ON MULTIPLICATIVE ZAGREB INDICES OF TWO NEW OPERATIONS OF GRAPHS

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Abstract. Recently, Wang et al. (2017) introduced two new operations of graphs. In this paper the upper bounds of the multiplicative Zagreb indices of the two newly proposed operations of graphs are derived.

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1. Introduction

Throughout the paper we consider only simple finite graphs. \( V(G) \) and \( E(G) \) are respectively the set of vertices and set of edges of a graph \( G \). The degree of a vertex \( u \in V(G) \) is denoted by \( d_G(u) \), if their is no confusion we simply write it as \( d(u) \). Two vertices \( u \) and \( v \) are called adjacent if there is an edge connecting them. The connecting edge is usually denoted by \( uv \). Any unexplained graph theoretic symbols and definitions may be found in [16].

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Topological indices are the numerical values which are associated with a graph structure. These graph invariants are utilized for modeling information of molecules in structural chemistry and biology. Over the years many topological indices are proposed and studied based on degree, distance and other parameters of graph. Some of them may be found in [5, 7]. Historically Zagreb indices can be considered as the first degree-based topological indices, which came into picture during the study of total $\pi$-electron energy of alternant hydrocarbons by Gutman and Trinajstić in 1972 [9]. Since these indices were coined, various studies related to different aspects of these indices are reported, for detail see the papers [4, 6, 8, 12, 17] and the references therein.

The first and second Zagreb indices of a graph $G$ are defined as

$$M_1(G) = \sum_{u \in V(G)} d_G^2(u) = \sum_{uv \in E(G)} (d_G(u) + d_G(v)),$$

$$M_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v).$$

The multiplicative versions of these Zagreb indices are proposed by Todeschini et al., in [14] and latter named as “Multiplicative Zagreb indices” by Gutman [10]. These indices can be defined as follows.

$$\prod_1(G) = \prod_{u \in V(G)} d_G^2(u)$$

$$\prod_2(G) = \prod_{uv \in E(G)} d_G(u)d_G(v).$$

Graph operations play a very important role in chemical graph theory, as some chemically interesting graphs can be obtained by different graph operations on some general or particular graphs. Like many other topological indices, studies related to Multiplicative Zagreb indices and coindices of various graph operations [1, 2, 3, 13].

In this paper we consider two new operations of graphs proposed by Wang et al. in [15] and derive the upper bounds on the multiplicative Zagreb indices of the two newly proposed operations of graphs. The rest of the paper is organized as follows. In section 2 we reproduce the two graph operations under consideration. In section 3 main results are presented.
2. Two new graph operations and some preliminaries

In this section we first reproduce the definitions of the two newly defined operations of graphs [15] and then some standard results are included.

The Cartesian product of graphs $G$ and $H$, denoted by $G \square H$, is the graph with vertex set $V(G) \square V(H) = \{(a, v) : a \in V(G), v \in V(H)\}$, and $(a, v)$ is adjacent to $(b, w)$ whenever $a = b$ and $(v, w) \in E(H)$, or $v = w$ and $(a, b) \in E(G)$. More detail on Cartesian product and some other operations of graphs may be found in [11]. In 2017, Wang et al., proposed the following two operations of graphs and also studied their adjacency spectrum. We reproduce the figure in [15] to make the discussion self-expository.

**Definition 2.1.** Let $G_1i = G_1$ and $G_2i = G_2$ $(1 \leq i \leq k)$ be $k$ copies of graphs $G_1$ and $G_2$, respectively, $G_j$ $(j = 3, 4)$ is an arbitrary graph.

- The first operation $G_1 \big[d (G_3 \square G_2)$ of $G_1, G_2$ and $G_3$ is obtained by making the Cartesian product of two graphs $G_3$ and $G_2$, thus produces $k$ copies $G_2i$ $(1 \leq i \leq k)$ of $G_2$, then makes $k$ joins $G_1i \cup G_2i, i = 1, 2, \ldots, k.$
The second operation \( (G_4 \square G_1)^k(G_3 \square G_2) \) of \( G_1, G_2, G_3 \) and \( G_4 \) is obtained by making the Cartesian product of two graphs \( G_3 \) and \( G_2 \), produces \( k \) copies \( G_{2i} \) \( (1 \leq i \leq k) \) of \( G_2 \) and making the Cartesian product of two graphs \( G_4 \) and \( G_1 \), produces \( k \) copies \( G_{1i} \) \( (1 \leq i \leq k) \) of \( G_1 \), then makes \( k \) joins \( G_{1i} \vee G_{2i}, i = 1, 2, \ldots, k. \)

Now we propose the following lemma which can easily be proved from the definition 2.1 of the two graph operations.

**Lemma 2.1.** Let \( G_1, G_2, G_3 \) and \( G_4 \) be four graphs with \( |V(G_i)| = n_i, |E(G_i)| = m_i \) where \( i = 1, 2, 3, 4. \) Then,

\[
d_{(G_1 \square_k (G_3 \square G_2))}(u) = \begin{cases} 
d_{G_1}(u) + n_2 & \text{if } u \in V(G_1) \\
d_{G_3}(u_3) + d_{G_2}(u_2) + n_1 & \text{if } u = (u_3, u_2) \in V(G_3 \square G_2) 
\end{cases}
\]

and

\[
d_{((G_4 \square G_1)^k(G_3 \square G_2))}(u) = \begin{cases} 
d_{G_4}(u_4) + d_{G_1}(u_1) + n_2 & \text{if } u = (u_4, u_1) \in V(G_4 \square G_1) \\
d_{G_3}(u_3) + d_{G_2}(u_2) + n_1 & \text{if } u = (u_3, u_2) \in V(G_3 \square G_2). 
\end{cases}
\]

For an example we consider \( G_1 = K_2, G_2 = P_2 \) and \( G_3 = G_4 = C_4 \) and hence obtain the graphs \( K_2^k(C_4 \square P_3) \) and \( (C_4 \times K_2)^k(C_4 \square P_3) \), which are shown in figure 1. It is clear from the definitions of the two operations that \( |E(G_1 \square_k (G_3 \square G_2))| = k(m_1 + m_2 + n_1 n_2) + n_2 m_3, \)

\( |E((G_4 \square G_1)^k(G_3 \square G_2))| = k(m_1 + n_1 n_2 + m_2) + n_1 m_4 + m_3 n_2 \) and \( |V(G_1 \square_k (G_3 \square G_2))| = |V((G_4 \square G_1)^k(G_3 \square G_2))| = k(n_1 + n_2). \) Also it is to be exclusively mentioned that \( |V(G_3)| = |V(G_4)| = k \) but \( G_3 \neq G_4 \) in general.

Here we highlight a standard lemma and two known results without proof. These results will be expedited in the coming section.

**Lemma 2.2.** (AM-GM Inequality) Let \( x_1, x_2, \ldots, x_n \) be nonnegative numbers. Then

\[
\frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n},
\]

where equality holds iff all the \( x_i \)'s are equal.

**Theorem 2.1.** [12] Let \( G_1, G_2, \ldots, G_n \) be graphs with \( V_i = V(G_i) \) and \( E_i = E(G_i), 1 \leq i \leq n, \) and \( V = V(\square_{i=1}^n G_i). \) Then \( M_1(\square_{i=1}^n G_i) = |V|\sum_{i=1}^n \frac{M_1(G_i)}{|V_i|} + 4|V|\sum_{i \neq j=1}^n \frac{|E_i| |E_j|}{|V_i||V_j|}. \)
\textbf{Theorem 2.2.} [12] Let $G_1, G_2, \ldots, G_n$ be graphs with $V_i = V(G_i)$ and $E_i = E(G_i)$, $1 \leq i \leq n$, and $V = V(\square_{i=1}^n G_i)$ and $E = E(\square_{i=1}^n G_i)$. Then $M_2(\square_{i=1}^n G_i) = |V| \sum_{i=1}^{n} \left( \frac{M_2(G_i)}{|V_i|} + 3M_1(G_i) \left( \frac{|E|}{|V_i|^2} \right) \right) + 4|V| \sum_{i,j,k=1}^{n} \left( E_i \cup E_j \cup E_k \right) \frac{|E_i||E_j||E_k|}{|V_i||V_j||V_k|}.

3. Main Results

\textbf{Theorem 2.3.} Let $G_1, G_2$ and $G_3$ be graphs with $|V_i| = |V(G_i)| = n_i$, $|E_i| = |E(G_i)| = m_i$, $1 \leq i \leq 3$ and $n_3 = k$. Then

$$
\prod_{1}^{n_1} (G_1 \square_k (G_3 \sqcup G_2)) \leq \left( \frac{M_1(G_1) + 4n_2m_1 + n_2^2n_1}{n_1} \right)^{n_1k} \times \left( \frac{kM_1(G_2) + n_2M_1(G_3) + 8m_2m_3 + n_2^2n_2 + 4n_1m_2 + 4n_1n_2m_3}{n_2} \right)^{n_2k},
$$

and

$$
\prod_{2}^{n_2} (G_1 \square_k (G_3 \sqcup G_2)) \leq \left( \frac{n_2M_1(G_1) + M_2(G_1) + n_2^2m_1}{m_1} \right)^{m_1k} \times \left( \frac{\gamma_1}{n_1n_2k} \right)^{n_1n_2k} \times \left( \frac{(3m_3 + n_1k)M_1(G_2) + (3m_2 + n_1n_2)M_1(G_3) + kM_2(G_2) + n_2M_2(G_3) + \gamma_2}{\beta} \right)^{\beta},
$$

where $\gamma_1 = 8n_1m_2m_3 + n_1^2n_2m_3 + n_1^2km_2$, $\gamma_2 = (n_2m_3 + km_2)(4m_1 + 2n_1n_2) + k(n_1^2n_2^2 + 2m_1n_1n_2)$ and $\beta = km_2 + n_2m_3$. Moreover equalities hold iff all $G_1$, $G_2$ and $G_3$ are regular graphs.

\textbf{Proof.} Since the set of vertices of $G_1 \square_k (G_3 \sqcup G_2)$ can be divided into two categories viz., $u \in V(G_1)$ and $u \in V(G_3 \sqcup G_2)$, so we have

$$
\prod_{1}^{n_1} (G_1 \square_k (G_3 \sqcup G_2)) = \prod_{u \in V(G_1)} d_{G_1 \square_k (G_3 \sqcup G_2)}^2(u) = \left( \prod_{u \in V(G_1)} d_{G_1 \square k (G_3 \sqcup G_2)}^2(u) \right)^k \prod_{u=(u_3,u_2) \in V(G_3 \sqcup G_2)} d_{G_1 \square_k (G_3 \sqcup G_2)}^2(u).
$$

Using lemma 2.1, expression (1) can be written as
\[
\prod_1^k (G_1 \square_k (G_3 \square G_2)) = \left( \prod_{u \in V(G_1)} (d_{G_1}(u) + n_2)^2 \right)^k \times \prod_{u=(u_3,u_2) \in V(G_3 \square G_2)} (d_{G_3}(u_3) + d_{G_2}(u_1) + n_1))^2.
\]

As \(d_{G_3 \square G_2}(u_3,u_2) = d_{G_3}(u_3) + d_{G_2}(u_2)\) the last expression is

(2) \[= \left( \prod_{u \in V(G_1)} (d_{G_1}(u) + n_2)^2 \right)^k \times \prod_{u \in V(G_3 \square G_2)} (d_{G_3 \square G_2}(u) + n_1)^2\]

and by lemma 2.2 we can write expression (2) as

(3) \[\prod_1^k (G_1 \square_k (G_3 \square G_2)) \leq \left( \sum_{u \in V(G_1)} \frac{(d_{G_1}^2(u) + 2n_2d_{G_1}(u) + n_2^2)}{n_1} \right)^{n_1k} \times \left( \sum_{u \in V(G_3 \square G_2)} \frac{(d_{G_3 \square G_2}(u) + 2n_1d_{G_3 \square G_2}(u) + n_1^2)}{n_2k} \right)^{n_2k}.\]

Now

(4) \[\sum_{u \in V(G_1)} (d_{G_1}^2(u) + 2n_2d_{G_1}(u) + n_2^2) = M_1(G_1) + 4n_2m_1 + n_2^2n_1,\]

and

(5) \[\sum_{u \in V(G_3 \square G_2)} (d_{G_3 \square G_2}(u) + 2n_1d_{G_3 \square G_2}(u) + n_1^2) = M_1(G_3 \square G_2) + n_1^2n_2k + 4n_1|E(G_3 \square G_2)|.\]

Using theorem 2.1 and the fact that \(|E(G_3 \square G_2)| = km_2 + n_2m_3\), we write the expression (5) as

(6) \[\sum_{u \in V(G_3 \square G_2)} (d_{G_3 \square G_2}(u) + 2n_1d_{G_3 \square G_2}(u) + n_1^2) = n_2M_1(G_3) + km_1(G_2) + 8m_2m_3 + n_1^2n_2k\]

\[+ 4n_1km_2 + 4n_1n_2m_3.\]

From (3), (4) and (6), we have the first inequality.

Moreover equality holds iff

\[d_{G_1}^2(u_i) + 2n_2d_{G_1}(u_i) + n_2^2 = d_{G_1}^2(u_j) + 2n_2d_{G_1}(u_j) + n_2^2 \quad (u_i, u_j \in V(G_1))\]

(7) \[\Rightarrow (d_{G_1}(u_i) - d_{G_1}(u_j))(d_{G_1}(u_i) + d_{G_1}(u_j) + 2n_2) = 0\]
and

\[ d_{G_3 \square G_2}^2(u) + 2n_1 d_{G_3 \square G_2}(u) + n_1^2 = d_{G_3 \square G_2}^2(u') + 2n_1 d_{G_3 \square G_2}(u') + n_1^2 \quad (u, u' \in V(G_3 \square G_2)) \]

(8) \[ \implies (d_{G_3 \square G_2}(u) - d_{G_3 \square G_2}(u'))(d_{G_3 \square G_2}(u) + d_{G_3 \square G_2}(u') + 2n_1) = 0. \]

So from (7) and (8), we can conclude that equality holds iff both \( G_1 \) and \( G_3 \square G_2 \) are regular.

Since the set of edges of \( G_1 \square_k (G_3 \square G_2) \) can again be divided into three categories viz., \( uv \in E(G_1), uv \in E(G_3 \square G_2) \), and \( uv \in E(G_1 \square (G_3 \square G_2)) \) s.t. \( u \in V(G_1) \) and \( v \in V(G_3 \square G_2) \), so we have

\[
\prod^2_k (G_1 \square_k (G_3 \square G_2)) = \left( \prod_{uv \in E(G_1)} (d_{G_1}(u) + n_2)(d_{G_1}(v) + n_2) \right)^k \\
\times \prod_{uv \in E(G_3 \square G_2)} (d_{G_3 \square G_2}(u) + n_1)(d_{G_3 \square G_2}(v) + n_1) \\
\times \prod_{uv \in E(G_1 \square (G_3 \square G_2)), u \in V(G_1) \text{ and } v \in V(G_3 \square G_2)} (d_{G_1}(u) + n_2)(d_{G_3 \square G_2}(v) + n_1)
\]

(9)

Again using lemma 2.1, expression (9) can be written as

\[
\prod^2_k (G_1 \square_k (G_3 \square G_2)) \leq \left( \frac{\sum_{uv \in E(G_1)} (d_{G_1}(u) + n_2)(d_{G_1}(v) + n_2)}{m_1} \right)^{m_1 k} \\
\times \left( \frac{\sum_{uv \in E(G_3 \square G_2)} (d_{G_3 \square G_2}(u) + n_1)(d_{G_3 \square G_2}(v) + n_1)}{km_2 + n_2 m_3} \right)^{km_2 + n_2 m_3} \\
\times \left( \frac{\sum_{uv \in E(G_1 \square (G_3 \square G_2))} (d_{G_1}(u) + n_2)(d_{G_3 \square G_2}(v) + n_1)}{n_1 n_2 k} \right)^{n_1 n_2 k}
\]

(10)

Now

\[
\sum_{uv \in E(G_1)} (d_{G_1}(u) + n_2)(d_{G_1}(v) + n_2) = k \left( M_2(G_1) + n_2 M_1(G_1) + n_1^2 m_1 \right),
\]

also using theorem 2.1 and theorem 2.2, we have

\[
\sum_{uv \in E(G_3 \square G_2)} (d_{G_3 \square G_2}(u) + n_1)(d_{G_3 \square G_2}(v) + n_1) \\
= kM_2(G_2) + n_2 M_2(G_3) + (3m_3 + n_1 k) M_1(G_2) + (3m_2 + n_1 n_2) M_1(G_3) \\
+ 8n_1 m_2 m_3 + n_1^2 n_2 m_3 + n_1^2 k m_2,
\]

(12)
\[ \sum_{uv \in E(G_1 \Box G_2)} (d_{G_1}(u) + n_2)(d_{G_2} \Box G_2)(v) + n_1) = (n_2m_3 + km_2)(4m_1 + 2n_1n_2) + k(n_1^2 n_2^2 + 2m_1n_1n_2). \]

(13)

Using (11), (12), (13) in (10), we have the second inequality.

From lemma 2.2 for connected graphs \(G_1, G_2\) and \(G_3\) equality holds iff

\[ (d_{G_1}(u) + n_2)(d_{G_1}(v) + n_2) = (d_{G_1}(u) + n_2)(d_{G_1}(v') + n_2) \quad \text{for any } uv, uv' \in E(G_1), \]

for any \(uv, uv' \in E(G_3 \Box G_2)\),

\[ d_{G_3 \Box G_2}(u) + n_1)d_{G_3 \Box G_2}(v) + n_1 = d_{G_3 \Box G_2}(u) + n_1)(d_{G_3 \Box G_2}(v') + n_1, \]

and for any \(uv, uv' \in E(G_1 \Box k(G_3 \Box G_2)\), \(u \in V(G_1)\) and \(v, v' \in V(G_3 \Box G_2)\),

\[ d_{G_1}(u) + n_2)(d_{G_3 \Box G_2}(v) + n_1) = d_{G_1}(u) + n_2)(d_{G_3 \Box G_2}(v') + n_1. \]

Hence equality holds in the second inequality iff all the graphs \(G_1, G_2\) and \(G_3\) are connected regular graphs. \(\square\)

Theorem 2.4. Let \(G_1, G_2, G_3\) and \(G_4\) be graphs with \(|V_i| = |V(G_i)| = n_i, |E_i| = |E(G_i)| = m_i, 1 \leq i \leq 3\) and \(n_3 = k = n_4\). Then

\[ \prod_{1}((G_4 \Box G_1) \Box (G_3 \Box G_2)) \leq \left( \frac{kM_1(G_1) + n_1M_1(G_4) + 8m_1m_4 + 4n_2(km_1 + n_1m_4) + n_2^2 kn_1}{n_1k} \right)^{n_1k}, \]

\[ \times \left( \frac{kM_1(G_2) + n_2M_1(G_3) + 8m_2m_3 + 4n_1(km_2 + n_2m_3) + n_1^2 kn_2}{n_2k} \right)^{n_2k}, \]

\[ \prod_{2}((G_4 \Box G_1) \Box (G_3 \Box G_2)) \leq \left( \frac{kM_2(G_1) + n_1M_2(G_4) + 3m_4M_1(G_1) + 3m_1M_1(G_4) + n_2(kM_1(G_1) + n_1M_1(G_4) + \delta_{14})}{m_1k + m_4n_1} \right)^{m_1k + m_4n_1}, \]

\[ \left( \frac{kM_2(G_2) + n_2M_2(G_3) + 3m_3M_1(G_2) + 3m_1M_1(G_3) + n_1(kM_1(G_2) + n_2M_1(G_3) + \delta_{23})}{m_2k + m_3n_2} \right)^{m_2k + m_3n_2}, \]

\[ \times \left( \frac{\gamma_2 + n_1n_2 \sum_{i=1}^{k} d_{G_4}(u_i^j) d_{G_3}(v_i^j)}{n_1n_2k} \right)^{n_1n_2k}, \]
where \( u^i_4 \in V(G_4), v^i_3 \in V(G_3) \) and for fixed \( i, (u^i_4, u_1) \) is adjacent to \((v^i_3, v_2)\) for any \( u_1 \in V(G_1)\), \( v_2 \in V(G_2)\) and \( \delta_{pq} = 8n_pm_q + n_pn_2m_q + kn_qm_p, \gamma_2 = 4n_1m_2m_4 + 2n_2^2m_2m_4 + 4n_2m_1m_3 + 4m_1m_2k + 2kn_1m_1n_2 + 2n_1n_2^2m_3 + 2kn_1n_2m_2 + kn_1^2n_2^2. \) Moreover equalities hold iff all \( G_1, G_2, G_3 \) and \( G_4 \) are regular graphs.

**Proof.** There are three two types of vertices present in \((G_4 \Box G_1) \square_k (G_3 \Box G_2)\), they are \( u \in G_4 \Box G_1 \) and \( u \in G_3 \Box G_2 \).

\[
\prod_1 (((G_4 \Box G_1) \square_k (G_3 \Box G_2))) \leq \left( \frac{\sum_{u \in V(G_4 \Box G_1)} (d_{(G_4 \Box G_1)}(u) + n_2)^2}{n_1 k} \right)^{n_1 k} \times \left( \frac{\sum_{u \in V(G_3 \Box G_2)} (d_{(G_3 \Box G_2)}(u) + n_1)^2}{n_2 k} \right)^{n_2 k}.
\]

(15)

Now

\[
\sum_{u \in V(G_4 \Box G_1)} (d_{(G_4 \Box G_1)}(u) + n_2)^2 = kM_1(G_1) + n_1M_1(G_4) + 8m_1m_4 + 4n_2(km_1 + n_1m_4) + n_2^2kn_1,
\]

and

\[
\sum_{u \in V(G_3 \Box G_2)} (d_{(G_3 \Box G_2)}(u) + n_1)^2 = kM_1(G_2) + n_2M_1(G_3) + 8m_2m_3 + 4n_1(km_2 + n_2m_3) + n_1^2kn_2.
\]

Hence we have the first inequality of the theorem.

For equality we must have

\[
(d_{(G_4 \Box G_1)}(u) + n_2)^2 = (d_{(G_4 \Box G_1)}(v) + n_2)^2 \quad (\text{for any } uv \in E(G_4 \Box G_1)),
\]

(16)
and

\[ (17) \quad (d(G_3 \Box G_2)(u) + n_1)^2 = (d(G_3 \Box G_2)(v) + n_1)^2 \quad \text{for any } uv \in E(G_3 \Box G_2). \]

From (16) and (17) it is easy to show that the equality holds iff all the graphs \( G_1, G_2, G_3 \) and \( G_4 \) are regular.

Now the set of edges of \((G_4 \Box G_1) \Box_k (G_3 \Box G_2)\) can again classified into three categories viz., \( uv \in E(G_4 \Box G_1), uv \in E(G_3 \Box G_2), \) and \( uv \in E((G_4 \Box G_1) \Box_k (G_3 \Box G_2)) \) where \( u \in V(G_4 \Box G_1), v \in V(G_3 \Box G_2). \) Hence we have

\[
\prod_1 ( (G_4 \Box G_1) \Box_k (G_3 \Box G_2) ) = \prod_{uv \in E(G_4 \Box G_1)} (d(G_4 \Box G_1)(u) + n_2)(d(G_4 \Box G_1)(v) + n_2) \\
\times \prod_{uv \in E(G_3 \Box G_2)} (d(G_3 \Box G_2)(u) + n_1)(d(G_3 \Box G_2)(v) + n_1) \\
\times \prod_{uv \in E((G_4 \Box G_1) \Box_k (G_3 \Box G_2))} (d(G_4 \Box G_1)(u) + n_2)(d(G_3 \Box G_2)(v) + n_1)
\]

(18)

Again using lemma 2.2, we have

\[
\prod_1 ( (G_4 \Box G_1) \Box_k (G_3 \Box G_2) ) \leq \left( \frac{\sum_{uv \in E(G_4 \Box G_1)} (d(G_4 \Box G_1)(u) + n_2)(d(G_4 \Box G_1)(v) + n_2)}{n_1 m_4 + k m_1} \right)^{n_1 m_4 + k m_1} \\
\times \left( \frac{\sum_{uv \in E(G_3 \Box G_2)} (d(G_3 \Box G_2)(u) + n_1)(d(G_3 \Box G_2)(v) + n_1)}{n_2 m_3 + m_2 k} \right)^{n_2 m_3 + m_2 k} \\
\times \left( \frac{\sum_{uv \in E((G_4 \Box G_1) \Box_k (G_3 \Box G_2))} (d(G_4 \Box G_1)(u) + n_2)(d(G_3 \Box G_2)(v) + n_1)}{k n_1 n_2} \right)^{kn_1 n_2}
\]

(19)

Hence we have the second inequality of the theorem from the following expressions.

\[
\sum_{uv \in E(G_4 \Box G_1)} (d(G_4 \Box G_1)(u) + n_2)(d(G_4 \Box G_1)(v) + n_2) = k M_2(G_1) + n_1 M_2(G_4) + 3 m_4 M_1(G_1) \\
+ 3 m_1 M_1(G_4) + n_2 (k M_1(G_1) + n_1 M_1(G_4) + 8 m_1 m_4 + n_1 n_2 m_4 + k m_1),
\]
The authors declare that there is no conflict of interests.

Conflict of Interests

From expression (20) and (21), we can easily see that equality holds iff all the graphs $G_1$, $G_2$, $G_3$ and $G_4$ are regular. □

References


