SOME COMPACT GENERALIZATIONS OF ENESTROM-KAKEYA TYPE

THEOREM FOR POLYNOMIALS

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Abstract. In this paper we present an interesting generalization of Enestrom-Kakeya Theorem which among other things yields a number of already known classical results by putting some restrictive conditions on the coefficients of the polynomials.

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1. INTRODUCTION AND STATEMENT OF RESULTS

The following result due to Enestrom-and Kakeya [7] is well-known in the theory of distribution of the zeros of polynomials

THEOREM A. If

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

is a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 \geq 0,$$

then \( P(z) \) does not vanish in \( |z| > 1 \).
Applying this result to \( P(tz) \), the following more general result is immediate.

**THEOREM B.** If

\[
P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0
\]

is a polynomial of degree \( n \) such that

\[
t^n a_n \geq t^{n-1} a_{n-1} \geq \ldots \geq t a_i \geq a_0 \geq 0
\]

then all the zeros of \( P(z) \) lie in \( |z| \leq t \).

In the literature \([1,2,4,5,8]\) there exist some extensions and generalizations of Enestrom-Kakeya Theorem, Joyal, Labellel and Rahman\([6]\) extended this theorem to polynomials whose coefficients were monotonic but not necessarily non-negative. Recently Aziz and Zarger\([3]\) relaxed the hypothesis in several ways and among other things proved the following interesting generalization of Theorem A:

**THEOREM C.** If

\[
P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0
\]

is a polynomial of degree \( n \) such that for some \( k \geq 1 \),

\[
ka_n \geq a_{n-1} \geq \ldots \geq a_i \geq a_0 \geq 0
\]

then \( P(z) \) has all its zeros in

\[(2) \quad |z + k - 1| \leq k\]

In this paper we start by proving the following result which includes Theorems A,B and C as special cases.

**THEOREM 1.** Let

\[
P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0
\]

be a polynomial of degree \( n \), if for some real number \( k \geq 1 \),

\[(3) \quad \max_{|z|=1} \left( |(ka_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \ldots + (a_i - a_0) z^{n-1} + a_0 z^n| \right) \leq M\]

then all the zeros of \( P(z) \) lie in the disk

\[(4) \quad |z + k - 1| \leq \frac{M}{|a_n|}\]

**REMARK 1.** Suppose \( P(z) \) satisfies the conditions of Theorem C, then clearly for \( |z| = 1 \),

\[
\left| (ka_n - a_{n-1}) + (a_{n-1} - a_{n-2}) z + \ldots + a_0 z^n \right|
\]
\[ \leq \left| (ka_n - a_{n-1}) \right| + \left| a_{n-1} - a_{n-2} \right| + \ldots + \left| a_1 - a_0 \right| + \left| a_0 \right| \]
\[ = ka_n - a_{n-1} + a_{n-1} - a_{n-2} + \ldots + a_1 - a_0 + a_0 \]
\[ = ka_n \]

We take \( M = ka_n \) in Theorem 1, it follows that all the zeros of \( P(z) \) satisfying the conditions of Theorem C lie in the circle
\[ \left| z + k - 1 \right| \leq k , \]
which is precisely the conclusion of Theorem C.

The following corollary follows by taking \( k = \frac{a_{n-1}}{a_n} \) in Theorem 1.

**COROLLARY 1.** If \( P(z) = \sum_{j=0}^{n} a_j z^j \), is a polynomial of degree \( n \), such that
\[ a_{n-1} \geq a_n \quad \text{and} \quad \max_{|z|=1} \left| (a_{n-1} - a_{n-2})z + \ldots + a_0 z^n \right| \leq M_1 \]
then all the zeros of \( P(z) \) lie in the disk.
\[ \left| z + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{M_1}{|a_n|} \]

**REMARK 2.** Let
\[ P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 \]
be a polynomial of degree \( n \) satisfying the condition
\[ 0 < a_n \leq a_{n-1} \geq \ldots \geq a_1 \geq a_0 \geq 0 \] (05)
then \( M_1 = \max_{|z|=1} \left| (a_{n-1} - a_{n-2}) + (a_{n-2} - a_{n-3})z + \ldots + a_0 z^n \right| \]
\[ \leq (a_{n-1} - a_{n-2}) + (a_{n-2} - a_{n-3}) + \ldots + a_0 \]
\[ = a_{n-1} . \]

Hence from Corollary 1 it follows that all the zeros of \( P(z) \) satisfying (5) lie in the circle.
\[ \left| z + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{a_{n-1}}{a_n} \]

This result was earlier proved by the authors in [3] Cor. 2.

Theorem 1 follows by taking \( t = 1 \) in the following general result:

**THEOREM 2.** Let
\[ P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 \]
be a polynomial of degree \( n \). If for some positive real numbers \( t \) and \( k \geq 1 \)

\[
(6) \quad \max_{|z|=1} |H(z)| \leq M, \text{ where } \quad H(z) = \left\{ (tka_n - a_{n-1}) + (ta_{n-1} - a_{n-2})z + \ldots + (ta_1 - a_0)z^{n-1} + ta_0z^n \right\}
\]

then all the zeros of \( P(z) \) lie in

\[
(7) \quad |z + t(k-1)| \leq \frac{M}{|a_n|}
\]

Since

\[
M \geq \max_{|z|=1} |H(z)| \geq \left| H\left(\frac{1}{t}\right) \right| \\
= \left| (tka_n - a_{n-1}) + (ta_{n-1} - a_{n-2})\frac{1}{t} + (ta_{n-2} - a_{n-3})\frac{1}{t^2} + \ldots + (ta_1 - a_0)\frac{1}{t^{n-1}} + ta_0\frac{1}{t^n} \right| \\
\geq kta_n
\]

Theorem 2 follows by taking \( R = \frac{1}{t} \) in the following more general result which yields a number of other interesting results for various choices of parameters \( R \) and \( t \):

**THEOREM 3.** Let

\[
P(z) = a_nz^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0
\]

be a polynomial of degree \( n \). If for some positive real numbers \( t \) and \( k \geq 1 \)

\[
(8) \quad \max_{|z|=R} |H(z)| \leq M, \text{ where } \quad H(z) = (ktka_n - a_{n-1}) + (ta_{n-1} - a_{n-2})z + \ldots + ta_0z^n.
\]

then all the zeros of \( P(z) \) lie in the circle

\[
(9) \quad |z + t(k-1)| \leq \frac{M}{|a_n|} \quad \text{if} \quad M \geq \left[ kt + \left( \frac{1}{R} - t \right) \right]|a_n|
\]

and in

\[
(10) \quad |z| \leq t(2k-1) + \left( \frac{1}{R} - t \right) \quad \text{if} \quad M < \left[ kt + \left( \frac{1}{R} - t \right) \right]|a_n|
\]

**PROOF OF THEOREM 3.** Consider the polynomial

\[
F(z) = (t - z)P(z) \\
= (t - z)(a_nz^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0)
\]
\[ = -a_n z^{n+1} + (ta_n - a_{n-1})z^n + (ta_{n-1} - a_{n-2})z^{n-1} + \ldots + (ta_1 - a_0)z + ta_0 \]

then we have

\[ G(z) = z^{n+1}F\left(\frac{1}{z}\right) \]

\[ = -a_n + (ta_n - a_{n-1})z + (ta_{n-1} - a_{n-2})z^2 + \ldots + (ta_1 - a_0)z^n + ta_0z^{n+1} \]

\[ = -a_n + ta_nz - kta_nz + (tka_n - a_{n-1})z + (ta_{n-1} - a_{n-2})z^2 + \ldots + ta_0z^{n+1} \]

\[ = -a_n + ta_nz - kta_nz + zH(z) \]

where

\[ H(z) = \{tka_n - a_{n-1}\} + (ta_{n-1} - a_{n-2})z + \ldots + ta_0z^n \}

Now for \( |z| \leq R \),

\[ (11) \quad |G(z)| \geq |a_n| |1(k - 1)z + 1| - |z| |H(z)| \]  

Since

\[ |H(z)| \leq M \quad \text{for} \quad |z| = R, \quad \text{and} \quad H(z) \quad \text{is analytical for} \quad |z| \leq R, \quad \text{therefore by Maximum Modulus Theorem} \]

\[ |H(z)| \leq M \quad \text{for} \quad |z| \leq R. \]

Using this fact in (11), we get

\[ |G(z)| \geq |a_n| |t(k - 1)z + 1| - |z|M. \]

\[ > 0, \quad \text{if} \]

\[ \frac{M}{a_n} |z| < |t(k - 1)z + 1| \quad \text{for} \quad |z| \leq R. \]

Thus in \( |z| \leq R \),

\[ |G(z)| > 0 \quad \text{for} \quad z \in E, \]

where

\[ E = \left\{ z : \frac{M}{|a_n|} |z| < |t(k - 1)z + 1| \right\} \]

we show if \( w \in E, \) then \( |w| \leq R \) if

\[ M \geq \left( kt + \frac{1}{R - t} \right)|a_n| \]

Let \( W \in E, \text{then} \)

\[
\frac{M}{|a_n|} |w| < |t(k-1)w + 1|
\]
\[
< t(k-1)|w| + 1
\]

This implies,
\[
\left\{ \frac{M}{|a_n|} - t(k-1) \right\} |w| < 1
\]
or
\[
|w| < \frac{|a_n|}{M - t(k-1)|a_n|} \leq R
\]

if
\[
|a_n| \leq MR - t(k-1)|a_n|R
\]
or if,
\[
M \geq \frac{|a_n|}{R} \{Rt(k-1) + 1\} = |a_n| \left\{ kt + \frac{1}{R} - t \right\}
\]

Thus if
\[
M \geq \left( kt + \left( \frac{1}{R} - t \right) \right) |a_n|
\]
then in \(|z| \leq R, \)
\[
|G(z)| > 0, \text{if}
\]
\[
\frac{M}{a_n} |z| < |t(k-1)z + 1|
\]

This shows that all the zeros of \(G(z)\) lie in the region defined by
\[
\frac{M}{|a_n|} |z| \geq |t(k-1)z + 1|
\]
Replacing \(z\) by \(\frac{1}{z}\) and noting that \(F(z) = z^{-n+1}G(z)\), it follows that all the zeros of \(F(z)\) lie in
\[
|z + t(k-1)| \leq \frac{M}{|a_n|}.
\]

Since all the zeros of \(P(z)\) are also the zeros of \(F(z)\), we conclude that all the zeros of \(P(z)\) lie in
(12) \[ |z + t(k - 1)| \leq \frac{M}{a_n} \quad \text{if} \quad M \geq \left\{ kt + \left( \frac{1}{R} - t \right) \right\} |a_n|. \]

We now assume that

\[ M < \left( \frac{Rt(k - 1) + 1}{R} \right) a_n \]

then for \( |z| \leq R \),

\[
\begin{align*}
|G(z)| &= |a_n + (ta_n - kta_n)z + zH(z)| \\
&\geq |a_n| \left\{ 1 - |z| \left( t(k - 1) + \left| \frac{H(z)}{a_n} \right| \right) \right\} \\
&\geq |a_n| - |z| \left( t(k - 1) + \frac{M}{|a_n|} \right) \\
&\geq |a_n| \left\{ 1 - |z| \left( t(k - 1) + \left( \frac{1}{R} - t \right) + tk \right) \right\} \\
&> 0, \quad \text{if} \quad |z| < \frac{1}{t(2k - 1) + \left( \frac{1}{R} - t \right)}(\leq R).
\end{align*}
\]

This shows that all the zeros of \( G(z) \) lie in the region

\[
|z| \geq \frac{1}{t(2k - 1) + \left( \frac{1}{R} - t \right)}
\]

Replacing \( z \) by \( \frac{1}{z} \) and as before noting that

\[ F(z) = z^{n+3} G \left( \frac{1}{z} \right) \]

it follows that all the zeros of \( F(z) \) and hence all the zeros of \( P(z) \) lie in the circle

(13) \[ |z| \leq t(2k - 1) + \left( \frac{1}{R} - t \right) \]

if

\[ M < \left( \left( \frac{1}{R} - t \right) + kt \right) |a_n| \]
From (12) and (13), the desired result follows.

REFERENCES