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GENERALIZED OSTROWSKI INEQUALITY WITH APPLICATIONS IN NUMERICAL INTEGRATION AND SPECIAL MEANS

MUHAMMAD AWAIS SHAIKH*, ASIF R. KHAN, NAZIA IRSHAD

Department of Mathematics, University of Karachi, Karachi 75270, Pakistan

Abstract. Generalization of Ostrowski inequality with applications in numerical integration and special means are recognized for differentiable functions up to second order whose second order derivatives are bounded and first derivatives are absolutely continuous.

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1. Introduction

In 1938, A. Ostrowski proved an inequality concerning function with bounded derivative which is known as Ostrowski inequality [10]. The inequality is stated as follows:

proposition 1.1. Let $\psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $(\rho, \flat)$ and let on $(\rho, \flat)$,

*Corresponding author

E-mail address: m.awaisshaikh2014@gmail.com

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$|\psi'(v)| \leq M$ for some positive real number M. Then for all $v \in [\rho, \flat]$,

$$\left|\psi(v) - \frac{1}{\flat - \rho}\int_{\rho}^{\flat} \psi(\dagger)d\dagger\right| \leq \left[\frac{1}{4} + \frac{(v - \frac{\rho+\flat}{2})^2}{(\flat - \rho)^2}\right](\flat - \rho)M. \tag{1.1}$$

The constant $\frac{1}{4}$ is the better possible in the sense that it cannot be replaced by the smaller one. The explanation of Ostrowski inequality can be demonstrate in two ways as follows:

(1) Estimate the deviation from its average value to functional value.
(2) Estimate the approximating area under the curve of a rectangle.

In last few decades, some work has been done on the generalizations of Ostrowski's inequality. Some examples are mentioned in [4,7,11,2]. In [3], an integral inequality has established of Ostrowski type by S. S. Dragomir et al. for mappings with bounded second derivatives. S. S. Dragomir et al. in [5], established a similar inequality. In [5], S. S. Dragomir and N. S. Barnett, an indicated Ostrowski type integral inequality give a same sense to that of [3] or [5].

In order to recall some results we need here some definitions which can be found in [15, pp. 125, 128].

Let $L_p[\rho, \flat](1 \leq p < 1)$ denotes the space of p-power integrable functions on the interval $[\rho, \flat]$ with the standard norm

$$\|\psi\|_p = \left(\int_{\rho}^{\flat} |\psi(\dagger)|^p d\dagger\right)^{\frac{1}{p}}$$

and $L_1[\rho, \flat]$ denotes the space of all essentially bounded functions on $[\rho, \flat]$ with the norm

$$\|\psi\|_p = ess \sup_{v \in [\rho, \flat]} |\psi'(v)| < \infty$$

In [12], S. S. Dragomir et al. proved the following generalization of Ostrowski in- equality:

proposition 1.2. Let $\psi : [\rho, \flat] \to \mathbb{R}$ be a function continuous on $[\rho, \flat]$ and differentiable on $(\rho, \flat)$. Assume that $|\psi'(v)| \leq$ M for $v \in (\rho, \flat)$ and M is positive real constant and denote

$$\|\psi'\|_\infty := \sup_{\dagger \in [\rho, \flat]} |\psi'(t)| < \infty$$

Then the inequality

$$\left|(\flat-\rho)\left[\lambda\frac{\psi(\rho)+\psi(\flat)}{2}+(1-\lambda)\psi(v)\right]-\int_{\rho}^{\flat}\psi(\dagger)d\dagger\right| \\ \leq\left[\frac{(\flat-\rho)^2}{4}(\lambda^2+(1-\lambda)^2)+\left(v-\frac{\rho+\flat}{2}\right)^2\right]\|\psi\|_{\infty}. \tag{1.2}$$

holds for all $\lambda\in[0,1]$ and $\rho+\lambda\frac{\flat-\rho}{2}\leq v\leq\flat-\lambda\frac{\flat-\rho}{2}$.

According to [13] and [14]

$$A_1(\psi):\frac{1}{\flat-\rho}\int_{\rho}^{\flat}\psi(v)dv\cong\psi\left(\frac{\rho+\flat}{2}\right)$$

$$A_2(\psi):\frac{1}{\flat-\rho}\int_{\rho}^{\flat}\psi(v)dv\cong\left(\frac{\psi(\rho)+\psi(\flat)}{2}\right)$$

$$A_3(\psi):\frac{1}{\flat-\rho}\int_{\rho}^{\flat}\psi(v)dv\cong\frac{1}{3}\left[\frac{\psi(\rho)+\psi(\flat)}{2}+2\left(\frac{\rho+\flat}{2}\right)\right]$$

$$A_4(\psi):\frac{1}{\flat-\rho}\int_{\rho}^{\flat}\psi(v)dv\cong\frac{3}{8}\left[\frac{\psi(\rho)+\psi(\flat)}{3}+\psi\left(\frac{2\rho+\flat}{3}\right)+\psi\left(\frac{\rho+2\flat}{3}\right)\right]$$

$$A_5(\psi):\frac{1}{\flat-\rho}\int_{\rho}^{\flat}\psi(v)dv\cong\frac{1}{2}\left[\psi(\flat)-\psi(\rho)+2\psi\left(\frac{\rho+\flat}{2}\right)\right)$$

$$A_6(\psi):\frac{1}{\flat-\rho}\int_{\rho}^{\flat}\psi(v)dv\cong\frac{1}{2}\left[\psi(\rho)-\psi(\flat)+2\psi\left(\frac{\rho+\flat}{2}\right)\right)$$

$$A_7(\psi):\frac{1}{\flat-\rho}\int_{\rho}^{\flat}\psi(v)dv\cong\psi(\rho)$$

$$A_8(\psi):\frac{1}{\flat-\rho}\int_{\rho}^{\flat}\psi(v)dv\cong\psi(\flat)$$

In [16], Fiza Zafar and Nazir Ahmad Mir proved the following generalization of Ostrowski in equality:

proposition 1.3. Let $\phi:[\rho,\flat]\to\mathbb{R}$ be a mapping whose first derivative is absolutely continuous on $[\rho,\flat]$ and suppose that the second derivative $\phi''\in L_{\infty}(\rho,\flat)$. Then, we have the inequality for all $v\in[\rho,\flat]$.

$$\left|\frac{1}{(\flat-\rho)}\int_{\rho}^{\flat}\phi(\dagger)d\dagger-\frac{1}{2}\left[(1-\lambda)\phi(v)+(1+\lambda)\left(\frac{\phi(\rho)+\phi(\flat)}{2}\right)\right.\right. \\ \left.-(1-\lambda)\left(v-\frac{\rho+\flat}{2}\right)\phi'(v)-\lambda\frac{(\flat-\rho)}{4}(\phi'(\flat)-\phi'(\rho))\right] \\ \leq\|\phi''\|_{\infty}\left[\frac{1}{3}\left|\left(v-\frac{\rho+\flat}{2}\right)^3\right|+\frac{(\flat-\rho)^3}{48}(3\lambda+2(1-\lambda)^3-1)\right]. \tag{1.3}$$

The objective of this paper is to further generalized Ostrowski inequality than inequality given in Proposition (1.3).

2. Main results

Theorem 2.1. *Let all the assumptions of Propositions (1.3) are valid. Then, we have the inequality.*

$$\begin{aligned}
&\left|\frac{1}{(\flat-\rho)}\int_{\rho}^{\flat}\phi(\dagger)d\dagger-\frac{1}{2}\left[(1-\lambda)\left(\frac{\phi(v)+\phi(\rho+\flat-v)}{2}\right)\right.\right.\\
&-(1-\lambda)\left(v-\frac{\rho+\flat}{2}\right)\left(\frac{\phi'(v)+\phi'(\rho+\flat-v)}{2}\right)-\lambda\frac{(\flat-\rho)}{4}(\phi'(\flat)-\phi'(\rho))\\
&\left.\left.+(1+\lambda)\left(\frac{\phi(\rho)+\phi(\flat)}{2}\right)\right]\right| \qquad (2.1)\\
&\leq\frac{\|\phi''\|_{\infty}}{(\flat-\rho)}\left[\frac{1}{3}\left(\frac{\rho+\flat}{2}-v\right)^{3}+\frac{(\flat-\rho)^{3}}{48}(3\lambda+2(1-\lambda)^{3}-1)\right.\\
&\left.+\frac{1}{48}\left((\flat-\rho)(3\lambda-1)+4(v+\rho)\right)\left(\frac{\rho+\flat}{2}-v\right)^{2}\right]
\end{aligned}$$

or equivalently,

$$\begin{aligned}
&\left|\int_{\rho}^{\flat}\phi(\dagger)d\dagger-\frac{(\flat-\rho)}{2}\left[(1-\lambda)\left(\frac{\phi(v)+\phi(\rho+\flat-v)}{2}\right)\right.\right.\\
&-(1-\lambda)\left(v-\frac{\rho+\flat}{2}\right)\left(\frac{\phi'(v)+\phi'(\rho+\flat-v)}{2}\right)-\lambda\frac{(\flat-\rho)}{4} \qquad (2.2)\\
&\left.\left.\times(\phi'(\flat)-\phi'(\rho))+(1+\lambda)\left(\frac{\phi(\rho)+\phi(\flat)}{2}\right)\right]\right|
\end{aligned}$$

$$\begin{aligned}
&\leq\|\phi''\|_{\infty}\left[\frac{1}{3}\left(\frac{\rho+\flat}{2}-v\right)^{3}+\frac{(\flat-\rho)^{3}}{48}(3\lambda+2(1-\lambda)^{3}-1)\right.\\
&\left.+\frac{1}{48}\left((\flat-\rho)(3\lambda-1)+4(v+\rho)\right)\left(\frac{\rho+\flat}{2}-v\right)^{2}\right].
\end{aligned}$$

$$\forall v\in\left[\rho+\lambda\frac{(\flat-\rho)}{2},\flat-\lambda\frac{(\flat-\rho)}{2}\right].$$

Proof. Consider the kernel $K : [\rho, \flat]^2 \to \mathbb{R}$, as defined in [4],

$$K(v,\dagger) = \begin{cases} \dagger - (\rho + \lambda \frac{(\flat-\rho)}{2}), & \text{if} \quad \dagger \in [\rho, v], \\ \\ \dagger - (\frac{\rho+\flat}{2}), & \text{if} \quad \dagger \in [v, \rho+\flat-v], \\ \\ \dagger - (\flat - \lambda \frac{(\flat-\rho)}{2}), & \text{if} \quad \dagger \in (\rho+\flat-v, \flat] \end{cases}$$

which gives,

$$\begin{aligned} &\int_\rho^\flat K(v,\dagger) f'(\dagger) d\dagger \\ &= (\flat - \rho)\left[\lambda \frac{\psi(\rho)+\psi(\flat)}{2} + (1-\lambda)\frac{\psi(v)+\psi(\rho+\flat-v)}{2}\right] \\ &- \int_\rho^\flat \psi(\dagger) d\dagger, \end{aligned} \tag{2.3}$$

which can be written as,

$$\begin{aligned} \left[\frac{\psi(v)+\psi(\rho+\flat-v)}{2}\right] &= \frac{1}{(1-\lambda)}\left[\frac{1}{(\flat-\rho)}\int_\rho^d \psi(\dagger) d\dagger \right. \\ &\left. -\frac{\lambda}{2}(\psi(\rho)+\psi(\flat))\right] + \frac{1}{(\flat-\rho)(1-\lambda)}\int_\rho^\flat K(v,\dagger)\psi'(\dagger) d\dagger, \end{aligned}$$

this implies that,

$$\begin{aligned} &\left[\frac{\psi(v)+\psi(\rho+\flat-v)}{2}\right](1-\lambda) \\ &= \frac{1}{(\flat-\rho)}\int_\rho^\flat \phi(\dagger) d\dagger - \frac{\lambda}{2}(f(\rho)+\psi(\flat)) + \frac{1}{(\flat-\rho)}\int_\rho^\flat K(v,\dagger)\psi'(\dagger) d\dagger, \end{aligned} \tag{2.4}$$

for all $v \in \left[\rho + \lambda \frac{\flat-\rho}{2}, \frac{\rho+\flat}{2}\right], \lambda \in [0,1]$ (provided ψ is absolutely continuous on $[\rho, \flat]$).

Now, we substitute

$$\psi(v) = \left(v - \frac{\rho+\flat}{2}\right)\phi'(v),$$

in (2.4) to get

$$\begin{aligned} &\frac{(1-\lambda)}{2}\left[\left(v - \frac{\rho+\flat}{2}\right)(\phi'(v)+\phi'(\rho+\flat-v))\right] \\ &= \frac{1}{(\flat-\rho)}\int_\rho^\flat \left(\dagger - \frac{\rho+\flat}{2}\right)\phi'(\dagger) d\dagger - \frac{\lambda}{4}(\flat-\rho)(\phi'(\flat)-\phi'(\rho)) \\ &+ \frac{1}{(\flat-\rho)}\int_\rho^\flat K(v,\dagger)\left[\phi'(\dagger) + \left(\dagger - \frac{\rho+\flat}{2}\right)\phi''(\dagger)\right] d\dagger. \end{aligned} \tag{2.5}$$

Integrating by parts, we have

$$\int_{\rho}^{\flat}\left(\dagger-\frac{\rho+\flat}{2}\right)\phi^{'}(\dagger)d\dagger=(\flat-\rho)\left(\frac{\phi(\rho)+\phi(\flat)}{2}\right)-\int_{\rho}^{\flat}\phi(\dagger)d\dagger. \tag{2.6}$$

Using (2.3) and (2.6) in (2.5), we get:

$$\begin{aligned}
&\frac{(1-\lambda)}{2}\left[\left(v-\frac{\rho+\flat}{2}\right)\left(\phi'(v)+\phi'(\rho+\flat-v)\right)\right]\\
&=\frac{\lambda}{2}[\phi(\rho)+\phi(\flat)]-\lambda\frac{(\flat-\rho)}{4}[\phi^{'}(\flat)-\phi^{'}(\rho)]\\
&+\frac{(1-\lambda)}{2}[\phi(v)+\phi(\rho+\flat-v)]\\
&-\frac{2}{(\flat-\rho)}\int_{\rho}^{\flat}\phi(\dagger)d\dagger+\frac{1}{(\flat-\rho)}\int_{\rho}^{\flat}K(v,\dagger)\left(\dagger-\frac{\rho+\flat}{2}\right)\phi''(\dagger)d\dagger
\end{aligned}$$

which can be written as

$$\begin{aligned}
&\frac{1}{(\flat-\rho)}\int_{\rho}^{\flat}\phi(\dagger)d\dagger=\frac{(1+\lambda)}{4}[\phi(\rho)+\phi(\flat)]-\lambda\left(\frac{\flat-\rho}{8}\right)[\phi'(\flat)-\phi'(\rho)]\\
&+\left(\frac{1-\lambda}{4}\right)[\phi(v)+\phi(\rho+\flat-v)]-\frac{(1-\lambda)}{4}\left(v-\frac{\rho+\flat}{2}\right)[\phi'(v)\\
&+\phi'(\rho+\flat-v)]-\frac{1}{2(\flat-\rho)}\int_{\rho}^{\flat}K(v,\dagger)\left(\dagger-\frac{\rho+\flat}{2}\right)\phi''(\dagger)d\dagger
\end{aligned}$$

and we obtain,

$$\begin{aligned}
&\left|\frac{1}{(\flat-\rho)}\int_{\rho}^{\flat}\phi(\dagger)d\dagger-\frac{1}{2}\left[-\lambda\left(\frac{\flat-\rho}{4}\right)\right.\right.\\
&\times\left[\phi'(\flat)-\phi'(\rho)\right]+\frac{1-\lambda}{2}[\phi(v)+\phi(\rho+\flat-v)]+\left(\frac{1+\lambda}{2}\right)[\phi(\rho)+\phi(\flat)]\\
&\left.\left.-\frac{1-\lambda}{2}\left(v-\frac{\rho+\flat}{2}\right)[\phi^{'}(v)+\phi'(\rho+\flat-v)]\right]\right|\\
&=\left|\frac{1}{2(\flat-\rho)}\int_{\rho}^{\flat}K(v,\dagger)\left(\dagger-\frac{\rho+\flat}{2}\right)\phi''(\dagger)d\dagger\right|\\
&\leq\frac{1}{2(\flat-\rho)}\int_{\rho}^{\flat}|K(v,\dagger)|\left|\dagger-\frac{\rho+\flat}{2}\right||\phi''(\dagger)|d\dagger,
\end{aligned} \tag{2.7}$$

also

$$\int_{\rho}^{\flat}|K(v,\dagger)|\left|\dagger-\frac{\rho+\flat}{2}\right||\phi''(\dagger)|d\dagger\leq\|\phi''\|_{\infty}\int_{\rho}^{\flat}|K(v,\dagger)|\left|\dagger-\frac{\rho+\flat}{2}\right|d\dagger \tag{2.8}$$

where

$$\|\phi''\|_\infty = \sup_{\dagger\in(\rho,\flat)} \in |\phi''| < \infty,$$

now, we let

$$I = \int_\rho^\flat |K(v,\dagger)| \left|\dagger - \frac{\rho+\flat}{2}\right| d\dagger,$$

or

$$\begin{aligned} I &= \int_\rho^v \left|\left(\dagger - \frac{\rho+\flat}{2}\right)\right| \left|\left(\dagger - \left(\rho + \lambda\left(\frac{\rho+\flat}{2}\right)\right)\right)\right| d\dagger \\ &+ \int_v^{\rho+\flat-v} \left|\left(\dagger - \frac{\rho+\flat}{2}\right)^2\right| d\dagger \\ &+ \int_{\rho+\flat-v}^b \left|\left(\dagger - \left(b - \lambda\left(\frac{\flat-\rho}{2}\right)\right)\right)\right| \left|\left(\dagger - \frac{\rho+\flat}{2}\right)\right| d\dagger. \end{aligned} \tag{2.9}$$

For $v \in [\rho + \lambda\frac{\flat-\rho}{2}, \frac{\rho+\flat}{2}]$ we get:

$$\begin{aligned} I &= \int_\rho^{\rho+\lambda\frac{\flat-\rho}{2}} \left(\left(\rho+\lambda\left(\frac{\flat-\rho}{2}\right)\right) - \dagger\right)\left(\frac{\rho+\flat}{2} - t\right) d\dagger \\ &+ \int_{\rho+\lambda\frac{\flat-\rho}{2}}^v \left(\dagger - \left(\rho+\lambda\left(\frac{\flat-\rho}{2}\right)\right)\right)\left(\frac{\rho+\flat}{2} - \dagger\right) d\dagger \\ &+ \int_v^{\rho+\flat-v} \left(\frac{\rho+\flat}{2} - \dagger\right)\left(\frac{\rho+\flat}{2} - \dagger\right) d\dagger \\ &+ \int_{\rho+\flat-v}^{\flat-\lambda\frac{(\flat-\rho)}{2}} \left(\left(b-\lambda\left(\frac{\flat-\rho}{2}\right)\right) - \dagger\right)\left(\dagger - \frac{\rho+\flat}{2}\right) d\dagger \\ &+ \int_{\flat-\lambda\frac{(\flat-\rho)}{2}}^b \left(\dagger - \left(b-\lambda\left(\frac{\flat-\rho}{2}\right)\right)\right)\left(\dagger - \frac{\rho+\flat}{2}\right) d\dagger. \end{aligned}$$

After some simple calculations,we get

$$\begin{aligned} I &= \frac{2}{3}\left(\frac{\rho+\flat}{2} - v\right)^3 + \frac{(\flat-\rho)^3}{24}[3\lambda + 2(1-\lambda)^3 - 1] \\ &+ \frac{1}{24}((\flat-\rho)(3\lambda-1) + 4(v+4))\left(\frac{\rho+\flat}{2} - v\right)^2 \\ &\forall\, v \in \left[\rho + \lambda\frac{(\flat-\rho)}{2}, \frac{(\rho+\flat)}{2}\right], \end{aligned} \tag{2.10}$$

Using (2.8), (2.9), and (2.10) in (2.7), we get

$$\left|\frac{1}{\flat-\rho}\int_\rho^\flat \phi(\dagger)d\dagger - \frac{1}{2}\left[(1-\lambda)\left(\frac{\phi(v)+\phi(\rho+\flat-v)}{2}\right)\right.\right.$$
$$-(1-\lambda)\left(v-\frac{\rho+\flat}{2}\right)\left(\frac{\phi'(v)+\phi'(\rho+\flat-v)}{2}\right)-\lambda\frac{(\flat-\rho)}{4}(\phi'(\flat)-\phi'(\rho))$$
$$\left.\left.+(1+\lambda)\left(\frac{\phi(\rho)+\phi(\flat)}{2}\right)\right]\right|$$

$$\leq \frac{\|\phi''\|_\infty}{2(\flat-\rho)}\left[\frac{2}{3}\left(\frac{\rho+\flat}{2}-v\right)^3+\frac{(\flat-\rho)^3}{24}(3\lambda+2(1-\lambda)^3-1)\right.$$
$$\left.+\frac{1}{24}((\flat-\rho)(3\lambda-1)+4(v+\rho))\left(\frac{\rho+\flat}{2}-v\right)^2\right].$$
$$=\frac{\|\phi''\|_\infty}{(\flat-\rho)}\left[\frac{1}{3}\left(\frac{\rho+\flat}{2}-v\right)^3+\frac{(\flat-\rho)^3}{48}(3\lambda+2(1-\lambda)^3-1)\right.$$
$$\left.+\frac{1}{48}((\flat-\rho)(3\lambda-1)+4(v+\rho))\left(\frac{\rho+\flat}{2}-v\right)^2\right].$$

This completes the proof.

Remark 2.2. *Putting* $\lambda = 0$ *in* (2.2)*, then we get.*

$$\begin{aligned}
&\left|\int_\rho^\flat \phi(\dagger)d\dagger - \frac{(\flat-\rho)}{2}\left[\left(\frac{\phi(v)+\phi(\rho+\flat-v)}{2}\right)\right.\right.\\
&\left.\left.-\left(v-\frac{\rho+\flat}{2}\right)\left(\frac{\phi'(v)+\phi'(\rho+\flat-v)}{2}\right)+\left(\frac{\phi(\rho)+\phi(\flat)}{2}\right)\right]\right|\\
&\leq \|\phi''\|_\infty\left[\frac{1}{3}\left(\frac{\rho+\flat}{2}-v\right)^3+\frac{(\flat-\rho)^3}{48}\right.\\
&\left.+\frac{1}{48}\left(4(v+\rho)-(\flat-\rho)\right)\left(\frac{\rho+\flat}{2}-v\right)^2\right].
\end{aligned} \tag{2.11}$$

Remark 2.3. *In (1.3), if we examine the estimates for the end points* $v=\rho, v=\flat$ *and the midpoint* $v=\frac{\rho+\flat}{2}$*, we find that the midpoint which gives the better approximation, so that from* *inequality (2.2), we have*

$$\begin{aligned}
&\left|\int_{\rho}^{\flat} \phi(\dagger)d\dagger - \frac{(\flat-\rho)}{2}\left[(1-\lambda)\left(\frac{\phi(v)+\phi(\rho+\flat-v)}{2}\right)\right.\right. \\
&-(1-\lambda)\left(v-\frac{\rho+\flat}{2}\right)\left(\frac{\phi'(v)+\phi'(\rho+\flat-v)}{2}\right) \\
&\left.\left.-\lambda\frac{(\flat-\rho)}{4}(\phi'(\flat)-\phi'(\rho))+(1+\lambda)\left(\frac{\phi(\rho)+\phi(\flat)}{2}\right)\right]\right| \\
&\leq \|\phi''\|_{\infty}\frac{(\flat-\rho)^3}{48}(3\lambda+2(1-\lambda)^3-1)
\end{aligned} \tag{2.12}$$

.

Remark 2.4. *Choosing $\lambda = \frac{1}{3}$ in the inequality (2.12) gives the better approximation:*

$$\begin{aligned}
&\left|\int_{\rho}^{\flat} \phi(\dagger)d\dagger - \frac{(\flat-\rho)}{2}\left[\frac{2}{3}\phi\left(\frac{\rho+\flat}{2}\right)-\frac{(\flat-\rho)}{12}(\phi'(\flat)-\phi'(\rho))\right.\right. \\
&\left.\left.+\frac{2}{3}(\phi(\rho)+\phi(\flat))\right]\right| \\
&\leq \|\phi''\|_{\infty}\frac{(\flat-\rho)^3}{81}
\end{aligned}$$

which has a new and a better approximation than the three-point quadrature inequalities. .

Remark 2.5. *If we select $\lambda = 1$ in the inequality (2.12), we get a perturbed trapezoid inequality as follows*

$$\begin{aligned}
&\left|\int_{\rho}^{\flat} \phi(\dagger)d\dagger - (\flat-\rho)\left(\frac{\phi(\rho)+\phi(\flat)}{2}\right)-\frac{(\flat-\rho)^2}{8}(\phi'(\flat)-\phi'(\rho))\right| \\
&\leq \|\phi''\|_{\infty}\frac{(\flat-\rho)^3}{24}.
\end{aligned} \tag{2.13}$$

which has a better approximation than the perturbed trapezoid quadrature inequalities mentioned in [5] *and* [12] *for $\|\cdot\|_{\infty}$ norm.*

Remark 2.6. *If we select $\lambda = \frac{1}{4}$ in the inequality* (2.12), *we get a perturbed trapezoid inequality as follows*

$$\begin{aligned}
&\left|\int_{\rho}^{\flat} \phi(\dagger)d\dagger - \frac{(\flat-\rho)}{8}\left[3\phi\left(\frac{\rho+\flat}{2}\right)+5\left(\frac{\phi(\rho)+\phi(\flat)}{2}\right)\right.\right. \\
&\left.\left.-\frac{(\flat-\rho)}{4}(\phi'(\flat)-\phi'(\rho))\right]\right| \\
&\leq \|\phi''\|_{\infty}\frac{19(\flat-\rho)^3}{1536}
\end{aligned} \tag{2.14}$$

which has a new and a better approximation than the perturbed trapezoid quadrature inequalities..

Remark 2.7.*If we select $\lambda = \frac{3}{4}$ in the inequality (2.12), we get a perturbed trapezoid inequality as follows*

$$\left| \int_{\rho}^{\flat} \phi(t) d\dagger - \frac{(\flat-\rho)}{2} \left[\frac{1}{4}\phi\left(\frac{\rho+\flat}{2}\right) - \frac{3(\flat-\rho)}{16}(\phi'(\flat)-\phi'(\rho)) \right.\right.$$
$$\left.\left. + \frac{7}{8}(\phi(\rho)+\phi(\flat)) \right] \right| \leq \|\phi''\|_{\infty} \frac{41(\flat-\rho)^3}{1536}$$

which has a new and a better approximation than the perturbed trapezoid quadrature inequalities. .

Remark 2.8.*If we select $\lambda = \frac{3}{10}$ in the inequality (2.12), we get a perturbed trapezoid inequality as follows*

$$\left| \int_{\rho}^{\flat} \phi(\dagger) \flat\dagger - \frac{(\flat-\rho)}{20} \left[7\phi\left(\frac{\rho+\flat}{2}\right) - \frac{3(\flat-\rho)}{4}(\phi'(\flat)-\phi'(\rho)) \right.\right.$$
$$\left.\left. + \frac{13}{2}(\phi(\rho)+\phi(\flat)) \right] \right| \leq \|\phi''\|_{\infty} \frac{293(\flat-\rho)^3}{24000}$$

which has a better approximation than the perturbed trapezoid quadrature inequalities presented in [2] *and* [5] $\|.\|_{\infty}$ *norm.*

3. Applications in numerical Integration

Using the inequality (2.1), we get the approximation of composite quadrature rules with smaller error found by the classical results.

Theorem 3.1. *Let $I_n : \rho = v_0 < v_1 < v_2 < \cdots\cdots < v_n - 1 < v_n = \flat$ be a partition of the interval* $[\rho, \flat]$,
$z_i = v_{i+1} - v_i, \lambda \in [0,1], v_i + \lambda\frac{z_i}{2} \leq \xi_i \leq v_i + 1 - \lambda\frac{z_i}{2}, i = 0, \cdots\cdots, n-1$, *then*

$$\int_{\rho}^{\flat} \phi(\dagger) d\dagger = S(\phi, \phi', I_n, \xi, \lambda) + R(\phi, \phi', I_n, \xi, \lambda),$$

where

$$
\begin{aligned}
&S(\phi,\phi',I_n,\xi,\lambda) \\
&= \frac{1}{2}\sum_{i=0}^{n-1}\left[(1-\lambda)\left(\frac{\phi(\xi_i)+\phi(v_i+v_{i+1}-\xi_i)}{2}\right)\right. \\
&-(1-\lambda)\left(\xi_i-\frac{v_i+v_{i+1}}{2}\right)\left(\frac{\phi'(\xi_i)+\phi'(v_i+v_{i+1}-\xi_i)}{2}\right) \\
&\left.-\ \lambda\frac{z_i}{4}(\phi'(v_{i+1})-\phi'(v_i))+(1+\lambda)\left(\frac{\phi(v_i)+\phi(v_{i+1})}{2}\right)\right] z_i
\end{aligned}
\tag{3.1}
$$

and

$$
\begin{aligned}
&|R(\phi,\phi',I_n,\xi,\lambda)| \\
&\leq \|\phi''\|_\infty\left[\sum_{i=0}^{n-1}\frac{1}{3}\left(\xi_i-\frac{v_i+v_{i+1}}{2}\right)^3\right. \\
&+\frac{z_i^3}{48}(3\lambda+2(1-\lambda)^3+1)+\frac{1}{48}(z_i(3\lambda-1) \\
&\left.+\ 4(\xi_i+v_i))\left(\xi_i-\frac{v_i+v_{i+1}}{2}\right)^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&|R(\phi,\phi',I_n,\xi,\lambda)| \\
&= \|\phi''\|_\infty\left[\sum_{i=0}^{n-1}\frac{1}{3}\left(\xi_i-\frac{v_i+v_{i+1}}{2}\right)^3+\frac{(3\lambda+2(1-\lambda)^3-1)}{48}\sum_{i=0}^{n-1}z_i^3\right. \\
&\left.+\frac{1}{48}\left(\sum_{i=0}^{n-1}z_i(3\lambda-1)\right)\ +4(\xi_i+v_i)\left(\xi_i-\frac{v_i+v_{i+1}}{2}\right)^2\right)\right]
\end{aligned}
\tag{3.2}
$$

Proof. Applying inequality (2.10) on $\xi_i \in [v_i+\lambda\frac{z_i}{2}, v_{i+1}-\lambda\frac{z_i}{2}]$ and summing over i from 0 to $n-1$ and using triangular inequality, we get (3.2).

Remark 3.2.*By choosing* $\lambda = 0$ *gives as a special case* [10]*, the modified version of approximations of composite quadrature rules.*

Corollary 3.3.*For* $\xi_i = \frac{v_i+v_{i+1}}{2}$*,* $(i=0,\cdots\cdots,n-1)$*, then we have the following quadrature rule:*

$$
\begin{aligned}
S(\phi,\phi',I_n,\lambda) &= \frac{1}{2}\sum_{i=0}^{n-1}\left[(1-\lambda)\left(\frac{v_i+v_{i+1}}{2}\right)\right. \\
&\left.-\ z_i\frac{\lambda}{4}(\phi'(v_{i+1})-\phi'(v_i))+(1+\lambda)\left(\frac{\phi(v_i)+\phi(v_{i+1})}{2}\right)\right] z_i
\end{aligned}
\tag{3.3}
$$

and

$$\begin{aligned} &|R(\phi,\phi',I_n,\lambda)| \\ &\leq \left(\frac{\|\phi''\|_\infty}{48}\right)(3\lambda+2(1-\lambda)^3+1)\sum_{i=0}^{n-1}(z_i)^3, \lambda\in[0,1]. \end{aligned} \tag{3.4}$$

Remark 3.4. *If we select $\lambda=0$ in (3.3) and (3.4), $(i=0,\cdots\cdots,n-1)$, then*

$$S(\phi,I_n)=\frac{1}{2}\sum_{i=0}^{n-1}\left[\left(\frac{v_i+v_{i+1}}{2}\right)+\left(\frac{\phi(v_i)+\phi(v_{i+1})}{2}\right)\right]z_i \tag{3.5}$$

and

$$R(\phi,I_n)\leq\frac{\|\phi''\|_\infty}{48}\sum_{i=0}^{n-1}z_i^3. \tag{3.6}$$

$S(\phi,I_n)$ is an arithmetic mean of the midpoint and trapezoidal quadrature rules.

Remark 3.5. *If we select $\lambda=1$ in (3.3) and (3.4), $(i=0,\cdots\cdots,n-1)$, then*

$$S(\phi,\phi',I_n)=\frac{1}{2}\sum_{i=0}^{n-1}[\phi(v_i)+\phi(v_{i+1})]z_i-\frac{1}{8}[\phi'(v_i)-\phi'(v_{i+1})]z_i \tag{3.7}$$

and

$$R(\phi,I_n)\leq\frac{1}{24}\|\phi''\|_\infty\sum_{i=0}^{n-1}z_i^3, \tag{3.8}$$

Which is a perturbed composite trapezoid inequality.

Remark 3.6. *If we select $\lambda=\frac{1}{3}$ in (3.3) and (3.4), $(i=0,\cdots\cdots,n-1)$, then*

$$\begin{aligned} S(\phi,\phi',I_n)=&\frac{1}{2}\sum_{i=0}^{n-1}\left[\frac{2}{3}\phi\left(\frac{v_i+v_{i+1}}{2}\right)\right.\\ &\left.+\frac{4}{3}\left(\frac{\phi(v_i+)+\phi(v_{i+1})}{2}\right)\right]z_i-\frac{1}{12}[\phi'(v_i)-\phi'(v_{i+1})]z_i \end{aligned} \tag{3.9}$$

and

$$R(\phi,I_n)\leq\frac{1}{81}\|\phi''\|_\infty\sum_{i=0}^{n-1}z_i^3, \tag{3.10}$$

which is a perturbed composite trapezoid inequality.

Remark 3.7. *If we select $\lambda=\frac{1}{4}$ in (3.3) and (3.4), $(i=0,\cdots\cdots,n-1)$, then*

$$\begin{aligned} S(\phi,\phi',I_n)=&\frac{1}{2}\sum_{i=0}^{n-1}\left[\frac{3}{4}\phi\left(\frac{v_i+v_{i+1}}{2}\right)\right.\\ &\left.+\frac{5}{8}(\phi(v_i)+\phi(v_{i+1}))\right]z_i-\frac{1}{16}[\phi'(v_i)-\phi'(v_{i+1})]z_i \end{aligned} \tag{3.11}$$

and

$$R(\phi, I_n) \leq \frac{19}{1536}\|\phi''\|_\infty \sum_{i=0}^{n-1} z_i^3, \tag{3.12}$$

which is a perturbed composite trapezoid inequality.

Remark 3.8. *If we select $\lambda = \frac{3}{4}$ in (3.3) and (3.4), $(i = 0, \cdots\cdots, n-1)$, then*

$$\begin{aligned} S(\phi, \phi', I_n) &= \frac{1}{2}\sum_{i=0}^{n-1}\left[\frac{1}{4}\phi\left(\frac{v_i + v_{i+1}}{2}\right)\right. \\ &\left. + \frac{7}{8}(\phi(v_i+) + \phi(v_{i+1}))\right] z_i - \frac{3}{16}[\phi'(v_i) - \phi'(v_{i+1})]z_i \end{aligned} \tag{3.13}$$

and

$$R(\phi, I_n) \leq \frac{41}{1536}\|\phi''\|_\infty \sum_{i=0}^{n-1} z_i^3, \tag{3.14}$$

which is a perturbed composite trapezoid inequality.

Remark 3.9. *If we select $\lambda = \frac{3}{10}$ in (3.3) and (3.4), $(i = 0, \cdots\cdots, n-1)$, then*

$$\begin{aligned} S(\phi, \phi', I_n) &= \frac{1}{20}\sum_{i=0}^{n-1}\left[7\phi\left(\frac{v_i + v_{i+1}}{2}\right) + \frac{13}{2}\phi(v_i) + \phi(v_{i+1})\right] z_i \\ &- \frac{3}{80}\sum_{i=0}^{n-1}[\phi'(v_{i+1} - \phi'(v_i))]z_i \end{aligned} \tag{3.15}$$

and

$$R(\phi, I_n) \leq \frac{293}{24000}\|\phi''\|_\infty \sum_{i=0}^{n-1} z_i^3, \tag{3.16}$$

This is Simpson's type perturbed composite three point quadrature inequality.

4. Applications for Probability Density Function

Let X be a continuous random variable having the probability density function $\phi : [\rho, \flat] \to \mathbb{R}_+$ and the cumulative distribution function $\Phi : [\rho, \flat] \to [0, 1]$, *i.e*,

$$\Phi(v) = \int_\rho^v \phi(\dagger)d\dagger, \quad v \in \left[\dagger - v + \frac{\flat - \rho}{2}, \dagger - v - \frac{\flat - \rho}{2}\right] \subset [\rho, \flat],$$

$$E(X) = \int_\rho^\flat \dagger\phi(\dagger)d\dagger,$$

is the expectation of the random variable x on the interval $[\rho;\flat]$.

$$\dagger - v + \frac{\flat-\rho}{2} \leq v \leq \dagger - v - \frac{\flat-\rho}{2}$$

is the expected of the random variable X on the interval $[\rho,\flat]$. Then we may have the following theorem.

Theorem 4.1. *Let the assumptions of Theorem (2.1) valid if probability density function belongs to $L_2[\rho,\flat]$ space, then we get the following inequality*

$$\begin{aligned}
&\left|\frac{(\flat-E(X))}{(\flat-\rho)} - \frac{1}{2}\left[(1-\lambda)\left(\frac{\phi(v)+\phi(\rho+\flat-v)}{2}\right)\right.\right.\\
&\left.\left.-(1-\lambda)\left(v-\frac{\rho+\flat}{2}\right)\left(\frac{\phi'(v)+\phi'(\rho+\flat-v)}{2}\right) + \frac{(1+\lambda)}{2}\right)\right]\Bigg|\\
&\leq \frac{\|\phi''\|_\infty}{(\flat-\rho)}\left[\frac{1}{3}\left(\frac{\rho+\flat}{2}-v\right)^3 + \frac{(\flat-\rho)^3}{48}(3\lambda+2(1-\lambda)^3-1)\right.\\
&\left.+\frac{1}{48}\left((\flat-\rho)(3\lambda-1)+4(v+\rho)\right)\left(\frac{\rho+\flat}{2}-v\right)^2\right]
\end{aligned} \tag{4.1}$$

for all $\xi \in [\rho,\flat]$.

Proof. Put $\phi = \Phi$ in (3) we get (4.1), by using these two identities

$$\int_\rho^\flat \Phi(\dagger) = \flat - E(X)$$

5. Applications for some Special Mean

The inequality (2.1) may some written as

$$\begin{aligned}
&\left|\frac{1-\lambda}{2}\left(\frac{\phi(v)+\phi(\rho+\flat-v)}{2}\right) - \frac{(1-\lambda)}{2}(v-A(\rho,\flat))\right.\\
&\left(\frac{\phi'(v)+\phi'(\rho+\flat-v)}{2}\right) - \lambda\frac{(\flat-\rho)}{8}(\phi'(\flat)-\phi'(\rho))\\
&+\frac{(1+\lambda)}{2}\left(\frac{\phi(\rho)+\phi(\flat)}{2}\right) - \frac{1}{(\flat-\rho)}\int_\rho^\flat \phi(\dagger)d\dagger|\\
&\leq \frac{\|\phi''\|_\infty}{(\flat-\rho)}\left[\frac{1}{3}(A(\rho,\flat)-v)^3 + \frac{(\flat-\rho)^3}{48}(3\lambda+2(1-\lambda)^3+1)\right.\\
&\left.+\frac{1}{48}\left((\flat-\rho)(3\lambda-1)+4(v+\rho)\right)(A(\rho,\flat)-v)^2\right].
\end{aligned}$$

where $A(\rho,\flat)=\frac{\rho+\flat}{2}$. Choosing $\lambda=0$ gives a special case, the modified version of the inequality in [10] as follows,

$$\left|\frac{1}{2}\left[\left(\frac{\phi(v)+\phi(\rho+\flat-v)}{2}\right)-(v-A(\rho,\flat))\left(\frac{\phi'(v)+\phi'(\rho+\flat-v)}{2}\right)\right.\right.$$
$$\left.\left.+\left(\frac{\phi(\rho)+\phi(\flat)}{2}\right)\right]-\frac{1}{(\flat-\rho)}\int_{\rho}^{\flat}\phi(\dagger)d\dagger\right|$$
$$\leq\|\phi''\|_{\infty}\left[\frac{1}{3(\flat-\rho)}(A(\rho,\flat)-v)^{3}+\frac{(\flat-\rho)^{2}}{48}\right.$$
$$\left.+\frac{1}{48}(4(v+\rho)-(\flat-\rho))(A(\rho,\flat)-v)^{2}\right].$$

applying (4.1), to infer some inequalities for special means using some particular mappings. The results of the special means are therefore as follows

Example no 1: Consider

$$\begin{aligned}\phi(\dagger)&=ln\dagger,\phi:(0,\infty)\rightarrow\mathbb{R},then\\ \frac{1}{\flat-\rho}\int_{\rho}^{\flat}\phi(\dagger)d\dagger&=lnI(\rho,\flat),\\ \frac{\phi(v)+\phi(\rho+\flat-v)}{2}&=lnG_{1}(v,\rho+\flat-v),\\ \frac{\phi(\rho)+\phi(\flat)}{2}&=lnG(\rho,\flat),\\ \phi'(\flat)-\phi'(\rho)&=-\frac{(\flat-\rho)}{G^{2}(\rho,\flat)},\\ \frac{\phi'(v)+\phi'(\rho+\flat-v)}{2}&=\frac{A(\rho,\flat)-v}{G^{2}(\rho,\flat)}\\ G(\rho,\flat)&=\sqrt{\rho\flat}\end{aligned}$$

and

$$\|\phi''\|_{\infty}=\sup_{\dagger\in(\rho,\flat)}\|\phi''\|(\dagger)\|=\frac{1}{\rho^{2}}$$

From (4.1), we have:

$$\left|(1-\lambda)lnG(v,\rho+\flat-v)+(1+\lambda)lnG(\rho,\flat)+\lambda\frac{(\flat-\rho)^2}{4G^2(\rho,\flat)}-2lnI(\rho,\flat)\right|$$
$$\leq\frac{2}{\rho^2}\left[\frac{1}{3(\flat-\rho)}(A(\rho,\flat)-v)+\frac{(\flat-\rho)^2}{48}\right.$$
$$\left.\times(3\lambda+2(1-\lambda)^3+1)+\frac{1}{48}((3\lambda-1)+4\left(\frac{v+c}{\flat-\rho}\right)(A(\rho,\flat)-v)^2\right]$$

from which we obtain the approximation at the centre $v=\frac{\rho+\flat}{2}=A(\rho,\flat)$, so that

$$\left|(1-\lambda)lnG(v,\rho+\flat-v)+(1+\lambda)lnG(\rho,\flat)+\lambda\frac{(\flat-\rho)^2}{4G^2(\rho,\flat)}-2lnI(\rho,\flat)\right|$$
$$\leq\left[\frac{(\flat-\rho)^2}{24\rho^2}(3\lambda+2(1-\lambda)^3-1)\right]$$

or

$$\left|ln\left(\frac{G^{(1+\lambda)}G_1^{(1-\lambda)}}{I^2}\right)+\lambda\frac{(\flat-\rho)^2}{4G^2}\right|$$
$$\leq\left[\frac{(\flat-\rho)^2}{24\rho^2}(3\lambda+2(1-\lambda)^3+1)\right]$$

from which we obtain the better approximation if we select $\lambda=\frac{3}{10}$, that is

$$\left|ln\left(\frac{G^{\frac{13}{10}}G_1^{\frac{7}{10}}}{I^2}\right)+\frac{3(\flat-\rho)^2}{40G^2}\right|\leq\left[\frac{293(\flat-\rho)^2}{12000\rho^2}\right]$$

For $\lambda=0$, we have

$$\left|ln\left(\frac{GG_1}{I^2}\right)\right|\leq\left[\frac{(\flat-\rho)^2}{24\rho^2}\right]$$

For $\lambda=1$, we have

$$\left|ln\left(\frac{G}{I}\right)^2+\frac{(\flat-\rho)^2}{4G^2}\right|\leq\left[\frac{(\flat-\rho)^2}{12\rho^2}\right]$$

For $\lambda=\frac{1}{4}$, we have

$$\left|ln\left(\frac{G^{\frac{5}{4}}G_1^{\frac{3}{4}}}{I^2}\right)+\frac{(\flat-\rho)^2}{16G^2}\right|\leq\left[\frac{19(\flat-\rho)^2}{768\rho^2}\right]$$

For $\lambda=\frac{3}{4}$, we have

$$\left|ln\left(\frac{G^{\frac{7}{4}}G_1^{\frac{1}{4}}}{I^2}\right)+\frac{3(\flat-\rho)^2}{16G^2}\right|\leq\left[\frac{41(\flat-\rho)^2}{768\rho^2}\right]$$

For $\lambda=\frac{1}{3}$, we have

$$\left|ln\left(\frac{G^{\frac{4}{3}}G_1^{\frac{2}{3}}}{I^2}\right)+\frac{(\flat-\rho)^2}{12G^2}\right|\leq\left[\frac{8(\flat-\rho)^2}{324\rho^2}\right]$$

Example no 2: Consider

$$\phi(\dagger) = \frac{1}{\dagger}, \phi : (0,\infty) \to (0,\infty), then$$

$$\frac{1}{\flat-\rho}\int_{\rho}^{\flat} \phi(\dagger)d\dagger = L^{-1}(\rho,\flat),$$

$$\frac{\phi'(v)+\phi'(\rho+\flat-v)}{2} = \frac{A(\rho,\flat)(A(\rho,\flat)-v)}{G_1^4(v,\rho+\flat-v)},$$

$$\frac{\phi(\rho)+\phi(\flat)}{2} = \frac{1}{H},$$

$$\phi'(\flat)-\phi'(\rho) = \frac{(\flat^2-\rho^2)}{G^4(\rho,\flat)},$$

$$\frac{\phi(v)+\phi(\rho+\flat-v)}{2} = \frac{A(\rho,\flat)}{G_1^2(v,\rho+\flat-v)}$$

$$H(\rho,\flat) = \frac{2\rho\flat}{\rho+\flat}$$

and

$$\|\phi''\|_\infty = \sup_{\dagger\in(\rho,\flat)} \|\phi''(\dagger)\| = \frac{2}{\rho^3}$$

From (4.1), we have:

$$\left|\frac{(1-\lambda)A}{2G_1^2} + \frac{(1+\lambda)}{2H} - \frac{(1-\lambda)}{2}(v-A)\right.$$
$$\left.\times\left(\frac{2A(A-t)}{G_1^4}\right) - \lambda\frac{(\flat-\rho)^2}{8G^4}(\flat+\rho) - 2L^{-1}\right|$$
$$\leq \frac{2}{(\flat-\rho)\rho^3}\left[\frac{1}{3}(A-v)^3 + \frac{(\flat-\rho)^3}{48}(3\lambda+2(1-\lambda)^3-1)\right.$$
$$\left.+\frac{1}{48}((\flat-\rho)(3\lambda-1+4(v+\rho))(A-v)^2)\right]$$

and the approximation at the centre $v = \frac{\rho+\flat}{2} = A(\rho,\flat)$, so that

$$\left|\frac{(1-\lambda)A}{2G_1^2} + \frac{(1+\lambda)}{2H} - \lambda\frac{(\flat-\rho)^2}{8G^4}(\flat+\rho) - L^{-1}\right|$$
$$\leq \frac{2}{(\flat-\rho)\rho^3}\left[\frac{(\flat-\rho)^3}{48}(3\lambda+2(1-\lambda)^3-1)\right]$$

from which we obtain the better approximation if we select $\lambda = \frac{3}{10}$, that is

$$\left| \frac{7A}{20G_1^2} + \frac{13}{20H} - \frac{3(\flat-\rho)^2}{40G^4}(\flat+\rho) - L^{-1} \right| \leq \frac{293(\flat-\rho)^2}{12000\rho^3}$$

For $\lambda = 0$, we have

$$\left| \frac{A}{2G_1^2} + \frac{1}{2H} - L^{-1} \right| \leq \frac{(\flat-\rho)^2}{24}$$

For $\lambda = 1$, we have

$$\left| \frac{1}{H} - \frac{(\flat-\rho)^2}{8G^4}(\flat+\rho) \right| \leq \frac{(\flat-\rho)^2}{12\rho^2}$$

For $\lambda = \frac{1}{4}$, we have

$$\left| \frac{3A}{8G_1^2} + \frac{5}{8H} - \frac{(\flat-\rho)^2}{32G^4}(\flat+\rho) - L^{-1} \right| \leq \frac{19(\flat-\rho)^2}{768\rho^3}$$

For $\lambda = \frac{1}{3}$, we have

$$\left| \frac{A}{3G_1^2} + \frac{5}{8H} - \frac{3(\flat-\rho)^2}{32G^4}(\flat+\rho) - L^{-1} \right| \leq \frac{2(\flat-\rho)^2}{81\rho^3}$$

For $\lambda = \frac{3}{4}$, we have

$$\left| \frac{A}{8G_1^2} + \frac{2}{3H} - \frac{(\flat-\rho)^2}{24G^4}(\flat+\rho) - L^{-1} \right| \leq \frac{41(\flat-\rho)^2}{768\rho^3}$$

Example no 3: Consider

$$\phi(\dagger) = \dagger^p, \phi : (0,\infty) \to (0,\infty),$$

where p $\in \mathbb{R}\backslash\{-1,0\}$ then for $\rho < \flat$

$$\frac{1}{(\flat-\rho)}\int_\rho^\flat \phi(\dagger)d\dagger = L_p^p(\rho,\flat),$$

$$\phi'(\flat) - \phi'(\rho) = p(p-1)(\flat-\rho)L_{p-2}^{p-2}(\rho,\flat),$$

$$\frac{\phi(v)+\phi(\rho+\flat-v)}{2} = A_1(\rho^p,(\rho+\flat-v)^p),$$

$$\frac{\phi(\rho)+\phi(\flat)}{2} = A(\rho^p,\flat^p)$$

and

$$\|\phi''\|_\infty = \begin{cases} \flat^{p-2}, & \text{if} \quad p \in [2,\infty) \\ \\ \rho^{p-2}, & \text{if} \quad p \in (-\infty,2]\backslash\{-1,0\}. \end{cases}$$

From (4.1), we obtain

$$\left|\frac{(1-\lambda)}{2}\left\{A_1+(v-A)p\left(\frac{X^{p-1}-(\rho+\flat-X)^{p-1}}{2}\right)\right\}\right.$$
$$\left.+\frac{(1+\lambda)}{2}A-\lambda\frac{(\flat-\rho)}{8}p(p-1)(\flat-\rho)L_{p-2}^{p-2}-L_p^p\right|$$
$$\leq |p(p-1)|\lambda_p\left[\frac{1}{3(\flat-\rho)}(A-X)^3+\frac{(\flat-\rho)^3}{48}(3\lambda+2(1-\lambda)^3-1)\right]$$

where

$$\lambda_p(\rho,\flat)=\begin{cases}\flat^{p-2}, & \text{if} \quad p\in[2,\infty)\\ \\ \rho^{p-2}, & \text{if} \quad p\in(-\infty,2]\backslash\{-1,0\}.\end{cases}$$

At $v=\frac{\rho+\flat}{2}=A(\rho,\flat)$, we get

$$\left|\frac{(1-\lambda)}{2}A_1+\frac{(1+\lambda)}{2}A_2-\lambda\frac{(\flat-\rho)^2}{8}p(p-1)L_{p-2}^{p-2}-L_p^p\right|$$
$$\leq |p(p-1)|\lambda_p\left[\frac{(\flat-\rho)^3}{48}(3\lambda+2(1-\lambda)^3-1)\right]$$

or

$$\left|(1-\lambda)A_1+(1+\lambda)A_2-\lambda\frac{(\flat-\rho)^2}{4}p(p-1)L_{p-2}^{p-2}-2L_p^p\right|$$
$$\leq |p(p-1)|\lambda_p\left[\frac{(\flat-\rho)^3}{24}(3\lambda+2(1-\lambda)^3-1)\right]$$

which gives the better approximation at $\lambda=\frac{3}{10}$

$$\left|\frac{7A_1}{10}+\frac{13A_2}{10}-\frac{3(\flat-\rho)^2}{40}p(p-1)(\flat-\rho)L_{p-2}^{p-2}-2L_p^p\right|$$
$$\leq |p(p-1)|\lambda_p\frac{293(\flat-\rho)^2}{12000}$$

Moreover, at $\lambda=0$

$$\left|\frac{7A_1}{10}+\frac{13A_2}{10}-\frac{3(\flat-\rho)}{40}p(p-1)(\flat-\rho)L_{p-2}^{p-2}-2L_p^p\right|$$
$$\leq |p(p-1)|\lambda_p\frac{(\flat-\rho)^2}{24}$$

For $\lambda = \frac{1}{4}$, we have

$$\left|\frac{3A_1}{4} + \frac{5A_2}{4} - \frac{(\flat - \rho)^2}{16} p(p-1)L_{p-2}^{p-2} - 2L_p^p\right|$$
$$\leq |p(p-1)|\lambda_p \frac{19(\flat - \rho)^2}{768}$$

For $\lambda = \frac{1}{3}$, we have

$$\left|\frac{2A_1}{3} + \frac{4A_2}{3} - \frac{(\flat - \rho)^2}{12} p(p-1)L_{p-2}^{p-2} - 2L_p^p\right|$$
$$\leq |p(p-1)|\lambda_p \frac{2(\flat - \rho)^2}{81}$$

For $\lambda = \frac{3}{4}$, we have

$$\left|\frac{A_1}{4} + \frac{7A_2}{4} - \frac{41(\flat - \rho)^2}{768} p(p-1)L_{p-2}^{p-2} - 2L_p^p\right|$$
$$\leq |p(p-1)|\lambda_p \frac{2(\flat - \rho)^2}{81}$$

Moreover, at $\lambda = 1$

$$\left|2A_1 - \frac{(\flat - \rho)^2}{4} p(p-1)L_{p-2}^{p-2} - 2L_p^p\right|$$
$$\leq |p(p-1)|\lambda_p \frac{(\flat - \rho)^2}{12}$$

6. Conclusion

In this paper, modified Peano Kernels are used for some remarkable Ostrowski type inequalities depending on the second derivatives are mentioned. Ostrowski type inequalities for twice differentiable functions have been lengthily mentioned. in the research paper [6] and [7]. We have mentioned a generalization and extension of the inequalities mentioned in [6] and [7]. we have mentioned a generalization (4) of the inequality (2) acquired in [1] for twice differentiable functions whose first derivatives are absolutely continuous and second derivative belong to $L_\infty(\rho, \flat)$ by introducing a parameter $\lambda \in [0,1]$. This generalization also results in attaining a three-point inequality for a specific value of λ as mentioned in remarks (2.4) to (2.8). The three-point inequality thus acquired has a useful bound than the three-point inequalities mentioned in [6] and [7] for $\|.\|_\infty$ -norm. Remarks (2.4) to (2.8) also shows that the perturbed trapezoid inequality that can be acquired from (2.1)is useful than the perturbed inequalities mentioned in [6] and [7] of perturbed trapezoid type for $\|.\|_\infty$ - norm. The inequality is then functional for a Divider of the interval $[\rho,\flat]$ to acquire many composite quadrature rules. The inequality is also functional to special means by Appropriately choosing selecting the function involved to get Certain direct association between different means.

Conflict of Interests

The authors declare that there is no conflict of interests.

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