ON SOME INTEGRAL INEQUALITIES FOR GENERALIZED FRACTIONAL INTEGRAL

ABDULLAH AKKURT¹,*, M. ESRA YILDIRIM², HÜSEYIN YILDIRIM¹

¹Department of Mathematics, Faculty of Science and Arts, University of Kahramanmaraş Sütçü İmam, 46100, Kahramanmaraş, Turkey
²Department of Mathematics, Faculty of Science, University of Cumhuriyet, 58140, Sivas, Turkey

Copyright © 2016 Akkurt, Yıldırım, and Yıldırım. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract. In this article, we obtain integral inequalities for generalized Riemann-Liouville fractional integrals and Chebyshev functional by using synchronous functions.

Keywords: integral inequalities; Riemann-Liouville fractional integral; Chebyshev functional.

2010 AMS Subject Classification: 26A33, 26D10, 26D15, 41A55.

1. Introduction

Let us consider the functional in [1]

\[ T(f,g) := \frac{1}{b-a} \int_a^b f(x)g(x)dx - \left( \frac{1}{b-a} \int_a^b f(x)dx \right) \left( \frac{1}{b-a} \int_a^b g(x)dx \right) \] (1)

where \( f \) and \( g \) are two synchronous and integrable functions on \( [a,b] \).

*Corresponding author

Received May 3, 2016
Over the last decade, this classical inequality has been improved and generalized in a number of ways; there have been a large number of research papers written on this subject, ([9]-[12], [15]) and the references therein.

In this paper, we obtain some integral inequalities for (1) type functional via generalized fractional integrals.

2. Preliminaries

Definition 2.1. Let \( a, b \in \mathbb{R}, a < b, \) and \( \alpha > 0. \) For \( f \in L_1(a, b) \)

\[
( J_{a^+}^{\alpha}f ) (x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0, \quad x > a
\]

and

\[
( J_{b^-}^{\alpha}f ) (x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad b > 0, \quad b > x.
\]

These integrals are called right-sided Riemann-Liouville fractional integral and left-sided Riemann-Liouville fractional integral respectively [2]-[7].

Definition 2.2. Let \( (a, b) \) be a finite interval of the real line \( \mathbb{R} \) and \( \alpha > 0. \) Also let \( h(x) \) be an increasing and a positive monotone function on \( (a, b] \), having a continuous derivative \( h'(x) \) on \( (a, b) \). The left- and right-sided fractional integrals of a function \( f \) with respect to another function \( h \) on \([a, b] \) are defined by [13]

\[
( J_{a^+}^{\alpha,h}f ) (x) := \frac{1}{\Gamma(\alpha)} \int_a^x [h(x) - h(t)]^{\alpha-1} h'(t) f(t) dt, \quad x \geq a,
\]

and

\[
( J_{b^-}^{\alpha,h}f ) (x) := \frac{1}{\Gamma(\alpha)} \int_x^b [h(t) - h(x)]^{\alpha-1} h'(t) f(t) dt, \quad x \leq b.
\]

For (4) and (5)

\[
( J_{a^+}^{\alpha,h}f ) (a) = ( J_{b^-}^{\alpha,h}f ) (b) = 0.
\]

If we take \( h(x) = x \) in (4) and (5), we will obtain

\[
J_{a^+}^{\alpha} = J_{a^+}^{\alpha} \text{ and } J_{b^-}^{\alpha,h} = J_{b^-}^{\alpha}.
\]
Also if we choose \( h(x) = \frac{x^{k+1}}{k+1} \) for \( k \geq 0 \), then the equalities (4) and (5) will be

\[
(J^{\alpha}_{a^{+},k}f)(x) = \frac{(k+1)^{1-\alpha}}{\Gamma(\alpha)} \int_{a}^{x} (\tau^{k+1})^{1-\alpha} \tau^{k} f(\tau) d\tau, \quad x > a
\]

and

\[
(J^{\alpha}_{b^{-},k}f)(x) = \frac{(k+1)^{1-\alpha}}{\Gamma(\alpha)} \int_{x}^{b} (\tau^{k+1})^{1-\alpha} \tau^{k} f(\tau) d\tau, \quad x < b
\]

respectively. This kind of generalized fractional integrals are studied in [5], [7], [14] and [16].

For \( a = 0 \) in (4), we can write

\[
(J^{\alpha}_{0^{+},h}f)(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (h(x) - h(t))^{\alpha-1} h'(t) f(t) dt, \quad x > 0
\]

and

\[
(J^{\alpha}_{0^{-},h}f)(x) = f(x).
\]

For the convenience of establishing the results, we give the semigroup property:

\[
J^{\alpha}_{a^{+},h}J^{\beta}_{a^{+},h}f(x) = J^{\alpha+\beta}_{a^{+},h}f(x), \quad \alpha \geq 0, \quad \beta \geq 0,
\]

which implies the commutative property:

\[
J^{\alpha}_{a^{+},h}J^{\beta}_{a^{+},h}f(x) = J^{\beta}_{a^{+},h}J^{\alpha}_{a^{+},h}f(x).
\]

From (8), when \( f(x) = h(x) \), we get:

\[
(J^{\alpha}_{0^{+},h}h)(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (h(x) - h(t))^{\alpha-1} h'(t) h(t) dt
\]

\[
= \frac{(h(x) - h(0))^{\alpha}}{\Gamma(\alpha + 2)} [h(x) + \alpha h(0)].
\]

Let \( \alpha = 0 \) in (9), then

\[
(J^{0}_{0^{+},h}h)(x) = h(x).
\]

From (8), when \( f(x) = x^\mu \) and \( h(x) = x^{k+1} \) we get:

\[
J^{\alpha}_{0^{+},h}(x^\mu)
\]

\[
= \frac{(k+1)^{-\alpha}\Gamma\left(\frac{k+\mu+1}{k+1}\right)}{\Gamma\left(\alpha + \frac{k+\mu+1}{k+1}\right)} t^{\alpha(k+1)+\mu}, \quad \alpha > 0; \quad k \geq 0, \quad \mu > -1, \quad t > 0.
\]
From (8), when \( f(x) = 1 \) and \( h(x) = x^{k+1} \) we get:

\[
J_{a^+, b}^{\alpha} (1) = \frac{(k + 1)^{-\alpha}}{\Gamma(\alpha + 1)} f^{\alpha(k+1)}; \quad \alpha > 0; \quad k \geq 0, \quad \mu > -1, \quad t > 0.
\]

3. Main results

**Theorem 3.1.** Let \( f \) and \( g \) be two synchronous functions on \([0, \infty)\). Also let \( h(x) \) be an increasing and a positive monotone function on \((a, b]\), having a continuous derivative \( h'(x) \) on \((a, b)\). Then for \( t > a, \alpha > 0; \)

\[
J_{a^+, b}^{\alpha} fg(t) \geq J_{a^+, b}^{\alpha} f(t)J_{a^+, b}^{\alpha} g(t).
\]

**Proof.** For \( f \) and \( g \) synchronous functions, we have

\[
(f(t) - f(\rho))(g(t) - g(\rho)) \geq 0.
\]

From (13) it can be written as following

\[
f(t)g(t) + f(\rho)g(\rho) \geq f(t)g(\rho) + f(\rho)g(t).
\]

If we multiply two sides of the (14) with \( \frac{(h(t) - h(\tau))^{\alpha-1}}{\Gamma(\alpha)} h'(\tau) f(\tau) g(\tau) + \frac{(h(t) - h(\tau))^{\alpha-1}}{\Gamma(\alpha)} h'(\tau) f(\rho) g(\rho) \)

\[
\geq \frac{(h(t) - h(\tau))^{\alpha-1}}{\Gamma(\alpha)} h'(\tau) f(\tau) g(\tau) + \frac{(h(t) - h(\tau))^{\alpha-1}}{\Gamma(\alpha)} h'(\tau) f(\rho) g(\tau).
\]

Then integrating (15) inequality over \((a, t)\), we obtain:

\[
\frac{1}{\Gamma(\alpha)} \int_a^t (h(t) - h(\tau))^{\alpha-1} h'(\tau) f(\tau) g(\tau) d\tau
\]

\[
+ \frac{1}{\Gamma(\alpha)} \int_a^t (h(t) - h(\tau))^{\alpha-1} h'(\tau) f(\rho) g(\rho) d\tau
\]

\[
\geq \frac{1}{\Gamma(\alpha)} \int_a^t (h(t) - h(\tau))^{\alpha-1} h'(\tau) f(\tau) g(\rho) d\tau
\]

\[
+ \frac{1}{\Gamma(\alpha)} \int_a^t (h(t) - h(\tau))^{\alpha-1} h'(\tau) f(\rho) g(\tau) d\tau.
\]
Consequently,
\[
J_{a^+,h}^\alpha (fg)(t) + f(\rho)g(\rho) \frac{1}{\Gamma(\alpha)} \int_a^t (h(t) - h(\tau))^{\alpha-1} h'(\tau) d\tau 
\geq g(\rho) \frac{1}{\Gamma(\alpha)} \int_a^t (h(t) - h(\tau))^{\alpha-1} h'(\tau) f(\tau) d\tau 
\]
\[
+ f(\rho) \frac{1}{\Gamma(\alpha)} \int_a^t (h(t) - h(\tau))^{\alpha-1} h'(\tau) g(\tau) d\tau.
\]

So we have
\[
J_{a^+,h}^\alpha (fg)(t) + f(\rho)g(\rho)J_{a^+,h}^\alpha (1) \geq g(\rho)J_{a^+,h}^\alpha f(t) + f(\rho)J_{a^+,h}^\alpha g(t).
\]

Now multiplying two sides of (18) by \( \frac{(h(t) - h(\rho))^{\alpha-1}}{\Gamma(\alpha)} h'(\rho) \), \( \rho \in (a, t) \), we obtain:
\[
\frac{(h(t) - h(\rho))^{\alpha-1}}{\Gamma(\alpha)} h'(\rho) J_{a^+,h}^\alpha (fg)(t)
\]
\[
\geq \frac{(h(t) - h(\rho))^{\alpha-1}}{\Gamma(\alpha)} h'(\rho) g(\rho) J_{a^+,h}^\alpha f(t)
\]
\[
+ \frac{(h(t) - h(\rho))^{\alpha-1}}{\Gamma(\alpha)} h'(\rho) f(\rho) J_{a^+,h}^\alpha g(t).
\]

By integrating to (19) over \((a, t)\), we get:
\[
J_{a^+,h}^\alpha (fg)(t) \int_a^t \frac{(h(t) - h(\rho))^{\alpha-1}}{\Gamma(\alpha)} h'(\rho) d\rho
\]
\[
+ \frac{J_{a^+,h}^\alpha (1)}{\Gamma(\alpha)} \int_a^t f(\rho)g(\rho)(h(t) - h(\rho))^{\alpha-1} h'(\rho) d\rho
\]
\[
\geq \frac{J_{a^+,h}^\alpha f(t)}{\Gamma(\alpha)} \int_a^t (h(t) - h(\rho))^{\alpha-1} h'(\rho) g(\rho) d\rho
\]
\[
+ \frac{J_{a^+,h}^\alpha g(t)}{\Gamma(\alpha)} \int_a^t (h(t) - h(\rho))^{\alpha-1} h'(\rho) f(\rho) d\rho.
\]
This inequality is can be written as the following at the same time,

\[
J_{\alpha,h}^{\alpha} (fg)(t) \geq \frac{1}{J_{\alpha,h}^{\alpha}(1)} J_{\alpha,h}^{\alpha} f(t) J_{\alpha,h}^{\alpha} g(t).
\]

So the proof is completed.

**Theorem 3.2.** Let \( f \) and \( g \) be two synchronous functions on \([a,b]\). Then for \( t > a, \alpha > 0, \) and \( \beta > 0, \)

\[
J_{\alpha,h}^{\beta}(1) J_{\alpha,h}^{\alpha} (fg)(t) + \frac{(h(t) - h(a))^\alpha}{\Gamma(\alpha + 1)} J_{\alpha,h}^{\beta} (fg)(t) \\
\geq J_{\alpha,h}^{\alpha} f(t) J_{\alpha,h}^{\beta} g(t) + J_{\alpha,h}^{\alpha} g(t) J_{\alpha,h}^{\beta} f(t).
\]

**Proof.** If we multiply two sides of (18) by \( \frac{(h(t) - h(\rho))^\beta - 1}{\Gamma(\beta)} h'(\rho), \) we obtain:

\[
\frac{(h(t) - h(\rho))^\beta - 1}{\Gamma(\beta)} h'(\rho) J_{\alpha,h}^{\alpha}(fg)(t) \\
+ \frac{(h(t) - h(\rho))^\beta - 1}{\Gamma(\beta)} h'(\rho) f(\rho) J_{\alpha,h}^{\alpha}(1)
\]

\[
\geq \frac{(h(t) - h(\rho))^\beta - 1}{\Gamma(\beta)} h'(\rho) g(\rho) J_{\alpha,h}^{\alpha} f(t) \\
+ \frac{(h(t) - h(\rho))^\beta - 1}{\Gamma(\beta)} h'(\rho) f(\rho) J_{\alpha,h}^{\alpha} g(t).
\]

Integrating to (22) over \((a, t), \) we get:

\[
\int_{a}^{t} \frac{(h(t) - h(\rho))^\beta - 1}{\Gamma(\beta)} h'(\rho) J_{\alpha,h}^{\alpha}(fg)(t) dt \\
+ \int_{a}^{t} \frac{(h(t) - h(\rho))^\beta - 1}{\Gamma(\beta)} h'(\rho) f(\rho) J_{\alpha,h}^{\alpha}(1) dt
\]

\[
\geq \int_{a}^{t} \frac{(h(t) - h(\rho))^\beta - 1}{\Gamma(\beta)} h'(\rho) g(\rho) J_{\alpha,h}^{\alpha} f(t) dt \\
+ \int_{a}^{t} \frac{(h(t) - h(\rho))^\beta - 1}{\Gamma(\beta)} h'(\rho) f(\rho) J_{\alpha,h}^{\alpha} g(t) dt.
\]
Consequently,

\[ J_{a^+,h}^\beta (1) J_{a^+,h}^\alpha (fg)(t) + J_{a^+,h}^\alpha (1) J_{a^+,h}^\beta (fg)(t) \]

(24)

\[ \geq J_{a^+,h}^\alpha f(t) J_{a^+,h}^\beta g(t) + J_{a^+,h}^\alpha g(t) J_{a^+,h}^\beta f(t). \]

This is the proof of the theorem.

**Remark 3.3.** Applying Theorem 3.2 for \( \alpha = \beta \), we obtain Theorem 3.1.

**Theorem 3.4.** Let \((f_i)_{i=1,...,n}\) be \( n \) positive increasing functions on \([0, \infty)\). Then for all \( t > a, \alpha > 0 \),

(25)

\[ J_{a^+,h}^\alpha \left( \prod_{i=1}^{n} f_i \right)(t) \geq \left( J_{a^+,h}^\alpha (1) \right)^{1-n} \left( \prod_{i=1}^{n} J_{a^+,h}^\alpha f_i \right)(t) \]

**Proof.** We will prove this theorem by induction. It is clear that for \( n = 1 \) and all \( t > 0, \alpha > 0 \), we have \( J_{a^+,h}^\alpha f_1(t) \geq J_{a^+,h}^\alpha f_1(t) \). And for \( n = 2 \), we obtain (12),

(26)

\[ J_{a^+,h}^\alpha (f_1f_2)(t) \geq \left( J_{a^+,h}^\alpha (1) \right)^{1-n} \left( J_{a^+,h}^\alpha f_1 \right)(t) \left( J_{a^+,h}^\alpha f_2 \right)(t) \]

Now assume that (induction hypothesis)

(27)

\[ J_{a^+,h}^\alpha \left( \prod_{i=1}^{n-1} f_i \right)(t) \geq \left( J_{a^+,h}^\alpha (1) \right)^{2-n} \left( \prod_{i=1}^{n-1} J_{a^+,h}^\alpha f_i \right)(t) \]

If \((f_i)_{i=1,...,n}\) are positive increasing functions, then \( \prod_{i=1}^{n-1} f_i \) (t) is an increasing function. So we can use Theorem 3.1 for functions \( \prod_{i=1}^{n-1} f_i = g, \) and \( f_n = f \), therefore we obtain

(28)

\[ J_{a^+,h}^\alpha \left( \prod_{i=1}^{n} f_i \right)(t) = J_{a^+,h}^\alpha (fg)(t) \geq \left( J_{a^+,h}^\alpha (1) \right)^{1-n} \left( \prod_{i=1}^{n-1} J_{a^+,h}^\alpha f_i \right)(t) \left( J_{a^+,h}^\alpha f_n \right)(t). \]

By (27)

(29)

\[ J_{a^+,h}^\alpha \left( \prod_{i=1}^{n} f_i \right)(t) \geq \left( J_{a^+,h}^\alpha (1) \right)^{2-n} \left( \prod_{i=1}^{n-1} J_{a^+,h}^\alpha f_i \right)(t) \left( J_{a^+,h}^\alpha f_n \right)(t). \]

This completes the proof.

**Theorem 3.5.** Let \( h(x) \) be an increasing and positive monotone function on \((a,b]\), having a continuous derivative \( h'(x) \) on \((a,b)\). If \( f \) is an increasing and \( g \) is a differentiable functions
and there exist a real number \( mh' (t) := \inf_{t \geq 0} g'(t) \) on \([0, +\infty)\). Then for all \( t \in [a, b] \) and \( \alpha > 0 \),

\[
J_{a^+, h}^{\alpha} (fg)(t)
\]

(30)

\[
\geq \left( J_{a^+, h}^{\alpha} (1) \right)^{-1} J_{a^+, h}^{\alpha} f(t) J_{a^+, h}^{\alpha} g(t) - \frac{mh(t)}{\alpha + 1} J_{a^+, h}^{\alpha} f(t) + m J_{a^+, h}^{\alpha} (hf)(t).
\]

**Proof.** Consider the given function \( H(t) = g(t) - mh(t) \). It is clear that \( H \) is an increasing function and differentiable on \([0, +\infty)\). Then using Theorem 3.1 we obtain

\[
J_{a^+, h}^{\alpha} (Hf)(t) = J_{a^+, h}^{\alpha} ((g(t) - mh(t)) f(t))
\]

\[
\geq \left( J_{a^+, h}^{\alpha} (1) \right)^{-1} J_{a^+, h}^{\alpha} f(t) \left[ J_{a^+, h}^{\alpha} g(t) - m J_{a^+, h}^{\alpha} h(t) \right]
\]

(31)

\[
\geq \left( J_{a^+, h}^{\alpha} (1) \right)^{-1} J_{a^+, h}^{\alpha} f(t) J_{a^+, h}^{\alpha} g(t) - \frac{m (J_{a^+, h}^{\alpha} (1))^{-1} (h(t) - h(a))^\alpha (h(t) + \alpha h(a))}{\Gamma(\alpha + 2)} J_{a^+, h}^{\alpha} f(t)
\]

Also,

\[
J_{a^+, h}^{\alpha} (Hf)(t)
\]

(32)

\[
= J_{a^+, h}^{\alpha} ((g(t) - mh(t)) f(t))
\]

\[
= J_{a^+, h}^{\alpha} (fg)(t) - m J_{a^+, h}^{\alpha} (hf)(t)
\]

From (31) and (32), we get:

\[
J_{a^+, h}^{\alpha} (fg)(t) \geq \left( J_{a^+, h}^{\alpha} (1) \right)^{-1} J_{a^+, h}^{\alpha} f(t) J_{a^+, h}^{\alpha} g(t) - \frac{m (h(t) + \alpha h(a))}{\alpha + 1} J_{a^+, h}^{\alpha} f(t) + m J_{a^+, h}^{\alpha} (hf)(t).
\]

(33)

This is the proof of theorem.
Corollary 3.6. Let \( h(x) \) be an increasing and positive monotone function on \((a, b)\), having a continuous derivative \( h'(x) \) on \((a, b)\). If \( f \) is an increasing and \( g \) is a differentiable functions on \([0, +\infty)\). Then for all \( t \in [a, b] \) and \( \alpha > 0 \),

I. If there exist real numbers \( m_1 h'(t) := \inf f'(x), \) and \( m_2 h'(t) := \inf g'(t) \). Then we have:

\[
J^{\alpha}_{a^+, h}(fg)(t) - m_1 J^{\alpha}_{a^+, h}(hg)(t) - m_2 J^{\alpha}_{a^+, h}(hf)(t) + m_1 m_2 J^{\alpha}_{a^+, h} h(t)^2
\]

(34)

\[
\geq \left( J^{\alpha}_{a^+, h}(1) \right)^{-1} \left[ J^{\alpha}_{a^+, h} f(t) J^{\alpha}_{a^+, h} g(t) - m_1 J^{\alpha}_{a^+, h} h(t) J^{\alpha}_{a^+, h} g(t) 
\right.
\]

\[
-m_2 J^{\alpha}_{a^+, h} h(t) J^{\alpha}_{a^+, h} f(t) + m_1 m_2 \left( J^{\alpha}_{a^+, h} h(t) \right)^2
\]

II. If there exist real numbers \( M_1 h'(t) := \sup f'(x), \) and \( M_2 h'(t) := \sup g'(t) \). Then we have:

\[
J^{\alpha}_{a^+, h}(fg)(t) - M_1 J^{\alpha}_{a^+, h}(hg)(t) - M_2 J^{\alpha}_{a^+, h}(hf)(t) + M_1 M_2 \left( J^{\alpha}_{a^+, h} h(t) \right)^2
\]

(35)

\[
\geq \left( J^{\alpha}_{a^+, h}(1) \right)^{-1} \left[ J^{\alpha}_{a^+, h} f(t) J^{\alpha}_{a^+, h} g(t) - M_1 J^{\alpha}_{a^+, h} h(t) J^{\alpha}_{a^+, h} g(t) 
\right.
\]

\[
-M_2 J^{\alpha}_{a^+, h} h(t) J^{\alpha}_{a^+, h} f(t) + M_1 M_2 \left( J^{\alpha}_{a^+, h} h(t) \right)^2
\]

Proof. Consider the given function \( F(t) = f(t) - m_1 h(t) \) and \( G(t) = g(t) - m_2 h(t) \). It is clear that \( F \) and \( G \) are an increasing function and differentiable on \([0, +\infty)\). Then using Theorem 3.1 we obtain

\[
J^{\alpha}_{a^+, h}(FG)(t) = J^{\alpha}_{a^+, h} \big(f(t) - m_1 h(t)\big) \big(g(t) - m_2 h(t)\big)
\]

\[
\geq \left( J^{\alpha}_{a^+, h}(1) \right)^{-1} J^{\alpha}_{a^+, h} \big(f(t) - m_1 h(t)\big) J^{\alpha}_{a^+, h} \big(g(t) - m_2 h(t)\big)
\]

\[
\geq \left( J^{\alpha}_{a^+, h}(1) \right)^{-1} \left[ J^{\alpha}_{a^+, h} f(t) J^{\alpha}_{a^+, h} g(t) - m_1 J^{\alpha}_{a^+, h} h(t) J^{\alpha}_{a^+, h} g(t) 
\right.
\]

\[
-m_2 J^{\alpha}_{a^+, h} f(t) J^{\alpha}_{a^+, h} h(t) + m_1 m_2 \left( J^{\alpha}_{a^+, h} h(t) \right)^2
\]
Therefore
\[
J_{a^+,h}^{\alpha}(fg)(t) - m_1 J_{a^+,h}^{\alpha}(hg)(t) - m_2 J_{a^+,h}^{\alpha}(hf)(t) + m_1 m_2 J_{a^+,h}^{\alpha}(h(t))^2
\]
\[
\geq \left(J_{a^+,h}^{\alpha}(1)\right)^{-1} \left[ J_{a^+,h}^{\alpha}(f(t)J_{a^+,h}^{\alpha}(g(t)) - m_1 J_{a^+,h}^{\alpha}(h(t)) J_{a^+,h}^{\alpha}(g(t))
\right.
\]
\[
- m_2 J_{a^+,h}^{\alpha}(f(t)J_{a^+,h}^{\alpha}(h(t)) + m_1 m_2 \left(J_{a^+,h}^{\alpha}(h(t))^2\right)\right].
\]
This is the proof of (I).

Consider the given function \(F(t) = f(t) - M_1 h(t), G(t) = g(t) - M_2 h(t)\). It is clear that \(F\) and \(G\) are an increasing function and differentiable on \([0, +\infty)\). Then using Theorem 3.1 we obtain
\[
J_{a^+,h}^{\alpha}(FG)(t) = J_{a^+,h}^{\alpha}(f(t) - M_1 h(t))(g(t) - M_2 h(t))
\]
\[
\geq \left(J_{a^+,h}^{\alpha}(1)\right)^{-1} J_{a^+,h}^{\alpha}(f(t) - M_1 h(t)) J_{a^+,h}^{\alpha}(g(t) - M_2 h(t))
\]
\[
\geq \left(J_{a^+,h}^{\alpha}(1)\right)^{-1} \left[ J_{a^+,h}^{\alpha}(f(t)J_{a^+,h}^{\alpha}(g(t)) - M_1 J_{a^+,h}^{\alpha}(h(t)) J_{a^+,h}^{\alpha}(g(t))
\right.
\]
\[
- m_2 J_{a^+,h}^{\alpha}(f(t)J_{a^+,h}^{\alpha}(h(t)) + M_1 M_2 \left(J_{a^+,h}^{\alpha}(h(t))^2\right)\right].
\]
Therefore
\[
J_{a^+,h}^{\alpha}(fg)(t) - M_1 J_{a^+,h}^{\alpha}(hg)(t) - M_2 J_{a^+,h}^{\alpha}(hf)(t) + M_1 M_2 J_{a^+,h}^{\alpha}(h(t))^2
\]
\[
\geq \left(J_{a^+,h}^{\alpha}(1)\right)^{-1} \left[ J_{a^+,h}^{\alpha}(f(t)J_{a^+,h}^{\alpha}(g(t)) - M_1 J_{a^+,h}^{\alpha}(h(t)) J_{a^+,h}^{\alpha}(g(t))
\right.
\]
\[
- M_2 J_{a^+,h}^{\alpha}(f(t)J_{a^+,h}^{\alpha}(h(t)) + M_1 M_2 \left(J_{a^+,h}^{\alpha}(h(t))^2\right)\right].
\]
This is the proof of (II).

**Conflict of Interests**

The authors declare that there is no conflict of interests.

**Acknowledgement**
M.E. Yildirim was partially supported by the Scientific and Technological Research Council of Turkey (TUBITAK Programme 2228-B).

REFERENCES