BOUNDS FOR THE GENERALIZATION OF TWO MAPPINGS RELATED TO THE HERMITE-HADAMARD INEQUALITY

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Abstract. In this paper, we give some results concerning the generalization of two mappings associated to the famous Hermite-Hadamard integral inequality for convex functions. As application, some new inequalities involving potential means are derived.

Keywords: convex functions; Hermite-Hadamard inequality; special means.

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1. Introduction

Let \( f \) be a convex function on \( [a, b] \subset \mathbb{R} \). The following inequality

\[
 f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2}
\]

is known in the literature as the integral Hermite-Hadamard inequality [16].

It is well known that the Hermite-Hadamard inequality plays an important role in nonlinear analysis. Over the last decade, this classical inequality has been improved and generalized in a number of ways; there has been a large number of research papers written on this subject, see

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[9], [10], [11], and [12] and the references therein.

It has many applications for special means (see [8], [13], [14] and [17]) and also provides necessary and sufficient condition for a function $f$ to be convex on $(a, b)$ (see [19]).

Dragomir introduced in 1991. the following associated mapping $H : [0, 1] \rightarrow \mathbb{R}$ defined by

$$H(t) = \frac{1}{b - a} \int_{a}^{b} f(t x + (1 - t) \frac{a + b}{2}) \, dx$$

for a given convex function $f : [a, b] \rightarrow \mathbb{R}$.

The corresponding double integral mapping $F : [0, 1] \rightarrow \mathbb{R}$ in connection with the Hermite-Hadamard inequalities is defined as

$$F(t) = \frac{1}{(b - a)^2} \int_{a}^{b} \int_{a}^{b} f(t x + (1 - t) y) \, dx \, dy.$$

For main properties of these mappings and some related results see [2], [5], [6], [7] and [18] and the references therein.

S.S.Dragomir [4] gave the following bounds for two mappings related to the Hermite-Hadamard inequality for convex functions:

**Theorem 1.1.** [4] Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function on the interval $[a, b]$. Then we have

\[
\frac{t}{b - a} \int_{a}^{b} f(x) \, dx + (1 - t) f\left(\frac{a + b}{2}\right) - H(t) \leq t(1 - t) \left[ \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \right]
\]

(2)

and

\[
\frac{1}{b - a} \int_{a}^{b} f(x) \, dx - F(t) \leq 2t(1 - t) \left[ \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \right]
\]

(3)

for any $t \in [0, 1]$.

In the present paper, we establish a weighted generalization of the above results involving a generalization of the two mappings associated to the Hermite-Hadamard inequality. Applications for potential means are also provided.
2. Preliminaries

Let \( f : [a, b] \to \mathbb{R} \) be a convex function on the interval \([a, b]\). Let \( p, g : [a, b] \to \mathbb{R} \) be integrable functions such that \( p \geq 0, \int_a^b p(x)dx = 1 \) and \( a \leq g(x) \leq b \) for any \( x \in [a, b] \) and let \( \bar{g} = \int_a^b p(x)g(x)dx \).

In order to state our results, we first need to introduce the following associated mapping \( H : [0, 1] \to \mathbb{R} \) defined by

\[
H(t; g) = \int_a^b p(x)f(tg(x) + (1-t)\bar{g})dx.
\]

Some of the main properties of the mapping \( H \) are:

1. \( H \) is convex on \([0, 1]\);
2. \( H \) increases monotonically on \([0, 1]\);
3. One has the bounds:
   \[
   \inf_{t \in [0, 1]} H(t; g) = H(0; g) = f(\bar{g})
   \]
   \[
   \sup_{t \in [0, 1]} H(t; g) = H(1; g) = \int_a^b p(x)f(g(x))dx.
   \]

We also need to introduce the corresponding double integral mapping \( F : [0, 1] \to \mathbb{R} \) defined by

\[
F(t; g) = \int_a^b \int_a^b p(x)p(y)f(tg(x) + (1-t)g(y))dxdy.
\]

Main results concerning this mapping are as follows:

1. \( F(\tau + \frac{1}{2}; g) = F\left(\frac{1}{2} - \tau; g\right) \) for every \( \tau \in [0, \frac{1}{2}] \)
2. \( F(t; g) = F(1-t; g) \) for every \( t \in [0, 1] \)
3. \( F \) is convex on \([0, 1]\)
4. \( F \) decreases monotonically on \([0, \frac{1}{2}] \) and increases monotonically on \([\frac{1}{2}, 1]\)
5. We have the bounds:
   \[
   \inf_{t \in [0, 1]} F(t; g) = F(0; g) = F(1; g) = \int_a^b p(x)f(g(x))dx
   \]
   \[
   \sup_{t \in [0, 1]} F(t; g) = F\left(\frac{1}{2}; g\right) = \int_a^b \int_a^b p(x)p(y)f\left(\frac{g(x) + g(y)}{2}\right)dxdy.
   \]
3. Main results

The following result gives us upper and lower bounds for the mappings $F$ and $H$ defined in the previous section.

**Theorem 3.1.** Let the conditions stated above hold. Then we have

$$0 \leq t \int_a^b p(x)f'(g(x))dx + (1-t)f(\bar{g}) - H(t;g)$$

and

$$0 \leq \int_a^b p(x)f(g(x))dx - H(t;g)$$

for any $t \in [0,1]$.

**Proof.** Function $f$ is convex, so the following inequality holds

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x,y \in [a,b]$ and for all $t \in [0,1]$. We can first replace $x$ with $g(x)$ and $y$ with $\bar{g}$, and then $x$ with $g(x)$ and $y$ with $g(y)$ in (6) because $g(x), g(y) \in [a,b]$ for all $x,y \in [a,b]$, and get respectively

$$f(tg(x) + (1-t)\bar{g}) \leq tf(g(x)) + (1-t)f(\bar{g})$$

and

$$f(tg(x) + (1-t)g(y)) \leq tf(g(x)) + (1-t)f(g(y)).$$

Multiplying the inequality (7) by $p(x) \geq 0$ and then integrating it over $x$ on $[a,b]$ we get the first inequality in (4) and multiplying the inequality (8) by $p(x) \geq 0$ and $p(y) \geq 0$ and then integrating it over $x$ and $y$ on $[a,b]$ we get the first inequality in (5).
Since the class of convex differentiable functions is dense in the uniform topology in the class of all convex functions defined on the interval \([a, b]\), we can assume that \(f\) is differentiable on \((a, b)\).

If we use the convexity of the function \(f\), we get the gradient inequality

\[
f(u) - f(v) \geq f'(v)(u - v)
\]

for any \(u, v \in (a, b)\).

Because \(tx + (1 - t)y \in (a, b)\) holds for any \(x, y \in (a, b)\) and \(t \in [0, 1]\), from (9) we get

\[
f(tx + (1 - t)y) - f(x) \geq (1 - t)f'(x)(y - x)
\]

and

\[
f(tx + (1 - t)y) - f(y) \geq -t f'(y)(y - x).
\]

Now, if we multiply (10) by \(t\) and (11) by \((1 - t)\), and add together the obtained inequalities, we get

\[
tf(x) + (1 - t)f(y) - f(tx + (1 - t)y)
\]

\[
\leq t(1 - t)[f'(y) - f'(x)](y - x)
\]

for any \(x, y \in (a, b)\) and \(t \in [0, 1]\).

Since \(a \leq g(x), \bar{g} \leq b\), we can replace \(x\) with \(g(x)\) and \(y\) with \(\bar{g}\) in (12), multiply the obtained inequality by \(p(x) \geq 0\) and then integrate it over \(x\) on \([a, b]\) and get

\[
t \int_a^b p(x)f(g(x))dx + (1 - t) \int_a^b p(x)f(\bar{g})dx - \int_a^b p(x)f(tg(x) + (1 - t)\bar{g})dx
\]

\[
\leq t(1 - t) \int_a^b p(x)[f'(\bar{g}) - f'(g(x))]|(\bar{g} - g(x))|dx,
\]

which is equivalent to (4).

Further more, if we replace \(x\) with \(g(x)\) and \(y\) with \(g(y)\) in (12), and then multiply that inequality by \(p(x) \geq 0\) and \(p(y) \geq 0\) and integrate it over \(x\) and \(y\) on \([a, b]\) we can obtain the
following inequality

\[
t \int_a^b \int_a^b p(x)p(y)f(g(x))dx dy + (1-t) \int_a^b \int_a^b p(x)p(y)f(g(y))dx dy
\]

\[
- \int_a^b \int_a^b p(x)p(y)f(tg(x) + (1-t)g(y))dx dy
\]

\[
\leq t(1-t) \int_a^b \int_a^b p(x)p(y)[f'(g(y)) - f'(g(x))](g(y) - g(x))dx dy.
\]

(14)

After some calculations, from (14) we easily get (5), and this completes the proof.

**Remark 3.2.** If we replace \( t \) with \( 1-t \) in (4), add together the obtained results, and then divide it by 2, we get the symmetric inequality

\[
\frac{1}{2} \left[ \int_a^b p(x)f(g(x))dx + f(\bar{g}) \right] - \frac{H(t; g) + H(1-t; g)}{2}
\]

\[
\leq t(1-t) \int_a^b p(x)f'(g(x))dx \left[ \frac{\int_a^b p(x)g(x)f'(g(x))dx}{\int_a^b p(x)f'(g(x))dx} - \bar{g} \right]
\]

(15)

for any \( t \in [0, 1] \).

**Remark 3.3.**

(i) Let the conditions of Theorem 2.1 hold. Then the integral version of the Slater inequality for convex functions found in [1] is valid:

\[
0 \leq \int_a^b p(x)f(g(x))dx - f(\bar{g}) \leq \int_a^b p(x)f'(g(x))(g(x) - \bar{g})dx.
\]

(16)

If we multiply the inequalities in (16) with \( 1-t \) and add it to (4), we get the following inequalities:

\[
0 \leq \int_a^b p(x)f(g(x))dx - H(t; g)
\]

\[
\leq (1-t^2) \int_a^b p(x)f'(g(x))(g(x) - \bar{g})dx.
\]

(17)

(ii) Now, if we subtract the inequalities in (5) from the inequalities in (17) we get

\[
0 \leq F(t; g) - H(t; g)
\]

\[
\leq (1-t^2) \int_a^b p(x)f'(g(x))(g(x) - \bar{g})dx.
\]

(18)
4. Application for potential means

Let \( f, w : [a, b] \to \mathbb{R} \) be positive integrable functions. The potential mean of order \( r \) of a function \( f \) with weight function \( w \) is given by

\[
M_r(f, w) = \left[ \frac{\int_a^b w(x)f(x)^r \, dx}{\int_a^b w(x) \, dx} \right]^{1/r}, \quad r \neq 0
\]

\[
M_0(f, w) = \exp \left[ \frac{\int_a^b w(x) \ln f(x) \, dx}{\int_a^b w(x) \, dx} \right], \quad r = 0
\]

(19)

Let us consider the convex mapping \( f : (0, \infty) \to \mathbb{R}, \ f(x) = x^p, \ p \in (-\infty, 0) \cup (1, \infty) \) and \( 0 < a < b \). We define the mapping

\[
H_p(t; g) = \frac{1}{W} \int_a^b w(x)(tg(x) + (1-t)\bar{g})^p \, dx, \quad t \in [0, 1],
\]

(20)

where \( W = \int_a^b w(x) \, dx \) and \( \bar{g} = \frac{1}{W} \int_a^b w(x)g(x) \, dx \).

It is obvious that \( H_p(0; g) = \frac{1}{W} \int_a^b w(x)\bar{g}^p \, dx = \bar{g}^p \) and \( H_p(1; g) = \frac{1}{W} \int_a^b w(x)g(x) \, dx = M_p^p(g, w) \), and for \( t \in (0, 1) \) and \( p \in \mathbb{N} \) we have

\[
H_p(t; g) = \frac{1}{W} \int_a^b w(x)(tg(x) + (1-t)\bar{g})^p \, dx = \sum_{k=0}^p \binom{p}{i} (tM_i(g, w))^i((1-t)\bar{g})^{p-i}.
\]

Now, consider the function

\[
F_p(t; g) = \frac{1}{W^2} \int_a^b \int_a^b w(x)w(y)(tg(x) + (1-t)g(y))^p \, dx \, dy, \quad t \in [0, 1].
\]

We observe that \( F_p(0; g) = F_p(1; g) = \frac{1}{W} \int_a^b w(x)g(x)^p \, dx = M_p^p(g, w) \) and we can calculate that for \( p \in \mathbb{N} \)

\[
F_p \left( \frac{1}{2}; g \right) = \frac{1}{W^2} \int_a^b \int_a^b w(x)w(y) \left( \frac{g(x) + g(y)}{2} \right)^p \, dx \, dy
\]

\[
= \frac{1}{2^p} \sum_{k=0}^p \binom{p}{i} M_{p-i}^p(g, w)M_i^i(g, w).
\]

Let \( g, w : [a, b] \to \mathbb{R} \) be positive integrable functions and let \( W = \int_a^b w(x) \, dx \) and \( \bar{g} = \frac{1}{W} \int_a^b w(x)g(x) \, dx \).

We define a new weight function \( p : [a, b] \to \mathbb{R} \) with \( p(x) = w(x)/W \). This is a positive, integrable function such that \( \int_a^b p(x) \, dx = 1 \).
Since the function \( f: (0, \infty) \to \mathbb{R}, f(x) = x^p \) is convex for all \( p \in (-\infty, 0) \cup (1, \infty) \), the conditions from Theorem 3.1 are satisfied, and we easily get the following result:

**Theorem 4.1.** Let \( w, g, f \) be as stated above. Then for all \( p \in (-\infty, 0) \cup (1, \infty) \) and for all \( t \in [0, 1] \) we have

\[
0 \leq tM_p^p(g, w) + (1-t)\bar{g}^p - H_p(t; g) \leq pt(1-t)(M_p^p(g, w) - \bar{g}M_p^{p-1}(g, w))
\]

and

\[
0 \leq M_p^p(g, w) - F_p(t; g) \leq 2pt(1-t)(M_p^p(g, w) - \bar{g}M_p^{p-1}(g, w)).
\]

In particular, if we choose \( t = \frac{1}{2} \), we get

\[
0 \leq A(M_p^p(g, w), \bar{g}^p) - H_p(\frac{1}{2}; g) \leq \frac{p}{4}(M_p^p(g, w) - \bar{g}M_p^{p-1}(g, w))
\]

and

\[
0 \leq M_p^p(g, w) - F_p(\frac{1}{2}; g) \leq \frac{p}{2}(M_p^p(g, w) - \bar{g}M_p^{p-1}(g, w)).
\]

where \( A(a, b) = \frac{a+b}{2} \) is the arithmetic mean of the numbers \( a \) and \( b \).

**Conflict of Interests**

The authors declare that there is no conflict of interests.

**References**


