \left( \frac{1}{b} + \frac{\lambda a+(1-\lambda)b}{ab} \right) \right]$$

$$= \frac{\lambda c + 1-\lambda d}{a}$$

Hence

$$(*) \quad \varphi \left( \frac{1}{\frac{ab}{b-a} \int_a^b \frac{dx}{x^2 f(x)}} \right) = \varphi \left( \frac{1}{\frac{\lambda}{c} + \frac{1-\lambda}{d}} \right) = \varphi \left( \frac{cd}{\lambda d + (1-\lambda)c} \right)$$

On the other hand we have

$$\frac{ab}{b-a} \int_a^b \frac{dx}{x^2 \varphi(f(x))} = \frac{ab}{b-a} \left[ \frac{\lambda a+(1-\lambda)b}{x^2 \varphi(c)} + \frac{\lambda a+(1-\lambda)b}{x^2 \varphi(d)} \right] = \frac{\lambda \varphi(c) + 1-\lambda \varphi(d)}{\varphi(c) \varphi(d)}$$

so

$$(**) \quad \frac{1}{\frac{ab}{b-a} \int_a^b \frac{dx}{x^2 \varphi(f(x))}} = \frac{1}{\frac{\lambda}{\varphi(c)} + \frac{1-\lambda}{\varphi(d)}} = \frac{\varphi(c) \varphi(d)}{\lambda \varphi(d) + (1-\lambda) \varphi(c)}$$

Now the (*), (**) and (5) show that $\varphi$ is HH-convex. \qed

**Example 2.4.** Let $X = \{x_1, x_2, \ldots, x_n\}$, $\mu(\{x_i\}) = \frac{1}{n}$ and $f(x_i) = a_i > 0$. Then (4) becomes

$$\varphi \left( \frac{1}{\frac{1}{n} \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right)} \right) \leq \frac{1}{\frac{1}{\varphi(a_1)} + \frac{1}{\varphi(a_2)} + \cdots + \frac{1}{\varphi(a_n)}}$$

or

$$(6) \quad \varphi \left( \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \right) \leq \frac{n}{\frac{1}{\varphi(a_1)} + \frac{1}{\varphi(a_2)} + \cdots + \frac{1}{\varphi(a_n)}}$$

Now we investigate this inequality for $\varphi(x) = x^\gamma$ and $\varphi(x) = e^{\frac{1}{x}}$. 


(i) \( \varphi(x) = x^\gamma \) is HH-convex on \((0, \infty)\) for \(0 \leq \gamma \leq 1\), because \( \frac{x^2 \varphi'(x)}{\varphi^2(x)} = x^{1-\gamma} \) is increasing (see [1]). The inequality (6) implies that
\[
\left( \frac{n}{a_1 + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \right)^\gamma \leq \frac{n}{a_1 + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}
\]
put \( \frac{1}{a_i} = p_i \) \((i = 1, 2, \ldots, n)\). It follows that
\[
\left( \frac{n}{p_1 + p_2 + \cdots + p_n} \right)^\gamma \leq \frac{n}{p_1 + p_2 + \cdots + p_n}
\]
or
\[
\frac{p_1 + p_2 + \cdots + p_n}{n} \geq \left( \frac{p_1^\gamma + p_2^\gamma + \cdots + p_n^\gamma}{n} \right)^{\frac{1}{\gamma}}
\]

The number \( C_\gamma = \left( \frac{p_1^\gamma + p_2^\gamma + \cdots + p_n^\gamma}{n} \right)^{\frac{1}{\gamma}} \) is termed the mean power of numbers \( p_1, p_2, \ldots, p_n \) of order \( \gamma \). Inequality (7) shows that for \(0 \leq \gamma \leq 1\), \( C_\gamma \leq C_1 \).

Now let \(0 \leq \gamma = \frac{\alpha}{\beta} \leq 1\), then (7) becomes,
\[
\frac{p_1 + p_2 + \cdots + p_n}{n} \geq \left( \frac{p_1^\alpha + p_2^\alpha + \cdots + p_n^\alpha}{n} \right)^{\frac{1}{\alpha}}
\]

Put \( p_i^{\frac{1}{\beta}} = q_i \) \((i = 1, 2, \ldots, n)\). It follows that
\[
\left( \frac{q_1^\beta + q_2^\beta + \cdots + q_n^\beta}{n} \right)^{\frac{1}{\beta}} \geq \left( \frac{q_1^\alpha + q_2^\alpha + \cdots + q_n^\alpha}{n} \right)^{\frac{1}{\alpha}}
\]
So if \(0 \leq \alpha \leq \beta\), then \( C_\alpha \leq C_\beta \). By HH-concavity of \( \varphi(x) = x^\gamma \) on \((0, \infty)\) for \( \gamma < 0 \), and \( \gamma > 1 \) and similar way we can prove for \( \alpha < 0 < \beta \) and \( \alpha < \beta < 0 \) we have
\[
C_\alpha < C_\beta
\]
Thus the mean power of order \( \gamma \) monotonically increasing together with \( \gamma \).

(ii) \( \varphi(x) = e^{\frac{x}{x}} \) is HH-concave on \((0, \infty)\), because \( \frac{x^2 \varphi'(x)}{\varphi^2(x)} = -e^{-\frac{1}{x}} \) is decreasing (see [1]). The inequality (6) follows that
\[
e^{\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \cdots + \frac{1}{\alpha_n}} \geq \frac{n}{e^{\frac{1}{\alpha_1}} + \frac{1}{e^{\frac{1}{\alpha_2}}} + \cdots + \frac{1}{e^{\frac{1}{\alpha_n}}}}
\]
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put $\frac{1}{e^n} = p_i$, $i = 1, 2, \ldots, n$. Hence

$$\sqrt[n]{\frac{1}{p_1 p_2 \cdots p_n}} \geq \frac{n}{p_1 + p_2 + \cdots + p_n}$$

so

$$\sqrt[n]{p_1 p_2 \cdots p_n} \leq \frac{p_1 + p_2 + \cdots + p_n}{n}$$

That is, the geometric mean of positive numbers is not greater than the arithmetic mean of the same numbers.

In the following theorem we obtain inequalities alike to Hermite-Hadamard inequality for HH-convex functions.

**Theorem 2.5.** Let $f : [a, b] \rightarrow (0, \infty)$ $(b > a > 0)$ be a HH-convex function. Then the following inequalities hold:

(i) $f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{dx}{\sqrt{x f(x)}} \leq \frac{2f(a)f(b)}{f(a)+f(b)}$

(ii) $f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b f(x)f\left(\frac{ax+bx-ab}{x+b}\right) \frac{dx}{x^2} \leq \frac{2f(a)f(b)}{f(a)+f(b)}$

**Proof.** (i) The inequality (5) follows that

$$\frac{1}{b-a} \int_a^b \frac{dx}{\sqrt{x f(x)}} \geq f\left(\frac{1}{b-a} \int_a^b \frac{dx}{x^2 f(x)}\right) = f\left(\frac{2ab}{a+b}\right)$$

on the other hand by change of variable $x = \frac{ab}{ta+(1-t)b} = \frac{ab}{t(a-b)+b}$, $dx = \frac{ab(b-a)}{(t(a-b)+b)^2} dt$ and HH-convexity of $f$ we get

$$\frac{ab}{b-a} \int_a^b \frac{dx}{x^2 f(x)} = \int_0^1 \frac{dt}{f\left(\frac{ab}{ta+(1-t)b}\right)} = \int_0^1 \frac{dt}{f\left(\frac{ta+1-tb}{b}\right)} \geq \int_0^1 \frac{dt}{f(b) \left(\frac{1}{b+a}\right)}$$

so

$$\frac{1}{b-a} \int_a^b \frac{dx}{x^2 f(x)} \leq \frac{2f(a)f(b)}{f(a)+f(b)}$$
(ii) Since \( f \) is HH-convex, we have
\[
\begin{align*}
  f \left( \frac{2ab}{a+b} \right) &= f \left( \frac{2}{1 + \frac{b}{a}} \right) = f \left( \frac{2}{\left( \frac{1}{a} + \frac{1}{b} \right) + \left( \frac{1}{b} + \frac{1}{a} \right)} \right) \\
  &\leq \frac{2f \left( \frac{ab}{tb+(1-t)a} \right)f \left( \frac{ab}{ta+(1-t)b} \right)}{f \left( \frac{ab}{ta+(1-t)b} \right) + f \left( \frac{ab}{tb+(1-t)a} \right)}
\end{align*}
\]

By integrating both sides and HH-convexity \( f \) we obtain
\[
\begin{align*}
  f \left( \frac{2ab}{a+b} \right) &\leq \int_0^1 \frac{2f \left( \frac{ab}{tb+(1-t)a} \right)f \left( \frac{ab}{ta+(1-t)b} \right)}{f \left( \frac{ab}{ta+(1-t)b} \right) + f \left( \frac{ab}{tb+(1-t)a} \right)} \\
  &= \int_0^1 \frac{1}{f \left( \frac{ab}{ta+(1-t)b} \right)} + \frac{1}{f \left( \frac{ab}{tb+(1-t)a} \right)} \\
  &\leq 2 \int_0^1 \frac{1}{f(a)f(b)}dt = \frac{2f(a)f(b)}{f(a) + f(b)}
\end{align*}
\]

On the other hand by change of variable
\[
\frac{ab}{ta+(1-t)b} = \frac{ab}{t(a-b) + b} = x, \quad \frac{ab(b-a)}{(t(a-b)+b)^2}dt = dx
\]

we see that
\[
\begin{align*}
  \int_0^1 \frac{2f \left( \frac{ab}{tb+(1-t)a} \right)f \left( \frac{ab}{ta+(1-t)b} \right)}{f \left( \frac{ab}{ta+(1-t)b} \right) + f \left( \frac{ab}{tb+(1-t)a} \right)}dt &= \frac{ab}{b-a} \int_a^b \frac{2f(x)f \left( \frac{abx}{x(a+b)-ab} \right)}{f(x) + f \left( \frac{abx}{x(a+b)-ab} \right)} dx \\
  &\leq \frac{2f(a)f(b)}{f(a) + f(b)}
\end{align*}
\]

The proof is complete. \( \square \)

**Corollary 2.6.** Let \( f : [a, b] \rightarrow (0, \infty) \) \( (b > a > 0) \) be a HH-convex function. Then the following inequalities hold:
\[
\begin{align*}
  f \left( \frac{2ab}{a+b} \right) &\leq \frac{1}{b-a} \int_a^b \frac{dx}{x^2f(x)} \\
  &\leq \frac{ab}{b-a} \int_a^b \frac{2f(x)f \left( \frac{abx}{x(a+b)-ab} \right)}{f(x) + f \left( \frac{abx}{x(a+b)-ab} \right)} dx \\
  &\leq \frac{2f(a)f(b)}{f(a) + f(b)}
\end{align*}
\]
Proof. By theorem 2.5 it is sufficient that prove the middle part.

By change of variable \( x = \frac{abt}{t(a+b)-ab} \), we see that

\[
\int_a^b \frac{dx}{x^2 f(x)} = \int_a^b \frac{dx}{x^2 f\left(\frac{abx}{x(a+b)-ab}\right)}.
\]

Hence

\[
\frac{ab}{b-a} \int_a^b \frac{dx}{x^2 f(x)} = \frac{ab}{2(b-a)} \left[ \int_a^b \frac{dx}{x^2 f(x)} + \int_a^b \frac{dx}{x^2 f\left(\frac{abx}{x(a+b)-ab}\right)} \right]
\]

\[
= \frac{ab}{2(b-a)} \int_a^b \left( \frac{1}{f(x)} + \frac{1}{f\left(\frac{abx}{x(a+b)-ab}\right)} \right) \frac{dx}{x^2}.
\]

Put \( h(x) = \frac{1}{f(x)} + \frac{1}{f\left(\frac{abx}{x(a+b)-ab}\right)} \), \( X = [a, b] \) and \( d\mu = \frac{dx}{x^2} \). Thus

\[
\frac{ab}{b-a} \int_a^b \frac{dx}{x^2 f(x)} = \frac{1}{2} \int_X h \, d\mu.
\]

On the other hand by these notations we see that

\[
\frac{ab}{b-a} \int_a^b \frac{2f(x) f\left(\frac{abx}{x(a+b)-ab}\right)}{f(x) + f\left(\frac{abx}{x(a+b)-ab}\right)} \frac{dx}{x^2} = \frac{2ab}{b-a} \int_a^b \left( \frac{1}{f(x)} + \frac{1}{f\left(\frac{abx}{x(a+b)-ab}\right)} \right) \frac{dx}{x^2} = 2 \int_X \frac{d\mu}{h}.
\]

By Holder’s inequality we have

\[
1 = \int_X d\mu = \int_X \sqrt{h} \frac{1}{\sqrt{h}} d\mu \leq \left( \int_X (\sqrt{h})^2 d\mu \right)^{\frac{1}{2}} \left( \int_X \left( \frac{1}{\sqrt{h}} \right)^2 d\mu \right)^{\frac{1}{2}} = \left( \int_X h \, d\mu \right)^{\frac{1}{2}} \left( \int_X \frac{d\mu}{h} \right)^{\frac{1}{2}},
\]

so

\[
1 \leq \int_X h \, d\mu \int_X \frac{d\mu}{h} \quad \text{or} \quad \frac{1}{\int_X h \, d\mu} \leq 2 \int_X \frac{d\mu}{h}.
\]

Thus

\[
\frac{1}{ab} \frac{dx}{x^2 f(x)} \leq \frac{ab}{b-a} \int_a^b \frac{2f(x) f\left(\frac{abx}{x(a+b)-ab}\right)}{f(x) + f\left(\frac{abx}{x(a+b)-ab}\right)} \frac{dx}{x^2}.
\]

\[\square\]
Conflict of Interests

The author declares that there is no conflict of interests.

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