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COMMON FIXED POINTS FOR TWO SET-VALUED MAPPINGS ON METRIC SPACES

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Abstract. In this paper, we introduce three classes of functions and construct two contractive conditions to discuss and obtain some new common fixed point theorems for two set-valued mappings on non-complete metric spaces.

Keywords: common fixed point; set-valued mapping; Hausdorff metric.

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1. INTRODUCTION AND PRELIMINARIES

Let (X, d) be a metric space and $CB(X)$ the family of all nonempty closed and bounded subset of X .

The following is the famous Banach's fixed point theorem^[1]:

Let (X, d) be a complete metric space and $f : X \rightarrow X$ a mapping. If f satisfies

$$d(fx, fy) \leq hd(x, y), \forall x, y \in X,$$

where $h \in [0, 1)$. Then f has a unique common fixed point in X .

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Later, many generalizations of Banach's fixed point theorem have appeared. For instance, if $f : X \rightarrow X$ is a mapping on a complete metric space (X, d) satisfying the following a quasi-contraction

$$d(fx, fy) \leq h \max\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}, \forall x, y \in X,$$

where $h \in [0, 1)$. Then f has a unique common fixed point in X . (see [2]).

In 1969, Nadlдер^[3] extended the famous Banach contraction Principle from a single-valued mapping to a set-valued mappings and gave the next fixed point theorem:

Theorem 1.1([3]) Let (X, d) be a complete metric space and $T : X \rightarrow CB(X)$. If there exists $h \in [0, 1)$ such that

$$\mathcal{H}(Tx, Ty) \leq h d(x, y), \forall x, y \in X,$$

where \mathcal{H} denote the Hausdorff metric on $CB(X)$ induced by d , that is,

$$\mathcal{H}(A, B) = \max\{\sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A)\}, \forall A, B \in CB(X),$$

where $D(x, B) = \inf_{z \in B} d(x, z)$. Then T has a fixed point in X .

Mizoguchi-Takahashi^[4] also gave the following fixed point theorem:

Theorem 1.2([4]) Let (X, d) be a complete metric space and $T : X \rightarrow CB(X)$. If

$$\mathcal{H}(Tx, Ty) \leq \xi(d(x, y)) d(x, y), \forall x, y \in X,$$

where $\xi : [0, \infty) \rightarrow [0, 1)$ satisfying $\limsup_{s \rightarrow t^+} \xi(s) < 1$ for all $t \in [0, \infty)$. Then T has a fixed point in X .

In 2011, Amini-Harandi generalized and improved the corresponding result in [2] from a single-valued mapping to a set-valued mapping, obtained the next result:

Theorem 1.3([5]) Let (X, d) be a complete metric space and $T : X \rightarrow CB(X)$ a k -set-valued quasi-contraction with $k < \frac{1}{2}$, that is,

$$\mathcal{H}(Tx, Ty) \leq k \max\{d(x, y), D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)\}, \forall x, y \in X.$$

Then T has a fixed point in X .

And in 2011, Chen^[6] introduced the following definition of ψ -contraction and obtained a fixed point theorem for set-valued mappings:

Theorem 1.4([6]) Let (X, d) be a complete metric space and $T : X \rightarrow CB(X)$ a set-valued ψ -contraction, that is,

$$\mathcal{H}(Tx, Ty) \leq \psi(d(x, y), D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)), \forall x, y \in X,$$

then T has a fixed point in X . Where $\psi : \mathbb{R}^{+5} \rightarrow \mathbb{R}^+$ is a function satisfying some particular conditions, see [6].

On the other hand, In 2009, Wu et al^[7] obtained the next common fixed point theorem for set-valued mappings $\{A_i\}$:

Theorem 1.5([7]) Let (X, d) be a complete metric space and $A_i : X \rightarrow CB(X)$ satisfy the condition: for any $i, j = 1, 2, \dots, x, y \in X$ and $u \in A_i x$, there exists $v \in A_j y$ such that

$$d(u, v) \leq \Phi(d(x, y), D(x, A_i x), D(y, A_j y), D(x, A_j y), D(y, A_i x)),$$

then $\{A_i\}$ have a common fixed point in X . Where $\Phi : \mathbb{R}^{+5} \rightarrow \mathbb{R}^+$ is a function satisfying some particular conditions, see [7].

The following is well-known results, see [7]:

Lemma 1.1 If (X, d) is a metric space, $A, B \in CB(X)$, then for any $\varepsilon > 0$ and any $a \in A$, there exists $b \in B$ such that $d(a, b) \leq \mathcal{H}(A, B) + \varepsilon$.

Lemma 1.2 If (X, d) is a metric space, $A \in CB(X)$, then $D(\cdot, A)$ is continuous. Moreover, we have that

- (i) $A = \{x \in X | d(x, A) = 0\}$;
- (ii) For any $x, y \in X$, $D(x, A) \leq d(x, y) + D(y, A)$.

In this paper, we use the method in [6] to obtain some common fixed point theorems for two set-valued mappings in metric spaces.

2. COMMON FIXED POINTS

Now, we begin with the following definition.

Let $\psi : \mathbb{R}^{+5} \rightarrow \mathbb{R}^+$ be a functions satisfying the following conditions, where $\mathbb{R}^+ = [0, +\infty)$:

(A1) ψ is non-decreasing and continuous in each coordinate;

(A2) for all $t > 0$, $\psi(t, t, t, 0, 2t) < t$, $\psi(t, t, t, 2t, 0) < t$, $\psi(0, 0, t, t, 0) < t$, $\psi(0, t, 0, 0, t) < t$ and $\psi(t, 0, 0, t, t) < t$.

Example 1.1 Let $\psi : \mathbb{R}^{+5} \rightarrow \mathbb{R}^+$ be $\psi(t_1, t_2, t_3, t_4, t_5) = a_1 t_1 + a_2 t_2 + a_3 t_3 + a_4 t_4 + a_5 t_5$, where a_1, a_2, a_3, a_4, a_5 are non-negative real numbers satisfying $a_1 + a_2 + a_3 + 2a_4 + 2a_5 < 1$, Then ψ satisfies (A1) and (A2).

Theorem 2.1 Let (X, d) be a metric space, $S, T : X \rightarrow CB(X)$ two set-valued mappings satisfying the following condition: for all $x, y \in X$,

$$\mathcal{H}(Sx, Ty) < \psi(d(x, y), D(x, Sx), D(y, Ty), D(x, Ty), D(y, Sx)). \quad (1)$$

If $S(X)$ or $T(X)$ is complete, then S and T have a common fixed point in X .

Proof. Note that for each $A, B \in CB(X)$, $a \in A$ and $\gamma > 0$ with $\mathcal{H}(A, B) < \gamma$, there exists $b \in B$ such that $d(a, b) < \gamma$ by Lemma 1.1.

Let $x_0 \in X$ and take any $x_1 \in Sx_0$, then for $x_1 \in Sx_0$, there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq \psi(d(x_0, x_1), D(x_0, Sx_0), D(x_1, Tx_1), D(x_0, Tx_1), D(x_1, Sx_0)).$$

Similarly, for $x_2 \in Tx_1$, there exists $x_3 \in Sx_2$ such that

$$d(x_3, x_2) \leq \psi(d(x_2, x_1), D(x_2, Sx_2), D(x_1, Tx_1), D(x_2, Tx_1), D(x_1, Sx_2)).$$

For $x_3 \in Sx_2$, there exists $x_4 \in Tx_3$ such that

$$d(x_3, x_4) \leq \psi(d(x_2, x_3), D(x_2, Sx_2), D(x_3, Tx_3), D(x_2, Tx_3), D(x_3, Sx_2)).$$

By the mathematical induction and the above observation, we can construct a sequence $\{x_n\}$ satisfying that for $x_{2n+1} \in Sx_{2n}$, there exists $x_{2n+2} \in Tx_{2n+1}$ such that

$$\begin{aligned} & d(x_{2n+1}, x_{2n+2}) \\ & \leq \psi(d(x_{2n}, x_{2n+1}), D(x_{2n}, Sx_{2n}), D(x_{2n+1}, Tx_{2n+1}), D(x_{2n}, Tx_{2n+1}), D(x_{2n+1}, Sx_{2n})) \\ & \leq \psi(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2}), (x_{2n+1}, x_{2n+1})) \\ & \leq \psi(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}), 0) \end{aligned} \quad (2)$$

and for $x_{2n+2} \in Tx_{2n+1}$, there exists $x_{2n+3} \in Sx_{2n+2}$ such that

$$\begin{aligned}
& d(x_{2n+3}, x_{2n+2}) \\
& \leq \psi(d(x_{2n+2}, x_{2n+1}), D(x_{2n+2}, Sx_{2n+2}), D(x_{2n+1}, Tx_{2n+1}), D(x_{2n+2}, Tx_{2n+1}), D(x_{2n+1}, Sx_{2n+2})) \\
& \leq \psi(d(x_{2n+2}, x_{2n+1}), d(x_{2n+2}, x_{2n+3}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+2}, x_{2n+2}), d(x_{2n+1}, x_{2n+3})) \\
& \leq \psi(d(x_{2n+2}, x_{2n+1}), d(x_{2n+2}, x_{2n+3}), d(x_{2n+1}, x_{2n+2}), 0, d(x_{2n+1}, x_{2n+2}) + d(x_{2n+2}, x_{2n+3})).
\end{aligned} \tag{3}$$

If $d(x_{2n}, x_{2n+1}) < d(x_{2n+1}, x_{2n+2})$ for some $n \in \mathbb{N}$, then $d(x_{2n+1}, x_{2n+2}) > 0$, hence by (2), (A1) and (A2),

$$\begin{aligned}
& d(x_{2n+1}, x_{2n+2}) \\
& \leq \psi(d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2}), 2d(x_{2n+1}, x_{2n+2}), 0) \\
& < d(x_{2n+1}, x_{2n+2}),
\end{aligned}$$

which is a contradiction. Hence

$$d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}), \forall n \in \mathbb{N}.$$

Similarly, by (A1), (A2) and (3),

$$d(x_{2n+3}, x_{2n+2}) \leq d(x_{2n+1}, x_{2n+2}), \forall n \in \mathbb{N}.$$

Therefore, for all $n \in \mathbb{N}$,

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n).$$

Let $c_m = d(x_m, x_{m+1})$ for all $m \in \mathbb{N}$, then $\{c_m\}$ is a decreasing sequence and bounded below, hence there exists $c \geq 0$ such that $\lim_{m \rightarrow \infty} c_m = c$. If $c > 0$, then using (2), we obtain

$$\begin{aligned}
c & \leq c_{2n+1} = d(x_{2n+1}, x_{2n+2}) \\
& \leq \psi(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}), 0) \\
& = \psi(c_{2n}, c_{2n}, c_{2n+1}, c_{2n} + c_{2n+1}, 0) \\
& \leq \psi(c_{2n}, c_{2n}, c_{2n+1}, 2c_{2n}, 0).
\end{aligned}$$

Let $n \rightarrow \infty$ on the two-sides of the above, then by (A1) and (A2), we obtain

$$c \leq \psi(c, c, c, 2c, 0) < c.$$

This contradiction shows $\lim_{m \rightarrow \infty} c_m = c = 0$.

Next, we will prove that $\{x_n\}$ is a Cauchy sequence. Otherwise, there exists $\gamma > 0$ such that for all $k \in \mathbb{N}$, there exist $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ satisfying

- (i) $m(k)$ is even and $n(k)$ is odd;
- (ii) $d(x_{m(k)}, x_{n(k)}) \geq \gamma$;
- (iii) $m(k)$ the smallest even number satisfying the conditions (i) and (ii).

By (iii), we have

$$d(x_{m(k)-2}, x_{n(k)}) < \gamma, \forall k = 1, 2, \dots$$

and

$$\gamma \leq d(x_{m(k)}, x_{n(k)}) \leq d(x_{n(k)}, x_{m(k)-2}) + c_{m(k)-2} + c_{m(k)-1} < \gamma + c_{m(k)-2} + c_{m(k)-1}.$$

Letting $k \rightarrow \infty$ on the above, we obtain

$$\gamma = \lim_{k \rightarrow \infty} d(x_{m(k)-2}, x_{n(k)}) = \lim_{k \rightarrow \infty} d(x_{m(k)}, x_{n(k)}). \quad (4)$$

We also have

$$\begin{aligned} & \gamma \\ & \leq d(x_{m(k)}, x_{n(k)}) \leq \mathcal{H}(Tx_{m(k)-1}, Sx_{n(k)-1}) = \mathcal{H}(Sx_{n(k)-1}, Tx_{m(k)-1}) \\ & \leq \Psi(d(x_{n(k)-1}, x_{m(k)-1}), D(x_{n(k)-1}, Sx_{n(k)-1}), D(x_{m(k)-1}, Tx_{m(k)-1}), D(x_{n(k)-1}, Tx_{m(k)-1}), D(x_{m(k)-1}, Sx_{n(k)-1})) \\ & \leq \Psi(d(x_{n(k)-1}, x_{m(k)-1}), d(x_{n(k)-1}, x_{n(k)}), d(x_{m(k)-1}, x_{m(k)}), d(x_{n(k)-1}, x_{m(k)}), d(x_{m(k)-1}, x_{n(k)})) \\ & \leq \Psi(c_{n(k)-1} + d(x_{n(k)}, x_{m(k)}) + c_{m(k)-1}, c_{n(k)-1}, c_{m(k)-1}, c_{n(k)-1} + d(x_{n(k)}, x_{m(k)}), c_{m(k)-1} + d(x_{m(k)}, x_{n(k)})). \end{aligned}$$

Letting $k \rightarrow \infty$ on the above, then using (4) and $c = 0$, we obtain the following contradiction

$$\gamma \leq \Psi(\gamma, 0, 0, \gamma, \gamma) < \gamma.$$

Hence $\{x_n\}$ is a Cauchy sequence.

Suppose that $S(X)$ is complete. Since $x_{2n+1} \in Sx_{2n} \subset S(X)$ for all $n \in \mathbb{N}$, there exists $u \in S(X)$ such that $x_{2n+1} \rightarrow u$ as $n \rightarrow \infty$. Hence

$$d(x_{2n+2}, u) \leq d(x_{2n+2}, x_{2n+1}) + d(x_{2n+1}, u) = c_{2n+1} + d(x_{2n+1}, u)$$

implies that $x_{2n+2} \rightarrow u$ as $n \rightarrow \infty$. By Lemma 1.2,

$$\begin{aligned} D(u, Tu) &= \lim_{n \rightarrow \infty} D(x_{2n+3}, Tu) \\ &\leq \lim_{n \rightarrow \infty} \mathcal{H}(Sx_{2n+2}, Tu) \\ &\leq \lim_{n \rightarrow \infty} \psi(d(x_{2n+2}, u), d(x_{2n+2}, x_{2n+3}), D(u, Tu), D(x_{2n+2}, Tu), d(u, x_{2n+3})). \end{aligned}$$

Let $n \rightarrow \infty$ on the two-sides of the above, then

$$D(u, Tu) \leq \psi(0, 0, D(u, Tu), D(u, Tu), 0),$$

hence $D(u, Tu) = 0$ by (A2), so $u \in Tu$ by Lemma 1.2 again. Similarly,

$$\begin{aligned} D(u, Su) &= \lim_{n \rightarrow \infty} D(x_{2n+2}, Su) \\ &\leq \lim_{n \rightarrow \infty} \mathcal{H}(Tx_{2n+1}, Su) \\ &= \lim_{n \rightarrow \infty} \mathcal{H}(Su, Tx_{2n+1}) \\ &\leq \lim_{n \rightarrow \infty} \psi(d(u, x_{2n+1}), D(u, Su), d(x_{2n+1}, x_{2n+2}), d(u, x_{2n+2}), d(x_{2n+1}, Su)). \end{aligned}$$

Let $n \rightarrow \infty$ on the two sides of the above, then

$$D(u, Su) \leq \psi(0, D(u, Su), 0, 0, D(u, Su)),$$

hence $D(u, Su) = 0$ by (A2), so $u \in Su$. Therefore, u is the common fixed point of S and T .

If $T(X)$ is complete, then since $x_{2n} \in T(X)$, there exists $u \in TX$ such that $x_{2n} \rightarrow u$ as $n \rightarrow \infty$.

But

$$d(x_{2n+1}, u) \leq d(x_{2n+1}, x_{2n}) + d(x_{2n}, u) = c_{2n} + d(x_{2n}, u),$$

hence $x_{2n+1} \rightarrow u$ as $n \rightarrow \infty$. Hence we are easy to prove the same conclusion for the case that $T(X)$ is complete.

Now, we consider another type common fixed point theorem.

Let $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a increasing and continuous function with $\phi(t) < \frac{1}{4}t$ for all $t > 0$ and $\phi(0) = 0$.

Let $\varphi : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}^+$ be a decreasing and continuous in each coordinate such that $\varphi(x, y) = 0$ if and only if $x = y = 0$.

Theorem 2.2 Suppose (X, d) is a metric space and two mappings $S, T : X \rightarrow CB(X)$ satisfy that for all $x, y \in X$,

$$\mathcal{H}(Sx, Ty) < \frac{1}{2}d(x, y) + \phi(D(x, Ty) + D(y, Sx)) - \varphi(D(x, Ty), D(y, Sx)). \quad (5)$$

If $S(X)$ or $T(X)$ is complete, Then S and T have an common fixed point in X .

Proof. Note that for each $A, B \in CB(X)$, $a \in A$ and $\gamma > 0$ with $\mathcal{H}(A, B) < \gamma$, there exists $b \in B$ such that $d(a, b) < \gamma$.

Let $x_0 \in X$ and take $x_1 \in Sx_0$, then there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq \frac{1}{2}d(x_0, x_1) + \phi(D(x_0, Tx_1) + D(x_1, Sx_0)) - \varphi(D(x_0, Tx_1), D(x_1, Sx_0));$$

For $x_2 \in Tx_1$, there exists $x_3 \in Sx_2$ such that

$$d(x_3, x_2) \leq \frac{1}{2}d(x_2, x_1) + \phi(D(x_2, Tx_1) + D(x_1, Sx_2)) - \varphi(D(x_2, Tx_1), D(x_1, Sx_2)).$$

Generally, for $x_{2n+1} \in Sx_{2n}$, there exists $x_{2n+2} \in Tx_{2n+1}$ such that

$$\begin{aligned} & d(x_{2n+1}, x_{2n+2}) \\ & \leq \frac{1}{2}d(x_{2n}, x_{2n+1}) + \phi(D(x_{2n}, Tx_{2n+1}) + D(x_{2n+1}, Sx_{2n})) - \varphi(D(x_{2n}, Tx_{2n+1}), D(x_{2n+1}, Sx_{2n})) \end{aligned}$$

and for $x_{2n+2} \in Tx_{2n+1}$, there exists $x_{2n+3} \in Sx_{2n+2}$ such that

$$\begin{aligned} & d(x_{2n+3}, x_{2n+2}) \\ & \leq \frac{1}{2}d(x_{2n+2}, x_{2n+1}) + \phi(D(x_{2n+2}, Tx_{2n+1}) + D(x_{2n+1}, Sx_{2n+2})) - \varphi(D(x_{2n+2}, Tx_{2n+1}), D(x_{2n+1}, Sx_{2n+2})). \end{aligned}$$

For any $n \in \mathbb{N}$,

$$\begin{aligned} & d(x_{2n+1}, x_{2n+2}) \\ & \leq \frac{1}{2}d(x_{2n}, x_{2n+1}) + \phi(d(x_{2n}, x_{2n+2})) - \varphi(d(x_{2n}, x_{2n+2}), 0) \\ & \leq \frac{1}{2}d(x_{2n}, x_{2n+1}) + \frac{1}{4}d(x_{2n}, x_{2n+2}) - \varphi(d(x_{2n}, x_{2n+2}), 0) \\ & \leq \frac{1}{2}d(x_{2n}, x_{2n+1}) + \frac{1}{4}d(x_{2n}, x_{2n+2}) \\ & \leq \frac{1}{2}d(x_{2n}, x_{2n+1}) + \frac{1}{4}[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})], \end{aligned} \quad (6)$$

hence we obtain that

$$d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}). \quad (7)$$

Similarly, since

$$\begin{aligned} & d(x_{2n+3}, x_{2n+2}) \\ & \leq \frac{1}{2}d(x_{2n+2}, x_{2n+1}) + \phi(d(x_{2n+1}, x_{2n+3})) - \phi(0, d(x_{2n+1}, x_{2n+3})) \\ & \leq \frac{1}{2}d(x_{2n+2}, x_{2n+1}) + \frac{1}{4}d(x_{2n+1}, x_{2n+3}) - \phi(0, d(x_{2n+1}, x_{2n+4})) \\ & \leq \frac{1}{2}d(x_{2n+2}, x_{2n+1}) + \frac{1}{4}[d(x_{2n+2}, x_{2n+1}) + d(x_{2n+3}, x_{2n+2})], \end{aligned} \quad (8)$$

so

$$d(x_{2n+2}, x_{2n+3}) \leq d(x_{2n+1}, x_{2n+2}). \quad (9)$$

Combining (7) and (9), we have that for all $n \in \mathbb{N}$,

$$d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}).$$

Let $c_m = d(x_m, x_{m+1})$ for all $m \in \mathbb{N}$, then $\{c_m\}$ is a non-increasing sequence and bounded below, so there exists $\xi \geq 0$ such that $\lim_{m \rightarrow \infty} c_m = \xi$.

In view of (6),

$$\begin{aligned} \xi & \leq c_{2n+1} = d(x_{2n+1}, x_{2n+2}) \\ & \leq \frac{1}{2}d(x_{2n}, x_{2n+1}) + \frac{1}{4}d(x_{2n}, x_{2n+2}) \\ & \leq \frac{1}{2}d(x_{2n}, x_{2n+1}) + \frac{1}{4}[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]. \end{aligned} \quad (10)$$

Let $n \rightarrow \infty$ on the above, then $\xi \leq \frac{1}{2}\xi + \frac{1}{4}\lim_{n \rightarrow \infty} d(x_{2n}, x_{2n+2}) \leq \frac{1}{2}\xi + \frac{1}{4}[\xi + \xi] = \xi$, hence we have that

$$\lim_{n \rightarrow \infty} d(x_{2n}, x_{2n+2}) = 2\xi.$$

By (6) again,

$$\xi \leq d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{2}d(x_{2n}, x_{2n+1}) + \frac{1}{4}d(x_{2n}, x_{2n+2}) - \phi(d(x_{2n}, x_{2n+2}), 0).$$

Let $n \rightarrow \infty$ on the above, then

$$\xi \leq \frac{1}{2}\xi + \frac{1}{4} \times 2\xi - \phi(2\xi, 0) \leq \xi,$$

so

$$\varphi(2\xi, 0) = 0,$$

hence

$$\xi = 0.$$

This prove that $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = \lim_{n \rightarrow \infty} c_n = 0$.

If $\{x_n\}$ is not a Cauchy sequence, then there exists $\gamma > 0$ such that for all $k \in \mathbb{N}$, there exist $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ satisfying (i),(ii) and (iii) in Theorem 2.1 and (4) holds.

We have

$$\begin{aligned} \gamma &\leq d(x_{m(k)}, x_{n(k)}) \\ &\leq \mathcal{H}(Tx_{m(k)-1}, Sx_{n(k)-1}) \\ &= \mathcal{H}(Sx_{n(k)-1}, Tx_{m(k)-1}) \\ &\leq \frac{1}{2}d(x_{n(k)-1}, x_{m(k)-1}) + \phi(D(x_{n(k)-1}, Tx_{m(k)-1}) + D(x_{m(k)-1}, Sx_{n(k)-1})) \\ &\quad - \phi(D(x_{n(k)-1}, Tx_{m(k)-1}), D(x_{m(k)-1}, Sx_{n(k)-1})) \\ &\leq \frac{1}{2}[c_{n(k)-1} + d(x_{n(k)}, x_{m(k)}) + c_{m(k)-1}] + \phi(d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)-1}, x_{n(k)})) \\ &\quad - \phi(d(x_{n(k)-1}, x_{m(k)}), d(x_{m(k)-1}, x_{n(k)})) \\ &\leq \frac{1}{2}[c_{n(k)-1} + d(x_{n(k)}, x_{m(k)}) + c_{m(k)-1}] + \phi([c_{n(k)-1} + d(x_{n(k)}, x_{m(k)})] + [d(x_{m(k)}, x_{n(k)}) + c_{m(k)-1}]) \\ &\quad - \phi([c_{n(k)-1} + d(x_{n(k)}, x_{m(k)})], [d(x_{m(k)}, x_{n(k)}) + c_{m(k)-1}]). \end{aligned}$$

Let $k \rightarrow \infty$ on the above, then we obtain that

$$\gamma \leq \frac{1}{2}\gamma + \phi(2\gamma) - \phi(\gamma, \gamma) \leq \frac{1}{2}\gamma + \frac{1}{4}(2\gamma) - \phi(\gamma, \gamma) \leq \gamma,$$

hence $\phi(\gamma, \gamma) = 0$, so $\gamma = 0$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence.

Since $S(X)$ or $T(X)$ is complete, there exists $z \in S(X)$ or $z \in T(X)$ such that $x_n \rightarrow z$ as $n \rightarrow \infty$ (for detail, see the proof process of Theorem 2.1).

Finally, by Lemma 1.2 and (5),

$$\begin{aligned}
D(z, Tz) &= \lim_{n \rightarrow \infty} D(x_{2n+1}, Tz) \\
&\leq \lim_{n \rightarrow \infty} \mathcal{H}(Sx_{2n}, Tz) \\
&\leq \lim_{n \rightarrow \infty} \left[\frac{1}{2}d(x_{2n}, z) + \phi(D(x_{2n}, Tz) + D(z, Sx_{2n})) - \phi(D(x_{2n}, Tz), D(z, Sx_{2n})) \right] \\
&\leq \lim_{n \rightarrow \infty} \left[\frac{1}{2}d(x_{2n}, z) + \phi(D(x_{2n}, Tz) + d(z, x_{2n+1})) - \phi(D(x_{2n}, Tz), d(z, x_{2n+1})) \right] \\
&= \phi(D(z, Tz)) - \phi(D(z, Tz), 0) \\
&\leq \frac{1}{4}D(z, Tz),
\end{aligned}$$

hence $D(z, Tz) = 0$, therefore $z \in Tz$ since Tz is closed.

Similarly, by Lemma 1.2 and (5),

$$\begin{aligned}
D(z, Sz) &= \lim_{n \rightarrow \infty} D(x_{2n+2}, Sz) \\
&\leq \lim_{n \rightarrow \infty} \mathcal{H}(Tx_{2n+1}, Sz) \\
&= \lim_{n \rightarrow \infty} \mathcal{H}(Sz, Tx_{2n+1}) \\
&\leq \lim_{n \rightarrow \infty} \left[\frac{1}{2}d(z, x_{2n+1}) + \phi(D(z, Tx_{2n+1}) + D(x_{2n+1}, Sz)) - \phi(D(z, Tx_{2n+1}), D(x_{2n+1}, Sz)) \right] \\
&\leq \lim_{n \rightarrow \infty} \left[\frac{1}{2}d(z, x_{2n+1}) + \phi(d(z, x_{2n+2}) + D(x_{2n+1}, Sz)) - \phi(d(z, x_{2n+2}), D(x_{2n+1}, Sz)) \right] \\
&= \phi(D(z, Sz)) - \phi(0, D(z, Sz)) \\
&\leq \frac{1}{4}D(z, Sz),
\end{aligned}$$

hence $D(z, Sz) = 0$, therefore $z \in Sz$ since Sz is closed. This complete that z is the common fixed point of S and T .

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

- [1] S. Banach, Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales, Fund. Math. 3(1922), 133-181.
- [2] L. B. Ćirić, A generalization of Banach's contraction principle, Proc. Amer. Math. Soc. 45(2)(1974): 133-181.

- [3] S. B. Jr. Nadler, Multi-valued contraction mappings, *Pac. J. Math.* 30(1969), 475-488.
- [4] N. Mizoguchi and W. Takahashi, Fixed point theorems for multi-valued mappings on complete metric spaces. *J. Math. Anal. Appl.* 141(1989), 177-188.
- [5] A. Amini-Harandi, Fixed point theory for set-valued quasi-contraction maps in metric spaces, *Appl. Math. Lett.* 24(11)(2011), 1791-1794.
- [6] C. M. Chen, Some new fixed point theorems for set-valued contractions in complete metric spaces, *Fixed Point Theory Appl.* 2011(2011), Article ID 72.
- [7] J. R. Wu and H. Y. Liu, Common fixed point theorems for sequences of Φ -type contraction set-valued mappings, *Chin. Q. J. Math.* 24(4)(2009), 504-510.