COMMON FIXED POINT THEOREM FOR TWO SELFMAPS OF AN S-METRIC SPACE WITH RATIONAL INEQUALITY

V. KIRAN*

Department of Mathematics, Osmania University, Hyderabad 500007, India

Abstract: A common fixed point theorem for two self maps of an S-metric space with rational inequality is proved in the present paper.

Keywords: S-metric space; fixed point; associated sequence of a point relative to two self maps; compatible mappings.

2010 AMS Subject Classification: 54H25, 47H10.

1. INTRODUCTION


*Corresponding author
E-mail address: kiranmathou@gmail.com
Received March 3, 2019
The purpose of this paper is to prove a common fixed point theorem for two self maps of an S-metric space with rational inequality.

2. Preliminaries

Definition 2.1 [5]: Let $X$ be a non empty set. By an $S$–metric we mean a function $S : X^3 \to [0, \infty)$ which satisfies the following conditions for all $x, y, z, w \in X$:

(a) $S(x, y, z) \geq 0$

(b) $S(x, y, z) = 0$ if and only if $x = y = z$.

(c) $S(x, y, z) \leq S(x, x, w) + S(y, y, w) + S(z, z, w)$

Also, the pair $(X, S)$ is called a $S$-metric space.

Example 2.2: Let $X = \mathbb{R}$ and $S : \mathbb{R}^3 \to [0, \infty)$ be defined by $S(x, y, z) = |y + z - 2x| + |y - z|$ for all $x, y, z \in X$, then $(X, S)$ is a $S$-metric space.

Remark 2.3: It was shown in ([5], Lemma 2.5) that $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.

Definition 2.4: Let $(X, S)$ be a $S$–metric space. A sequence $\{x_n\}$ in $X$ is said to be Convergent, if there is a $x \in X$ such that $S(x_n, x_n, x) \to 0$ as $n \to \infty$; that is, for each $\varepsilon > 0$, there exists a $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $S(x_n, x_n, x) < \varepsilon$ and we write in this case that $\lim_{n \to \infty} x_n = x$.

Definition 2.5: Let $(X, S)$ be a $S$–metric space. A sequence $\{x_n\}$ in $X$ is said to be a Cauchy sequence, if for each $\varepsilon > 0$, there exists a $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \varepsilon$ for each $n, m \geq n_0$. 
It is easy to see that (in fact proved in [5], Lemma 2.10 and Lemma 2.11), if \{x_n\} converges to \(x\) in \((X,S)\) then \(x\) is unique and \{x_n\} is a Cauchy sequence in \((X,S)\). However, a Cauchy sequence in \((X,S)\) need not be convergent as shown in the following example

**Example 2.6**: Let \(X=(0,1] \) and \(S(x,y,z)=|x-y|+|y-z|+|z-x|\) for \(x,y,z \in X\), so that \((X,S)\) is a \(S\)-metric space. Taking \(x_n=\frac{1}{n}\) for \(n=1,2,3,...\) then \(S(x_n,x_n,x_m)=2\left|\frac{1}{n}-\frac{1}{m}\right|\) so that \(S(x_n,x_n,x_m) \to 0\) as \(n,m \to \infty\) proving that \{x_n\} is a Cauchy sequence in \((X,S)\) but \{x_n\} does not converge to any point in \(X\).

**Definition 2.7**: Let \((X,S)\) be an \(S\)-metric space. If there exists sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that \(\lim_{n \to \infty} x_n = x\) and \(\lim_{n \to \infty} y_n = y\) then \(\lim_{n \to \infty} S(x_n,x_n,y_n) = S(x,x,y)\). Then we say that \(S(x,y,z)\) is continuous in \(x\) and \(y\).

**Definition 2.8**: If \(g\) and \(f\) are selfmaps of a \(S\)-metric space \((X,S)\) such that for every sequence \(\{x_n\}\) in \(X\) with \(\lim_{n \to \infty} gx_n = \lim_{n \to \infty} fx_n = t\) for some \(t \in X\) we have

\[
\lim_{n \to \infty} S(gx_n,fx_n,fgx_n) = 0
\]
then \(g\) and \(f\) are said to be compatible.

Trivially commuting self maps of a \(S\)-metric space are compatible but not conversely. As an example we have the following.

**Example 2.9**: Let \(X=[0,1]\) with \(S(x,y,z)=|x-y|+|y-z|+|z-x|\) for \(x,y,z \in X\). Then \(S\) is a \(S\)-metric on \(X\). Define \(g:X \to X, f:X \to X\) by \(gx = \frac{x^2}{2}\) and \(fx = \frac{x^2}{3}\) for \(x \in X\). we now prove that \(g, f\) are compatible.
Let \( \{x_n\} \) be a sequence in \( X \) with \( \lim g x_n = \lim f x_n = t \) for some \( t \in X \). Then
\[
\lim \frac{x_n^2}{2} = t = \lim \frac{x_n^2}{3}
\]
so that \( 3t = 2t \) which shows that \( t = 0 \). Also since \( g f x_n = \frac{x_n^4}{18} \) and then we have
\[
\lim S( g f x_n, g f x_n, f g x_n ) = \lim S \left( \frac{x_n^4}{18}, \frac{x_n^4}{18}, \frac{x_n^4}{12} \right) = \lim \frac{x_n^4}{18} = 0
\]
showing that \((g, f)\) is a pair of compatible self maps. But \( g f (1) = \frac{1}{18} \) and \( f g (1) = \frac{1}{12} \)
proves that \( g f (1) \neq f g (1) \) showing that that \( g \) and \( f \) are not commutative

**Definition 2.10:** Let \( g \) and \( f \) be self maps of a \( S \)-metric space such that \( g(X) \subseteq f(X) \). For any \( x_0 \in X \), if \( \{x_n\} \) is a sequence in \( X \) such that \( f x_n = g x_{n-1} \) for \( n \geq 0 \). Then \( \{x_n\} \) is called an associated sequence of \( x_0 \) relative to the two self maps \( g \) and \( f \).

**3. Main Results**

Before stating main Theorem, we prove an essential Lemma.

**Lemma 3.1:** Let \( f \) and \( g \) be compatible self maps of an \( S \)-metric space \((X, S)\). Suppose
\[
\lim f x_n = \lim g x_n = x \quad \text{for some} \quad x \in X \quad \text{and some sequence} \quad \{x_n\} \quad \text{in} \quad X,
\]
then \( \lim g f x_n = f x \) if \( f \) is continuous.

**Proof:** suppose \( f \) and \( g \) are compatible mappings and \( \lim f x_n = \lim g x_n = x \) for some \( x \in X \). Then
\[
(3.1.1) \quad \lim S( g f x_n, g f x_n, g f x_n ) = 0
\]
since \( f \) is continuous and \( g x_n \rightarrow x \) as \( n \rightarrow \infty \) we have
\[
(3.1.2) \quad \lim g f x_n = f x.
\]
From (3.1.1) and (3.1.2) we get
\[
\lim S( f x, f x, g f x_n ) = 0 \quad \text{which imply} \quad \lim g f x_n = f x.
\]
S-METRIC SPACE WITH RATIONAL INEQUALITY

Proving the lemma.

**Theorem 3.2.** Let \( f \) and \( g \) be self maps of a \( S \)-metric space \( (X,S) \) satisfying

1. \( g(X) \subseteq f(X) \)
2. \( S(gx, gx, gy) \leq \frac{\alpha S(fx, fx, gy)[1 + S(fx, fx, gx)]}{1 + S(fx, fx, fy)} + \beta S(fx, fx, fy) \)

for all \( x, y \in X \) where \( \alpha, \beta \geq 0, \alpha + \beta < 1 \)

3. one of \( f \) and \( g \) is continuous
4. \( f \) and \( g \) are compatible
5. an associated sequence \( \{x_n\} \) of a point \( x_0 \in X \) relative to the self maps \( f \) and \( g \) is such that \( \{fx_n\} \) converges to \( t \) for some point \( t \in X \),

Then \( t \) is the common fixed point of \( f \) and \( g \).

**Proof:** From (v), the associated sequence \( \{x_n\} \) of \( x_0 \) relative to the selfmaps \( f \) and \( g \) such that \( fx_n = gx_{n-1} \) for \( n \geq 1 \) and \( fx_n \rightarrow t \) as \( n \rightarrow \infty \) it follows that \( gx_n \rightarrow t \) as \( n \rightarrow \infty \)

**Case(i):** If \( f \) is continuous, then we have by Lemma 3.1 that

\[ \lim_{{n \to \infty}} \alpha S(fx_n, fx_n, gy) = \alpha \]

and also

\[ \lim_{{n \to \infty}} f^2 x_n = ft \]

Now from (ii) we get

\[ S(gfx_n, gfx_n, gx_{n-1}) \leq \frac{S(f^2 x_n, f^2 x_n, gx_{n-1})[1 + S(f^2 x_n, f^2 x_n, gfx_n)]}{1 + S(f^2 x_n, f^2 x_n, fx_{n-1})} + \beta[S(f^2 x_n, f^2 x_n, fx_{n-1})] \]

Where \( \alpha, \beta \geq 0, \alpha + \beta < 1 \)

on letting \( n \rightarrow \infty \) in the above inequality and using (3.2.1) and (3.2.2), we get
\[ S(ft, ft, t) \leq \frac{\alpha S(ft, ft, t)[1 + S(ft, ft, t)]}{1 + S(ft, ft, t)} + \beta S(ft, ft, t) \]

i.e \[ S(ft, ft, t) \leq \frac{\alpha S(ft, ft, t)}{1 + S(ft, ft, t)} + \beta S(ft, ft, t) \]

\[ S(ft, ft, t) < (\alpha + \beta)S(ft, ft, t) \]

which implies \( S(ft, ft, t) = 0 \) and hence \( ft = t \)

Also from (ii), we get
\[ S(gt, gt, gx_{n-1}) \leq \frac{\alpha S(ft, ft, gx_{n-1})[1 + S(ft, ft, gt)]}{1 + S(ft, ft, fx_{n-1})} + \beta S(ft, ft, gx_{n-1}) \]

where \( \alpha, \beta \geq 0, \ \alpha + \beta < 1 \)

Letting \( n \to \infty \) in the above inequality, we obtain
\[ S(gt, gt, t) \leq \frac{\alpha S(ft, ft, t)[1 + S(ft, ft, t)]}{1 + S(ft, ft, t)} + \beta S(ft, ft, t) \]

since \( ft = t \), we get \( S(gt, gt, t) = 0 \) which implies \( gt = t \), showing that \( t \) is a common fixed point of \( f \) and \( g \).

**Case(ii)**: Now suppose that \( g \) is a continuous, then we have by Lemma 3.1, that

(3.2.3) \[ \lim_{n \to \infty} fgx_n = gt \]

(3.2.4) \[ \lim_{n \to \infty} g^2x_n = gt \]

Now from (ii), we get
\[ S(g^2x_n, g^2x_n, gx_{n-1}) \leq \frac{S(fgx_n, fgx_n, gx_{n-1})[1 + S(fgx_n, fgx_n, g^2x_n)]}{1 + S(fgx_n, fgx_n, fx_{n-1})} + \beta[S(fgx_n, fgx_n, fx_{n-1})] \]

where \( \alpha, \beta \geq 0, \ \alpha + \beta < 1 \)

on letting \( n \to \infty \) in the above inequality and using (3.2.3) and (3.2.4), we get
S-METRIC SPACE WITH RATIONAL INEQUALITY

\[ S(gt, gt, t) \leq \frac{\alpha S(gt, gt, t)(1 + S(gt, gt, gt))}{1 + S(gt, gt, t)} + \beta S(gt, gt, t) \]
\[ < (\alpha + \beta)S(gt, gt, t) \]

which implies \( S(gt, gt, t) = 0 \) and hence \( gt = t \)

(since \( \alpha + \beta < 1 \), \( 1 + S(gt, gt, t) > 1 \Rightarrow \frac{1}{1 + S(gt, gt, t)} < 1 \))

From (i), we can find \( w \in X \) such that \( gt = fw \). Now from (ii) we have

\[ S(g^2 x_n, g^2 x_n, gw) \leq \frac{S(fgx_n, fgx_n, gw)(1 + S(fgx_n, fgx_n, g^2 x_n))}{1 + S(fgx_n, fgx_n, fw) + \beta[S(fgx_n, fgx_n, fw)]} \]

where \( \alpha, \beta \geq 0, \alpha + \beta < 1 \)

Letting \( n \to \infty \) in the above inequality and using (3.2.3) and (3.2.4), we obtain

\[ S(gt, gt, gw) \leq \frac{\alpha S(gt, gt, gw)(1)}{1} \]

That is, \( S(gt, gt, gw) \leq \alpha S(gt, gt, gw) \)

which implies that \( gt = gw \) since \( \alpha \in (0,1) \)

thus \( t = gt = gw = fw \).

Now put \( y_n = w \) for \( n = 0, 1, 2, 3... \) then \( fy_n \to fw \) and \( gy_n \to gw \) as \( n \to \infty \) since \( \lim S(fgx_n, fgx_n, gfw) = 0 \) giving \( S(fgw, fgw, gfw) = 0 \) which implies that \( fgw = gfw \)

since \( fw = gw = t \) we get \( ft = gt \) and since \( gt = t \), it follows that \( ft = gt = t \), showing that \( t \) is a common fixed point of \( f \) and \( g \).
Finally to prove the uniqueness of common fixed point \( f \) and \( g \).

Suppose \( u = fu = gu \) and \( v = fv = gv \) for some \( u, v \in X \)

From (ii), we get

\[
S(u, u, v) = S(gu, gu, gv) \leq \frac{\alpha S(fu, fu, gv)[1 + S(fu, fu, gu)]}{1 + S(fu, fu, fv)} + \beta S(fu, fu, fv)
\]

where \( \alpha, \beta \geq 0, \alpha + \beta < 1 \)

\[
S(u, u, v) \leq \frac{\alpha S(u, u, v) [1 + S(u, u, u)]}{1 + S(u, u, v)} + \beta S(u, u, v)
\]

\[
= \frac{[\alpha + \beta]S(u, u, v)}{1 + S(u, u, v)}
\]

which implies that \( S(u, u, v) = 0 \) since \( \frac{S(u, u, v)}{1 + S(u, u, v)} < 1 \) and hence \( u = v \), proving the theorem completely.

3.3 Example: Let \( X = [0, 1) \) and \( S(x, y, z) = d(x, y) + d(x, z) + d(y, z) \) for all \( x, y, z \in X \)

and \( d(x, y) = |x - y| \), then \( (X, S) \) is a \( S \)-metric space. Define \( f : X \rightarrow X \) and \( g : X \rightarrow X \)

by \( f(x) = x, \ g(x) = \frac{x}{2} \) for all \( x \in X \) Then \( g(X) = [0, \frac{1}{2}] \subset [0, 1) = f(X) \), clearly \( fg = gf \),

so that \( f \) and \( g \) are compatible. Also an associated sequence of \( x_0 = 0 \) relative to the self

maps \( f \) and \( g \) is given by \( x_n = 0 \) for \( n \geq 0 \) and since \( \{fx_n\} \) is a constant sequence converging

to ‘0’, which is a point in \( X \) taking \( \alpha = 0, \beta = \frac{1}{2} \) then \( f \) and \( g \) satisfy the inequality (ii).

Thus the conditions (iii) and (v) of Theorem 3.2 are satisfied.

Hence by Theorem 3.2, ‘0’ is the unique common fixed point of \( f \) and \( g \).
S-METRIC SPACE WITH RATIONAL INEQUALITY

REFERENCES


