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COMMON FIXED POINT FOR SIX SELF MAPS IN METRIC SPACE

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Abstract: In this paper we obtain a common fixed point theorem for six self maps in a metric space without completeness. We also give an example in support of our result.

Keywords: associated sequence relative to six self maps; fixed point; self maps and weakly compatible mappings.

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1. Introduction and Preliminaries

In 1986, G. Jungck [1] introduced the concept of compatible maps as follows.

1.1 Compatible mappings [1]: Two self maps E and F of a metric space (X, d) are said to be compatible mappings if $\lim_{n\to\infty} d(EFx_n, FEx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ex_n = \lim_{n\to\infty} Fx_n = t$ for some $t \in X$.

Further Jungck and Rhoades [4] defined weaker class of maps called weakly compatible maps and is defined as follows.

1.2 Weakly Compatible mappings [4]: Two self maps E and F of a metric space (X, d) are said to be weakly compatible if they commute at their coincidence point. i.e, if Eu = Fu for some $u \in X$ then EFu = FEu.

It is clear that every pair of compatible maps is weakly compatible but not conversely.

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1.3 Example: Let X= (-1, 1] with the usual metric $d(x,y)=|x-y|$ for all x, y∈X. Define self mappings E and F of X by

$$E(x)=\begin{cases}\frac{1}{9} & if\ -1<x<\frac{1}{8}\\ \frac{1}{8} & if\ \frac{1}{8}\le x\le 1\end{cases},\qquad F(x)=\begin{cases}\frac{1}{8} & if\ -1<x\le\frac{1}{8}\\ \frac{1}{9} & if\ \frac{1}{8}<x\le 1\end{cases}$$

$$E\left(\frac{1}{8}\right)=F\left(\frac{1}{8}\right)=\left(\frac{1}{8}\right),\qquad EF\left(\frac{1}{8}\right)=E\left(\frac{1}{8}\right)=\frac{1}{8}\ \text{and}\ FE\left(\frac{1}{8}\right)=F\left(\frac{1}{8}\right)=\frac{1}{8}$$

Hence E and F are weakly compatible.

Hence $EF(x_n)\neq FE(x_n)$ therefore E and F are but not

compatible.

$$\lim_{n\to\infty}EF(x_n)=\lim_{n\to\infty}EF\left(\frac{1}{8}+\frac{1}{2n}\right)=E\left(\frac{1}{9}\right)=\frac{1}{9}\ and\ \lim_{n\to\infty}FE(x_n)=\lim_{n\to\infty}FE\left(\frac{1}{8}+\frac{1}{2n}\right)=F\left(\frac{1}{8}\right)=\frac{1}{8}$$

1.4 Associated sequence[6]: Suppose E,F,G,H,I and J are six self maps of a metric space (X,d) such that $E(X)\subseteq IJ(X)\ and\ F(X)\subseteq GH(X)$ .Then for an arbitrary $x_0\in X$ we have $Ex_0\in E(X)$. since $E(X)\subseteq IJ(X)$, there exists $x_1\in X$ such that $Ex_0=IJx_1$. for this point x_1, there is a point $x_2\in X$ such that $Fx_1=GHx_2$ and so on. Repeating this process to obtain a sequence $\{y_n\}$ in X such that $y_{2n}=Ex_{2n}=IJx_{2n+1}\ and\ y_{2n+1}=Fx_{2n+1}=GHx_{2n+2}$ for $n\ge 0$ we shall call this sequence $\{y_n\}$ an associated sequence of x_0 relative to the six self maps E,F,G,H,I and J.

2. Lemma: Let E, F, G, H, I and J are six self maps of a metric space (X,d) satisfying

$$E(X)\subseteq IJ(X)\ and\ F(X)\subseteq GH(X) \tag{2.1}$$

$$d(Ex,Fy)\le\alpha\frac{d(IJy,Fy)[1+d(GHx,Ex)]}{[1+d(GHx,IJy)]}+\beta d(GHx,IJy) \tag{2.2}$$

for all x,y in X where $\alpha,\beta\ge 0,\alpha+\beta<1$.

Furtherif X is complete, then for any $x_0\in X$ and for any of its associated sequence $Ex_0,Fx_1,Ex_2,Fx_3,.........Ex_{2n},Fx_{2n+1}.....$ converges to some point p in X .

Proof: From the conditions (2.1) and (2.2) we have

$$d(y_{2n},y_{2n+1})=d(Ex_{2n},Fx_{2n+1})$$

$$\leq \alpha\frac{d(IJx_{2n+1},Fx_{2n+1})[1+d(GHx_{2n},Ex_{2n})]}{[1+d(GHx_{2n},IJy_{2n+1})]}+\beta d(GHx_{2n},IJy_{2n+1})$$

$$=\alpha\frac{d(y_{2n},y_{2n+1})[1+d(y_{2n-1},y_{2n})]}{[1+d(y_{2n-1},y_{2n})]}+\beta d(y_{2n-1},y_{2n})$$

$$=\alpha d(y_{2n,}y_{2n+1})+\beta d(y_{2n-1},y_{2n})$$ and so that

$$(1-\alpha)d(y_{2n},y_{2n+1})\leq \beta d(y_{2n-1},y_{2n})$$

$$d(y_{2n},y_{2n+1})\leq\frac{\beta}{(1-\alpha)}d(y_{2n-1},y_{2n})=hd(y_{2n-1},y_{2n}),\ \text{where } h=\frac{\beta}{1-\alpha}$$

That is $d(y_{2n,}y_{2n+1})\leq h(y_{2n-1},y_{2n})$ (2.3)

Similarly, we can prove that $d(y_{2n+1},y_{2n+2})\leq hd(y_{2n},y_{2n+1})$. (2.4)

Hence, from (2.3) and (2.4), we get

$$d(y_n,y_{n+1})\leq hd(y_{n-1},y_n)\leq h^2d(y_{n-2},y_{n-1})\leq.......\leq h^nd(y_0,y_1). \quad (2.5)$$

Now for any positive integer k, we have

$$d(y_n,y_{n+k})\leq d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+........+d(y_{n+k-1},y_{n+k})$$

$$\leq h^nd(y_0,y_1)+h^{n+1}d(y_0,y_1)+........+h^{n+k-1}d(y_0,y_1)$$

$$=(h^n+h^{n+1}+.........+h^{n+k-1})d(y_0,y_1)$$

$$=h^n(1+h+h^2+.......+h^{k-1})d(y_0,y_1)$$

$$<\frac{h^n}{1-h}d(y_0,y_1)\to 0\quad \text{as}\ \ \text{n}\to\infty,\text{since h<1}.$$

So that $d(y_n,y_{n+k})\to 0.$

Thus the sequence $\{y_n\}$ is a Cauchy sequence in X. Since X is a complete, it converges to some pointp in X.

2.6 Remark: The converse of the above Lemma is not true. That is, if E, F, G, H, I and J are self maps of metric space(X, d) satisfying (2.1), (2.2) and even if for any x_0 in X and for any of its associated sequence of converges. The metric space need not be complete. This can be seen from the following example.

2.7 Example: Let X=(-1, 1] with the usual metric $d(x,y)=|x-y|$ for all x, y∈X.Define self mappings E, F, G, H, I and J of X by

$$E(x)=F(x)=\begin{cases}\frac{1}{8} & if\ -1<x\le\frac{1}{8}\\ \frac{1}{9} & if\ \ \frac{1}{8}<x\le 1\end{cases}\ ,\quad J(x)=\begin{cases}\frac{1}{9} & if\ -1<x<\frac{1}{8}\\ \frac{1}{4}-x & if\ \frac{1}{8}\le x\le 1\end{cases}$$

$$I(x)=G(x)=x\ \ if\ -1<x\le 1\ \ ,\quad H(x)=\begin{cases}\frac{1}{9} & if\ -1<x<\frac{1}{8}\\ \frac{8x+7}{64} & if\ \frac{1}{8}\le x\le 1\end{cases}$$

Then

$$IJ(x)=\begin{cases}\frac{1}{9} & if\ -1<x<\frac{1}{8}\\ \frac{1}{4}-x & if\ \frac{1}{8}\le x\le 1\end{cases}\ ,\quad GH(x)=\begin{cases}\frac{1}{9} & if\ -1<x<\frac{1}{8}\\ \frac{8x+7}{64} & if\ \frac{1}{8}\le x\le 1\end{cases}.$$

$$E(x)=F(x)=\left\{\frac{1}{8},\frac{1}{9}\right\},\ \ J(x)=\left[\frac{-3}{4},\frac{1}{8}\right],\ \ IJ(x)=\left[\frac{-3}{4},\frac{1}{8}\right]$$

and

$$\text{H(x)}=\left\{\frac{1}{9}\right\}\cup\left[\frac{-3}{4},\frac{1}{8}\right],\quad GH(x)=\left\{\frac{1}{9}\right\}\cup\left[\frac{-3}{4},\frac{1}{8}\right]$$

Clearly $E(X)\subseteq IJ(X),\ F(X)\subseteq GH(X)$. Also the inequality (2.2) can easily be verified for appropriate values of $\alpha,\beta\ge 0,\ \alpha+\beta<1.$ Moreover if we take $x_n=\frac{1}{8}+\frac{1}{2n}$ for $n\ge 1$ then the associated sequence $Ex_0,Fx_1,Ex_2,Fx_3,....Ex_{2n},Fx_{2n+1}.....$ converges to $\frac{1}{8}$. Note that (X, d) is not complete.

The following theorem was proved in [5].

2.8 Theorem: Let P, Q, S and T be self mappings from a complete metric space (X,d) into itself satisfying the following conditions

$$S(X)\subseteq Q(X)\ and\ T(X)\subseteq P(X) \tag{2.8.1}$$

$$d(Sx,Ty)\le\alpha\frac{d(Qy,Ty)[1+d(Px,Sx)]}{[1+d(Px,Qy)]}+\beta d(Px,Qy) \tag{2.8.2}$$

for all $x,y\ in\ X\ where\ \alpha,\beta\ge 0,\alpha+\beta<1.$

one of P, Q, S and T is continuous and (2.8.3)

the pairs (S, P) and (T, Q) are compatible on X. (2.8.4)

Then P, Q, S and T have a unique common fixed point in X.

Now we extend and generalize the above Theorem to six self maps as follows.

3. Main result

3.1 Theorem: If E, F, G, H, I and J are self maps of a metric space (X,d) satisfying the conditions

$$E(X) \subseteq IJ(X) \text{ and } F(X) \subseteq GH(X) \tag{3.1.1}$$

$$d(Ex,Fy) \leq \alpha \frac{d(IJy,Fy)[1+d(GHx,Ex)]}{[1+d(GHx,IJy)]} + \beta d(GHx,IJy) \tag{3.1.2}$$

for all $x, y \text{ in } X \text{ where } \alpha, \beta \geq 0, \alpha+\beta<1.$

IJ=JI, GH=HG, HE=EH, FJ=JF, (GH)E=E(GH) and (IJ)F=F(IJ) (3.1.3)

the pairs (E, GH) and (F, IJ) are weakly compatible on X (3.1.4)

IJ(x) and GH(x) are closed in X (3.1.5)

Further if there is a point $x_0 \in X$ and its associated sequence

$\{y_n\} = \{Ex_0, Fx_1, Ex_2, Fx_3..............\}$ relative to six self maps E, F, G, H, I and J converges to some point p∈X, then p is a unique common fixed point of E, F, G, H, I and J. (3.1.6)

Proof: From (3.1.6), we have $Ex_{2n} \to p,\ IJx_{2n+1} \to p,\ Fx_{2n+1} \to p,\ and\ GHx_{2n+2} \to p\ as\, n \to \infty.$ (3.1.7)

Suppose IJ(x) is closed in X. Then there exists $u \in X$ such that

p = IJu = $\lim_{n\to\infty} IJx_{2n+1}$ (3.1.8)

Now from (3.1.2), we obtain

$$d(Ex_{2n},Fu) \leq \alpha \frac{d(IJu,Fu)\left[1+d(GHx_{2n},Ex_{2n})\right]}{\left[1+d(GHx_{2n},IJu)\right]} + \beta d(GHx_{2n},IJu)$$

$$d(p,Fu) \leq \alpha \frac{d(p,Fu)\left[1+d(p,p)\right]}{\left[1+d(p,p)\right]} + \beta d(p,p)$$

$(1-\alpha)\, d(p,Fu) \leq 0$

$d(p,Fu) \leq 0$ since $\alpha, \beta \geq 0, \alpha+\beta<1$ and this implies Fu = p.

Hence p = Fu = IJu. (3.1.9)

Since (F, IJ) is weakly compatible, we have (IJ)Fu = F(IJ)u. Thus IJp =Fp. (3.1.10)

Again from (3.1.2), we obtain

$$d(Ex_{2n},Fp)\le\alpha\frac{d(IJp,Fp)\left[1+d(GHx_{2n},Ex_{2n})\right]}{\left[1+d(GHx_{2n},IJp)\right]}+\beta d(GHx_{2n},IJp)$$

$$d(p,Fp)\le\alpha\frac{d(p,Fp)\left[1+d(p,p)\right]}{\left[1+d(p,Fp)\right]}+\beta d(p,Fp)$$

(1 - β) d (p, Fp) ≤ 0,

(p, Fp) ≤ 0, since α, β ≥ 0, α + β < 1 and this implies Fp = p.

Hence p= Fp = IJp. (3.1.11)

So from equation (3.1.2), we obtain

$$d(Ex_{2n},FJp)\le\alpha\frac{d((IJ)Jp,FJp)\left[1+d(GHx_{2n},Ex_{2n})\right]}{\left[1+d(GHx_{2n},(IJ)Jp)\right]}+\beta d(GHx_{2n},(IJ)Jp)$$

$$d(p,Jp)\le\alpha\frac{d(Jp,Jp)\left[1+d(p,p)\right]}{\left[1+d(p,Jp)\right]}+\beta d(p,Jp)$$

$(1-\beta)d(p,Jp)\le 0$,

$d(p,Jp)\le 0$, since α, β ≥ 0, α + β < 1 and this implies Jp = p.

Thus IJp = p ⇒ Ip =p.

Hence Fp = Jp = Ip = p. (3.1.12)

Now since GH(X) is closed, we can find $v\in X$ such that

P = GHv = $\lim_{n\to\infty} GHx_{2n+1}$ (3.1.13)

So from (3.1.2), we obtain

$$d(Ev,Fx_{2n+1})\le\alpha\frac{d(IJx_{2n+1},Fx_{2n+1})\left[1+d(GHv,Ev)\right]}{\left[1+d(GHv,IJx_{2n+1})\right]}+\beta d(GHv,IJx_{2n+1})$$

$$d(Ev,p)\le\alpha\frac{d(p,p)\left[1+d(p,Ev)\right]}{\left[1+d(p,p)\right]}+\beta d(p,p)$$

$d(Ev,p)\le$ 0,

$d(Ev,p)\le$ 0 since α, β ≥ 0, α + β < 1 and so that Ev = p.

Hence p = Ev = GHv. (3.1.14)

Since (E, GH) is weakly compatible, we have (GH)Ev = E(GH)v. Thus GHp =Ep. (3.1.15)

Therefor from (3.1.2), we obtain

$$d(Ep,Fx_{2n+1})\le\alpha\frac{d(IJx_{2n+1},Fx_{2n+1})\left[1+d(GHp,Ep)\right]}{\left[1+d(GHp,IJx_{2n+1})\right]}+\beta d(GHp,IJx_{2n+1})$$

$$d(Ep,p) \le \alpha \frac{d(p,p)\left[1+d(Ep,Ep)\right]}{\left[1+d(Ep,p)\right]} + \beta d(Ep,p)$$

$(1-\beta)d(Ep,p) \le 0$,

$d(Ep,p) \le 0$,since α, β ≥ 0, α + β < 1 and so that Ep = p.

Hence p = Ep = GHp. (3.1.16)

So from equation (3.1.2), we obtain

$$d(EHp,Fx_{2n+1}) \le \alpha \frac{d(IJx_{2n+1},Fx_{2n+1})\left[1+d((GH)Hp,EHp)\right]}{\left[1+d((GH)Hp,IJx_{2n+1})\right]} + \beta d((GH)Hp,IJx_{2n+1})$$

$$d(Hp,p) \le \alpha \frac{d(p,p)\left[1+d(Hp,Hp)\right]}{\left[1+d(Hp,p)\right]} + \beta d(Hp,p)$$

$d(Hp,p) \le \beta d(Hp,p)$

$(1-\beta)d(Hp,p) \le 0$,

$d(Hp,p) \le 0$,since α , β ≥ 0, α + β < 1 and so that Hp = p.

Thus GHp = p ⇒ Gp = p.

Hence Ep = Hp = Gp = p.

Therefore Ep = Fp = Gp = Hp = Ip = Jp =p, showing that p is a common fixed point of E, F, G, H, I and J. The uniqueness of fixed point can be proved easily.

If I = J and G = H, we get the following result.

3.2 Corollary: Let E, F, G and I be self mappings from a metric space(X, d) into it self satisfying the following conditions.

$E(X) \subseteq I(X)$ *and* $F(X) \subseteq G(X)$ (3.2.1)

$$d(Ex,Fy) \le \alpha \frac{d(Iy,Fy)[1+d(Gx,Ex)]}{[1+d(Gx,IJy)]} + \beta d(Gx,Iy) \quad (3.2.2)$$

for all x, y *in* X *where* $\alpha, \beta \ge 0, \alpha + \beta < 1.$

the pairs (E, G) and (F, I) are weakly compatible on X (3.2.3)

I(x) and G(x) are closed in X (3.2.4)

Further if there is a point $x_0 \in X$ and its associated sequence

$\{y_n\} = \{Ex_0, Fx_1, Ex_2, Fx_3..............\}$ relative to four self maps E, F, G and I converges to some point p∈X, then p is a unique common fixed point of E, F, G and I.

3.3 Remark

In the example (2.7), the self maps E, F, G, H, I and J satisfy all the conditions of the Theorem (3.1). It may be noted that '$\frac{1}{8}$' is the unique common fixed point of E, F, G, H, I and J.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] G.Jungck, Compatible mappings and common fixed points, Int. J. Math. Math. Sci, 9(1986), 771-778.

[2] R.P.Pant, A Common fixed point theorem under a new condition, Indian. Appl. Math., 30(2) (1999), 147-152.

[3] Jungck, Compatible mappings and common fixed points (2), Int. J. Math. Math. Sci., 11(1988), 285-288.

[4] G.Jungck and B.E. Rhoades, Fixed point for set valued functionswithout continuity, Indian Pure. Appl. Math. 29(3) (1998), 227-238.

[5] P.C.Lohani and V.H.Badshah, Compatible mappings and common fixed point for four mappings, Bull. Cal. Math. Soc., (1998), 301-308.

[6] V.Nagaraju and V.Srinivas, Common fixed point theorem on compatible mappings of type (P), Gen. Math. Notes, 21(2) (2014), 87-94.

[7] Pathak H.K. and Khan M.S., A comparison of various types of Compatible maps and common fixed points, Indian J. Pure Appl. Math, 28(4) (1997), 477-485.

[8] Jungck G, Murthy P.P and Cho Y.J., Compatible mappings of type (A) and common fixed points, Math. Japan, 38(1993), 381-391.

[9] Khan M.S, Pathak H.K and George Reny, Compatible mappings of type (A-1) and type(A-2) and common fixed points in fuzzy metric spaces, Int. Math. Forum, 2(11)(2007),515-524.

[10] V.Srinivas and Ravi S, Generation of a common fixed point theorem using A-compatible and B-compatible mappings of type(E), Glob. J. Pure Appl. Math., 13(6) (2017), 1735-1744.

[11] A. Djoudi, A common fixed point theorem for compatible mappingsoftype (B) in complete metric spaces, Demonstr.Math., XXXVI (2) (2003), 463-470.

[12] B.Singh and S.Chauhan,On common fixed points of four mappings, Bull. Call. Math. Soc.,88(1998),301-308.

[13] H.K.Pathak and M.S.Khan, compatible mappings of type(B) and common fixed point of Gregustype, Czech Math. J., 45 (120) (1995), 685-698.

[14] H.K.Pathak,Y.J.Cho, S,S.Chang and S.M.Kang, compatible mappings of type(P) and fixed point theorem in metric spaces and Probabilistic metricspaces, Novi Sad. J. Math. 26(2)(1996), 87-109.

[15] Ravi Sriramula and V.Srinivas, Extraction of a common fixed point theoremusing semi compatible mappings, Indian J. Math. Math. Sci.,13 (2), (2017), 433-443.