A COMMON FIXED POINT THEOREM IN INTUITIONISTIC FUZZY METRIC SPACE USING IMPLICIT RELATION

PREETI MALVIYA¹*, VANDNA GUPTA² AND V.H. BADSHAH³

¹Govt. New Science College, Dewas (M.P.), India
²Govt. Kalidas Girl's College, Ujjain (M.P.), India
³School of Studies in Mathematics, Vikram University, Ujjain (M.P.), India

Abstract. In this paper, we prove a common fixed point theorem for subsequential continuous and compatible mappings in Intuitionistic fuzzy metric space using Implicit relation.

Keywords: common fixed point; intuitionistic fuzzy metric space; compatible mapping; reciprocal continuity; subcompatible mapping.

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction


Subramanyam [4], Vasuki[5], Pant and Jha [6] obtained some analogous results proved by Balasubraman et.al.[7].

Jungck [8] introduced the notion of compatible maps for a pair of self maps. Popa ([9],[10]) introduced the idea of implicit function to prove a common fixed point theorem in metric space, Singh and Jain [11] extened the result of Popa ([9],[10]) in fuzzy metric space.Recently in 2009, using the concept of subcompatible maps, H.Bouhadjera et.al.[12] proved common fixed point theorems.

*Corresponding author
Received March 7, 2016
Atanassov [13] introduced and studies the concept of intuitionistic fuzzy sets. In 2004, Park [14] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norm and continuous t-conorm. In 2009, Bouhadjera and Godet-Thobie [15] introduced new notions of subcompatibility and subsequential continuity which are respectively weaker than owc and reciprocally continuity. In 2010 and 2011, B. Singh et al. ([16],[17],[18]) proved fixed point theorem in fuzzy metric space and Menger space using the concept of semi-compatibility, weak compatibility and compatibility of type (β) respectively.

2. Preliminaries

**Definition 2.1** [19] A binary operation *:[0,1] x[0,1]→ [0,1] is a continuous t-norm if it satisfies the following conditions

(i) *is associative and commutative,
(ii) *is continuous,
(iii) a*1 = a, for all a ∈[0,1]
(iv) a*b ≤ c*d, whenever a ≤ c and b ≤ d, for all a,b,c,d ∈ [0,1].

**Definition 2.2** [19] A binary operation ◊:[0,1] x[0,1]→ [0,1] is a continuous t-conorm if it satisfies the following conditions:

(i) ◊is associative and commutative,
(ii) ◊is continuous,
(iii) a◊1 = a, for all a ∈[0,1]
(iv) a◊b ≤ c◊d, whenever a ≤ c and b ≤ d, for all a,b,c,d ∈ [0,1].

**Definition 2.3** [20] A 5-tuple (X, M, N, *, ◊) is said to be an Intuitionistic fuzzy metric space if X is an arbitrary set, * is a continuous t-norm, ◊ is a continuous t-conorm and M,N are fuzzy sets on X² x [0,∞) satisfying the following conditions.

(i) M(x,y,t) + N(x,y,t) ≤ 1, for all x,y ∈ X and t > 0,
(ii) M(x,y,0) = 0, for all x,y ∈ X,
(iii) M(x,y,t) = 1, for all x,y ∈ X and t > 0, iff x = y,
(iv) M(x,y,t) = M(y,x,t), for all x,y ∈ X and t > 0,
(v) M(x,y,t)*M(y,z,s) ≤ M(x,z,t+s), for all x,y ∈ X and t,s > 0,
(vi) for all x,y ∈ X, M(x,y,.) : [0,∞) → [0,1] is left continuous,
(vii) lim_{t→∞} M(x,y,t) = 1, for all x,y ∈ X and t > 0,
(viii) $N(x,y,0) = 1$, for all $x,y \in X$,
(ix) $N(x,y,t) = 0$, for all $x,y \in X$ and $t > 0$, iff $x = y$,
(x) $N(x,y,t) = N(y,x,t)$, for all $x,y \in X$ and $t > 0$,
(xi) $N(x,y,t)N(y,z,s) \leq N(x,z,t+s)$, for all $x,y \in X$ and $t,s > 0$,
(xii) for all $x,y \in X$, $N(x,y,.) : [0,\infty) \rightarrow [0,1]$ is right continuous,
(xiii) $\lim_{t \to \infty} N(x,y,t) = 0$, for all $x,y \in X$.

**Remark 2.1[20]**
In intuitionistic metric fuzzy space $(X,M,\ast)$ is an intuitionistic fuzzy space of the form $(X,M,1-M,\ast,\Diamond)$, such that $t$-norm $\ast$ and $t$-conorm $\Diamond$ are associated as $x\Diamond y = 1-(1-x) \ast (1-y)$ for all $x,y \in X$.

**Remark 2.2[20]**
In intuitionistic fuzzy metric space $(X,M,N,\ast,\Diamond)$, $M(x,y,\ast)$ is non-decreasing and $N(x,y,\Diamond)$ is Non-increasing for all $x,y \in X$.

**Example 2.1** – Let $(X,d)$ be a metric space. Define $a \ast b = ab$ and $a \Diamond b = \min \{1,a + b\}$ for all $a,b \in [0,1]$ and let $M_d$ and $N_d$ be a fuzzy sets on $X^2 \times (0,\infty)$ defined as

$$M_d(x,y,t) = t / t + d(x,y), \quad N_d(x,y,t) = d(x,y) / t + d(x,y)$$

Then $(X, M_d,N_d,\ast,\Diamond)$ is an intuitionistic fuzzy metric space.

**Definition 2.4[20]** Let $(X,M,N,\ast,\Diamond)$ be an intuitionistic fuzzy metric space. Then

(1) A sequence $\{x_n\}$ in $X$ is set to be convergent to a point $x$ in $X$ iff $\lim_{n \to \infty} M(x_n,x,t) = 1$ and

$$\lim_{n \to \infty} N(x_n,x,t) = 0$$

for all $t > 0$.

**Definition 2.5[20]** A pair of self mappings $(P,Q)$ of a intuitionistic fuzzy metric space $(X,M,N,\ast,\Diamond)$ is said to be compatible if $\lim_{n \to \infty} M(PQx_n, QPx_n, t) = 1$ and $\lim_{n \to \infty} N(PQx_n, QPx_n, t) = 0$, for all $t > 0$, Whenever $\{x_n\}$ is a sequence in $X$ such that

$$\lim_{n \to \infty} PQx_n = \lim_{n \to \infty} Qx_n = z$$

for some $z \in X$.

**Definition 2.6[20]** A pair of self mappings $(P,Q)$ of a intuitionistic fuzzy metric space $(X,M,N,\ast,\Diamond)$ is said to be semi compatible if $\lim_{n \to \infty} PQx_n = Qx$, When ever $\{x_n\}$ is a sequence in $X$ such that

$$\lim_{n \to \infty} PQx_n = \lim_{n \to \infty} Qx_n = x$$

for some $x \in X$. 


Definition 2.7[21] A pair \((P,Q)\) of self mappings defined on an Intuitionistic fuzzy metric space \((X,M,N,*\),\(\odot)\) is said to be Subcompatible if and only if there exists a sequence \(\{x_n\}\) such that \(\lim_{n \to \infty} P x_n = \lim_{n \to \infty} Q x_n = z\), for some \(z \in X\), and
\[
\lim_{n \to \infty} M(P Q x_n, Q P x_n, t) = 1\text{, for all } t > 0.
\]

Definition 2.8 [4] A pair \((P,Q)\) of self mappings defined on an Intuitionistic fuzzy metric space \((X,M,N,*\),\(\odot)\) is said to be reciprocally continuous if for a sequence \(\{x_n\}\) in \(X\), \(\lim_{n \to \infty} P Q x_n = P z\) and \(\lim_{n \to \infty} Q P x_n = Q z\). Whenever
\[
\lim_{n \to \infty} P x_n = z = \lim_{n \to \infty} Q x_n\text{ for some } z \in X.
\]

Definition 2.9 [21] A pair \((P,Q)\) of self mappings defined on an Intuitionistic fuzzy metric space \((X,M,N,*\),\(\odot)\) is said to be Subsequentially continuous if and only if there exists a sequence \(\{x_n\}\) such that \(\lim_{n \to \infty} P x_n = \lim_{n \to \infty} Q x_n = z\), for some \(z \in X\), and
\[
\lim_{n \to \infty} P Q x_n = P z\text{ and } \lim_{n \to \infty} Q P x_n = Q z.
\]

Implicit Relation 2.10 - Let \(\Phi\) be the set of all real continuous function \(\emptyset : [0,1]^4 \to R\), non-decreasing in first argument, and satisfying the following conditions :
(i). For \(u,v \geq 0\), \(\emptyset(u,v,u,v) \geq 0\) or \(\emptyset(u,v,v,u) \geq 0\) implies that \(u \geq v\),
(ii). \(\emptyset(u,u,1,1) \geq 0\) implies that \(u \geq 1\).

Example Define \(\emptyset(a,b,c,d) = 15a-13b+5c-7d\). Then \(\emptyset \in \Phi\).

3. Main Result

Theorem 3.1- Let \(P\), \(Q\), \(S\) and \(T\) be four self maps of an Intuitionistic Fuzzy metric space \((X,M,N,*\),\(\odot)\) with continuous t-norm * and continuous t-conorm \(\odot\) defined by \(t^*t \geq t\), and \((1-t)\odot(1-t) \leq (1-t)\) respectively for all \(t \in [0,1]\). If the pairs \((P,S)\) and \((Q,T)\) are subsequential continuous and compatible mappings, then

(a). For any \(x,y \in X\), \(\emptyset \gamma \emptyset \in \Phi\), and for all \(t > 0\),
\[
\emptyset[M(P x, Q y, t), M(P x, T y, t), M(S x, P x, t), M(T y, Q y, t)] \geq 0.\text{ and }
\gamma[N(P x, Q y, t), N(P x, T y, t), N(S x, P x, t), N(T y, Q y, t)] \leq 0.
\]

then the pairs \((P,S)\) and \((Q,T)\) have a point of coincidence. Then \(P,Q,S\) and \(T\) have a unique common fixed point.

Proof - Since the pairs \((P,S)\) and \((Q,T)\) are subsequential continuous and compatible mappings, then there exists two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that
\[
\lim_{n \to \infty} P x_n = \lim_{n \to \infty} S x_n = z\text{, for some } z \in X\text{, and } \lim_{n \to \infty} M(P S x_n, S P x_n, t) = 1\text{, for all } t > 0.
\]
A COMMON FIXED POINT THEOREM

\[
\lim_{n \to \infty} Qy_n = \lim_{n \to \infty} Ty_n = w, \text{ for some } w \in X, \text{ and } \lim_{n \to \infty} M(Qt_n, TQy_n, t) = 1, \text{ for all } t > 0.
\]

Therefore \( Pz = Sz \) and \( Qw = Tw \), \( \ldots(1) \)

that is \( z \) is the coincidence point of the pair \((P,S)\) and \( w \) is the coincidence point of the pair \((Q,T)\).

Now, we will prove that \( z = w \).

Put \( x = x_n \) and \( y = y_n \) in inequality 3.1(a), we get

\[
\emptyset \left[ M(Px_n, Qy_n, t), M(Px_n, Ty_n, t), M(Sx_n, Px_n, t), M(Ty_n, Qy_n, t) \right] \geq 0, \text{ and}
\]

\[
\psi \Psi \left[ N(Px_n, Qy_n, t), N(Px_n, Ty_n, t), N(Sx_n, Px_n, t), N(Ty_n, Qy_n, t) \right] \leq 0
\]

Taking \( \lim n \to \infty \), we get

\[
\emptyset \left[ M(z, w, t), M(z, w, t), M(z, z, t), M(w, w, t) \right] \geq 0, \text{ and}
\]

\[
\psi \Psi \left[ N(z, w, t), N(z, w, t), N(z, z, t), N(w, w, t) \right] \leq 0
\]

In view of 2.10(ii), \( z = w \).

Again, we claim that \( Pz = z \).

By putting \( x = z \) and \( y = y_n \), in inequality 3.1(a), we get

\[
\emptyset \left[ M(Pz, Qy_n, t), M(Pz, Ty_n, t), M(Sz, Pz, t), M(Ty_n, Qy_n, t) \right] \geq 0, \text{ and}
\]

\[
\psi \Psi \left[ N(Pz, Qy_n, t), N(Pz, Ty_n, t), N(Sz, Pz, t), N(Ty_n, Qy_n, t) \right] \leq 0
\]

Taking limit \( n \to \infty \), we get

\[
\emptyset \left[ M(Pz, w, t), M(Pz, w, t), M(Sz, Pz, t), M(w, w, t) \right] \geq 0, \text{ and}
\]

\[
\psi \Psi \left[ N(Pz, w, t), N(Pz, w, t), N(Sz, Pz, t), N(w, w, t) \right] \leq 0
\]

Since \( w = z \), then we get

\[
\emptyset \left[ M(Pz, z, t), M(Pz, z, t), M(z, z, t), M(z, z, t) \right] \geq 0, \text{ and}
\]

\[
\Psi \Psi \left[ N(Pz, z, t), N(Pz, z, t), N(z, z, t), N(z, z, t) \right] \leq 0.
\]

Using (1) and (2), we get

\[
Pz = z = Sz \ldots(2)
\]

Now, again we claim that \( Qw = z \).

Substitute \( x = z \) and \( y = w \) in inequality 3.1(a), we get

\[
\emptyset \left[ M(Pz, Qw, t), M(Pz, Tw, t), M(Sz, Pz, t), M(Tw, Qw, t) \right] \geq 0, \text{ and}
\]

\[
\psi \Psi \left[ N(Pz, Qw, t), N(Pz, Tw, t), N(Sz, Pz, t), N(Tw, Qw, t) \right] \leq 0
\]

Using (1) and (2), we get

\[
\emptyset \left[ M(z, Qw, t), M(z, Qw, t), M(Pz, Pz, t), M(Qw, Qw, t) \right] \geq 0, \text{ and}
\]
\[\psi[N(z,Q_w,t), N(z,Q_w,t), N(P_z,P_z,t),N(Q_w,Q_w,t)] \leq 0.\]
\[\emptyset[M(z,Q_w,t), M(z,Q_w,t), 1,1] \geq 0, \text{ and } \psi[N(z,Q_w,t),N(z,Q_w,t),1,1] \leq 0.\]

From 2.10 (ii), we get \( Q_w = z \). Since \( w = z \), then we get \( Q_z = z \).

Since \( Q_z = T_z \), hence from this we conclude that \( Q_z = z = T_z \).

Therefore, combining all the results, we get
\[
z = P_z = S_z = Q_z = T_z.
\]
That is \( z \) is a common fixed point of \( P, Q, S \) and \( T \).

**Uniqueness** – Let \( u \) be another common fixed point of \( P, Q, S \) and \( T \). Then
\[
Pu = Su = Qu = Tu = u.
\]
Put \( x = z \) and \( y = u \) in inequality 3.1(a), we get
\[
\emptyset[M(P_z,Qu,t), M(P_z,Tu,t), M(S_z,P_z,t), M(Tu,Qu,t)] \geq 0, \text{ and } \psi[N(P_z,Qu,t),N(P_z,Tu,t),N(S_z,P_z,t),N(Tu,Qu,t)] \leq 0.
\]
\[
\emptyset[M(z,u,t), M(z,u,t), M(z,z,t), M(u,u,t)] \geq 0, \text{ and } \psi[N(z,u,t), N(z,u,t), N(z,z,t), N(u,u,t)] \leq 0.
\]

From 2.10 (ii), we get \( z = u \).

Therefore, uniqueness follows.

**Corollary 3.2** Let \( P, Q \) and \( S \) be three self-maps of an Intuitionistic fuzzy metric space \( (X,M,N,*,\Diamond) \) with continuous
\[t\text{-norm } * \text{ and continuous } t\text{-conorm } \Diamond \text{ defined by } t^*t \geq t \text{ and } (1-t)^*(1-t) \leq (1-t), \text{ for all } t \in [0,1].\]
If the pairs \( (P,S) \) and
\[\text{(Q,T)}\]
are subsequential continuous and compatible mappings, then for some \( \emptyset, \Psi \in \Phi \) and for all \( x,y \in X, \) and every \( t > 0, \)
\[
\emptyset[M(Px,Qy,t), M(Sx,Px,t),M(Sy,Qy,t),M(Sx,Qy,t),M(Sy,Px,t)] \geq 0, \text{ and } \psi\Psi[N(Px,Qy,t), N(Sx, Px, t), N(Sy, Qy, t), N(Sx, Qy, t), N(Sy, Px, t)] \leq 0.
\]
then \( (P,S) \) and \( (Q,S) \) have a coincidence points. Then \( P, Q \) and \( S \) have a unique common fixed point.

**Corollary 3.3** Let \( P,Q,S \) and \( T \) be four self maps of an Intuitionistic fuzzy metric space \( (X,M,N,*,\Diamond) \) with continuous \( t\text{-norm } * \) and continuous \( t\text{-conorm } \Diamond \text{ defined by } t^*t \geq t \text{ and } (1-t)^*(1-t) \leq (1-t), \text{ for all } t \in [0,1].\)
If the pairs \( (P,S) \) and \( (Q,T) \) are subsequential continuous and compatible mappings, then for some \( \emptyset, \Psi \in \Phi \) and for all \( x,y \in X, \) and every \( t > 0, \)

∅ \{ M(Px,Qy,t) , 1/2 \{ M(Sx,Px,t) + M(Ty,Qy,t) \} , M(Sx,Qy,t), M(Ty,Px,t) \} \geq 0 , \text{ and} \\
\psi[N(Px,Qy,t), 1/2\{ N(Sx,Px,t) + N(Ty,Qy,t) \} , N(Sx,Qy,t), N(Ty,Px,t)] \leq 0 .

Then \((P,S)\) and \((Q,T)\) have a coincidence points. Then \(P,Q,S\) and \(T\) a unique common fixed point.

Conflicts of Interests
The author declares that there is no conflict of interests

REFERENCES


[18] B. Singh, S. Jain, Weak compatibility and fixed point theorems in fuzzy metric space, Ganita 56(2) (2005), 167-176.

