COMMON FIXED POINTS FOR TWO SET-VALUED MAPPINGS ON METRIC SPACES

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Abstract. In this paper, we introduce three classes of functions and construct two contractive conditions to discuss and obtain some new common fixed point theorems for two set-valued mappings on non-complete metric spaces.

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1. INTRODUCTION AND PRELIMINARIES

Let \((X, d)\) be a metric space and \(CB(X)\) the family of all nonempty closed and bounded subset of \(X\).

The following is the famous Banach’s fixed point theorem\(^1\):

Let \((X, d)\) be a complete metric space and \(f : X \to X\) a mapping. If \(f\) satisfies

\[
d(fx, fy) \leq hd(x, y), \forall x, y \in X,
\]

where \(h \in [0, 1)\). Then \(f\) has a unique common fixed point in \(X\).

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Later, many generalizations of Banach’s fixed point theorem have appeared. For instance, if $f : X \rightarrow X$ is a mapping on a complete metric space $(X, d)$ satisfying the following a quasi-contraction $d(fx, fy) \leq h \max\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, f y)\}, \forall x, y \in X,$ where $h \in [0, 1)$. Then $f$ has a unique common fixed point in $X$ (see [2]).

In 1969, Nadler[3] extended the famous Banach contraction Principle from a single-valued mapping to a set-valued mappings and gave the next fixed point theorem:

**Theorem 1.1** ([3]) Let $(X, d)$ be a complete metric space and $T : X \rightarrow CB(X)$. If there exists $h \in [0, 1)$ such that $H(Tx, Ty) \leq hd(x, y), \forall x, y \in X,$

where $H$ denote the Hausdorff metric on $CB(X)$ induced by $d$, that is, $H(A, B) = \max\{\sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A)\}, \forall A, B \in CB(X),$ where $D(x, B) = \inf_{z \in B} d(x, z).$ Then $T$ has a fixed point in $X$.

Mizoguchi-Takahashi[4] also gave the following fixed point theorem:

**Theorem 1.2** ([4]) Let $(X, d)$ be a complete metric space and $T : X \rightarrow CB(X)$. If $H(Tx, Ty) \leq \xi(d(x, y))d(x, y), \forall x, y \in X,$

where $\xi : [0, \infty) \rightarrow [0, 1)$ satisfying $\limsup_{s \rightarrow t^+} \xi(s) < 1$ for all $t \in [0, \infty)$. Then $T$ has a fixed point in $X$.

In 2011, Amini-Harandi generalized and improved the corresponding result in [2] from a single-valued mapping to a set-valued mapping, obtained the next result:

**Theorem 1.3** ([5]) Let $(X, d)$ be a complete metric space and $T : X \rightarrow CB(X)$ a $k$-set-valued quasi-contraction with $k < \frac{1}{2}$, that is,

$H(Tx, Ty) \leq k \max\{d(x, y), D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)\}, \forall x, y \in X.$

Then $T$ has a fixed point in $X$.

And in 2011, Chen[6] introduced the following definition of $\psi$-contraction and obtained a fixed point theorem for set-valued mappings:
Theorem 1.4([6]) Let \((X,d)\) be a complete metric space and \(T : X \to CB(X)\) a set-valued \(\psi\)-contraction, that is,
\[
H(T x, Ty) \leq \psi(d(x,y), D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)), \forall x, y \in X,
\]
then \(T\) has a fixed point in \(X\). Where \(\psi : \mathbb{R}^+ \to \mathbb{R}^+\) is a function satisfying some particular conditions, see [6].

On the other hand, In 2009, Wu et al[7] obtained the next common fixed point theorem for set-valued mappings \(\{A_i\}\):

Theorem 1.5([7]) Let \((X,d)\) be a complete metric space and \(A_i : X \to CB(X)\) satisfy the condition: for any \(i, j = 1, 2, \cdots, x, y \in X\) and \(u \in A_i x\), there exists \(v \in A_j y\) such that
\[
d(u,v) \leq \Phi(d(x,y), D(x, A_i x), D(y, A_j y), D(x, A_j y), D(y, A_i x)),
\]
then \(\{A_i\}\) have a common fixed point in \(X\). Where \(\Phi : \mathbb{R}^+ \to \mathbb{R}^+\) is a function satisfying some particular conditions, see [7].

The following is well-known results, see [7]:

Lemma 1.1 If \((X,d)\) is a metric space, \(A,B \in CB(X)\), then for any \(\varepsilon > 0\) and any \(a \in A\), there exists \(b \in B\) such that \(d(a,b) \leq H(A, B) + \varepsilon\).

Lemma 1.2 If \((X,d)\) is a metric space, \(A \in CB(X)\), then \(D(\cdot, A)\) is continuous. Moreover, we have that
(i) \(A = \{x \in X | d(x,A) = 0\}\);
(ii) For any \(x, y \in X\), \(D(x,A) \leq d(x,y) + D(y,A)\).

In this paper, we use the method in [6] to obtain some common fixed point theorems for two set-valued mappings in metric spaces.

2. COMMON FIXED POINTS

Now, we begin with the following definition.

Let \(\psi : \mathbb{R}^+ \to \mathbb{R}^+\) be a functions satisfying the following conditions, where \(\mathbb{R}^+ = [0, +\infty)\):
(A1) \(\psi\) is non-decreasing and continuous in each coordinate;
(A2) for all \(t > 0\), \(\psi(t,t,t,0,2t) < t\), \(\psi(t,t,t,2t,0) < t\), \(\psi(0,0,t,t,0) < t\), \(\psi(0,t,0,t) < t\) and \(\psi(t,0,0,t,t) < t\).
Example 1.1 Let \( \psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) be \( \psi(t_1,t_2,t_3,t_4,t_5) = a_1t_1 + a_2t_2 + a_3t_3 + a_4t_4 + a_5t_5 \), where \( a_1, a_2, a_3, a_4, a_5 \) are non-negative real numbers satisfying \( a_1 + a_2 + a_3 + 2a_4 + 2a_5 < 1 \), Then \( \psi \) satisfies (A1) and (A2).

Theorem 2.1 Let \( (X, d) \) be a metric space, \( S, T : X \rightarrow CB(X) \) two set-valued mappings satisfying the following condition: for all \( x, y \in X \),

\[
\mathcal{H}(Sx, Ty) < \psi(d(x,y), D(x,Sx), D(y,Ty), D(x,Ty), D(y,Sx)).
\]

(1)

If \( S(X) \) or \( T(X) \) is complete, then \( S \) and \( T \) have a common fixed point in \( X \).

Proof. Note that for each \( A, B \in CB(X) \), \( a \in A \) and \( \gamma > 0 \) with \( \mathcal{H}(A,B) < \gamma \), there exists \( b \in B \) such that \( d(a,b) < \gamma \) by Lemma 1.1.

Let \( x_0 \in X \) and take any \( x_1 \in Sx_0 \), then for \( x_1 \in Sx_0 \), there exists \( x_2 \in Tx_1 \) such that

\[
d(x_1,x_2) \leq \psi(d(x_0,x_1), D(x_0,Sx_0), D(x_1,Tx_1), D(x_0,Tx_1), D(x_1,Sx_0)).
\]

Similarly, for \( x_2 \in Tx_1 \), there exists \( x_3 \in Sx_2 \) such that

\[
d(x_3,x_2) \leq \psi(d(x_2,x_1), D(x_2,Sx_2), D(x_1,Tx_1), D(x_2,Tx_1), D(x_1,Sx_2)).
\]

For \( x_3 \in Sx_2 \), there exists \( x_4 \in Tx_3 \) such that

\[
d(x_3,x_4) \leq \psi(d(x_2,x_3), D(x_2,Sx_2), D(x_3,Tx_3), D(x_2,Tx_3), D(x_3,Sx_2)).
\]

By the mathematical induction and the above observation, we can construct a sequence \( \{x_n\} \) satisfying that for \( x_{2n+1} \in Sx_{2n} \), there exists \( x_{2n+2} \in Tx_{2n+1} \) such that

\[
d(x_{2n+1},x_{2n+2})
\]

\[
\leq \psi(d(x_{2n},x_{2n+1}), D(x_{2n},Sx_{2n}), D(x_{2n+1},Tx_{2n+1}), D(x_{2n},Tx_{2n+1}), D(x_{2n+1},Sx_{2n}))
\]

\[
\leq \psi(d(x_{2n},x_{2n+1}), d(x_{2n},x_{2n+1}), d(x_{2n+1},x_{2n+2}), d(x_{2n},x_{2n+2}), (x_{2n+1},x_{2n+1}))
\]

\[
\leq \psi(d(x_{2n},x_{2n+1}), d(x_{2n},x_{2n+1}), d(x_{2n+1},x_{2n+2}), d(x_{2n},x_{2n+1}) + d(x_{2n+1},x_{2n+2}), 0)
\]

(2)
and for $x_{2n+2} \in T x_{2n+1}$, there exists $x_{2n+3} \in S x_{2n+2}$ such that
\[
d(x_{2n+3}, x_{2n+2})
\leq \psi(d(x_{2n+2}, x_{2n+1}), D(x_{2n+2}, S x_{2n+2}), D(x_{2n+1}, T x_{2n+1}), D(x_{2n+2}, T x_{2n+1}), D(x_{2n+1}, S x_{2n+2}))
\leq \psi(d(x_{2n+2}, x_{2n+1}), d(x_{2n+2}, x_{2n+3}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+2}, x_{2n+2}), d(x_{2n+1}, x_{2n+3}))
\leq \psi(d(x_{2n+2}, x_{2n+1}), d(x_{2n+2}, x_{2n+3}), d(x_{2n+1}, x_{2n+2}), 0, d(x_{2n+1}, x_{2n+2}) + d(x_{2n+2}, x_{2n+3})).
\tag{3}
\]

If $d(x_{2n}, x_{2n+1}) < d(x_{2n+1}, x_{2n+2})$ for some $n \in \mathbb{N}$, then $d(x_{2n+1}, x_{2n+2}) > 0$, hence by (2), (A1) and (A2),
\[
d(x_{2n+1}, x_{2n+2})
\leq \psi(d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2}), 2d(x_{2n+1}, x_{2n+2}), 0)
\leq d(x_{2n+1}, x_{2n+2}),
\]
which is a contradiction. Hence
\[
d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}), \quad \forall \ n \in \mathbb{N}.
\]

Similarly, by (A1), (A2) and (3),
\[
d(x_{2n+3}, x_{2n+2}) \leq d(x_{2n+1}, x_{2n+2}), \quad \forall \ n \in \mathbb{N}.
\]

Therefore, for all $n \in \mathbb{N}$,
\[
d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n).
\]

Let $c_m = d(x_m, x_{m+1})$ for all $m \in \mathbb{N}$, then \{$c_m$\} is a decreasing sequence and bounded below, hence there exists $c \geq 0$ such that $\lim_{m \to \infty} c_m = c$. If $c > 0$, then using (2), we obtain
\[
c \leq c_{2n+1} = d(x_{2n+1}, x_{2n+2})
\leq \psi(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}), 0)
= \psi(c_{2n}, c_{2n}, c_{2n+1}, c_{2n} + 2c_{2n+1}, 0)
\leq \psi(c_{2n}, c_{2n}, 2c_{2n}, 0).
\]

Let $n \to \infty$ on the two-sides of the above, then by (A1) and (A2), we obtain
\[
c \leq \psi(c, c, 2c, 0) < c.
\]
This contradiction shows $\lim_{m \to \infty} c_m = c = 0$.

Next, we will prove that $\{x_n\}$ is a Cauchy sequence. Otherwise, there exists $\gamma > 0$ such that for all $k \in \mathbb{N}$, there exist $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ satisfying

(i) $m(k)$ is even and $n(k)$ is odd;

(ii) $d(x_{m(k)}, x_{n(k)}) \geq \gamma$;

(iii) $m(k)$ the smallest even number satisfying the conditions (i) and (ii).

By (iii), we have

$$d(x_{m(k) - 2}, x_{n(k)}) < \gamma, \forall k = 1, 2, \ldots$$

and

$$\gamma \leq d(x_{m(k)}, x_{n(k)}) \leq d(x_{n(k)}, x_{m(k) - 2}) + c_{m(k) - 2} + c_{m(k) - 1} < \gamma + c_{m(k) - 2} + c_{m(k) - 1}.$$

Letting $k \to \infty$ on the above, we obtain

$$\gamma = \lim_{k \to \infty} d(x_{m(k) - 2}, x_{n(k)}) = \lim_{k \to \infty} d(x_{m(k)}, x_{n(k)}). \quad (4)$$

We also have

$$\gamma$$

$$\leq d(x_{m(k)}, x_{n(k)}) \leq \mathcal{H}(Tx_{m(k)} - 1, Sx_{n(k)} - 1) = \mathcal{H}(Sx_{n(k)} - 1,Tx_{m(k)} - 1)$$

$$\leq \psi(d(x_{n(k)}, - 1, x_{m(k)} - 1), D(x_{m(k)}, - 1, Sx_{n(k)} - 1), D(x_{n(k)} - 1, Tx_{m(k)} - 1), D(x_{m(k)} - 1, Sx_{n(k)} - 1))$$

$$\leq \psi(d(x_{n(k)} - 1, x_{m(k)} - 1), d(x_{n(k)}, x_{m(k)} - 1), d(x_{m(k)}, - 1, Sx_{n(k)} - 1), D(x_{n(k)} - 1, Tx_{m(k)} - 1))$$

$$\leq \psi(c_{n(k) - 1} + d(x_{n(k)}, x_{m(k)}), c_{m(k) - 1}, c_{n(k) - 1}, c_{m(k)} - 1, d(x_{n(k)}, x_{m(k)}), c_{m(k) - 1} + d(x_{m(k)}, x_{n(k)})).$$

Letting $k \to \infty$ on the above, then using (4) and $c = 0$, we obtain the following contradiction

$$\gamma \leq \psi(\gamma, 0, 0, \gamma) < \gamma.$$ 

Hence $\{x_n\}$ is a Cauchy sequence.

Suppose that $S(X)$ is complete. Since $x_{2n+1} \in Sx_{2n} \subset S(X)$ for all $n \in \mathbb{N}$, there exists $u \in S(X)$ such that $x_{2n+1} \to u$ as $n \to \infty$. Hence

$$d(x_{2n+2}, u) \leq d(x_{2n+2}, x_{2n+1}) + d(x_{2n+1}, u) = c_{2n+1} + d(x_{2n+1}, u)$$
implies that \(x_{2n+2} \to u\) as \(n \to \infty\). By Lemma 1.2,
\[
D(u, Tu) = \lim_{n \to \infty} D(x_{2n+3}, Tu)
\leq \lim_{n \to \infty} \mathcal{H}(Sx_{2n+2}, Tu)
\leq \lim_{n \to \infty} \psi(d(x_{2n+2}, u), d(x_{2n+2}, x_{2n+3}), D(u, Tu), D(x_{2n+2}, Tu), d(u, x_{2n+3})).
\]
Let \(n \to \infty\) on the two-sides of the above, then
\[
D(u, Tu) \leq \psi(0, 0, D(u, Tu), D(u, Tu), 0),
\]

hence \(D(u, Tu) = 0\) by (A2), so \(u \in Tu\) by Lemma 1.2 again. Similarly,
\[
D(u, Su) = \lim_{n \to \infty} D(x_{2n+2}, Su)
\leq \lim_{n \to \infty} \mathcal{H}(Tx_{2n+1}, Su)
= \lim_{n \to \infty} \mathcal{H}(Su, Tx_{2n+1})
\leq \lim_{n \to \infty} \psi(d(u, x_{2n+1}), D(u, Su), d(x_{2n+1}, x_{2n+2}), d(u, x_{2n+2}), d(x_{2n+1}, Su)).
\]
Let \(n \to \infty\) on the two sides of the above, then
\[
D(u, Su) \leq \psi(0, D(u, Su), 0, 0, D(u, Su)),
\]

hence \(D(u, Su) = 0\) by (A2), so \(u \in Su\). Therefore, \(u\) is the common fixed point of \(S\) and \(T\).

If \(T(X)\) is is complete, then since \(x_{2n} \in T(X)\), there exists \(u \in TX\) such that \(x_{2n} \to u\) as \(n \to \infty\).
But
\[
d(x_{2n+1}, u) \leq d(x_{2n+1}, x_{2n}) + d(x_{2n}, u) = c_{2n} + d(x_{2n}, u),
\]
hence \(x_{2n+1} \to u\) as \(n \to \infty\). Hence we are easy to prove the same conclusion for the case that \(T(X)\) is complete.

Now, we consider another type common fixed point theorem.

Let \(\phi : \mathbb{R}^+ \to \mathbb{R}^+\) be a increasing and continuous function with \(\phi(t) < \frac{1}{4} t\) for all \(t > 0\) and \(\phi(0) = 0\).

Let \(\varphi : (\mathbb{R}^+)^2 \to \mathbb{R}^+\) be a decreasing and continuous in each coordinate such that \(\varphi(x, y) = 0\) if and only if \(x = y = 0\).
Theorem 2.2 Suppose \((X, d)\) is a metric space and two mappings \(S, T : X \to CB(X)\) satisfy that for all \(x, y \in X\),
\[
\mathcal{H}(Sx, Ty) < \frac{1}{2}d(x, y) + \phi(D(x, Ty) + D(y, Sx)) - \varphi(D(x, Ty), D(y, Sx)).
\]

If \(S(X)\) or \(T(X)\) is complete, then \(S\) and \(T\) have an common fixed point in \(X\).

**Proof.** Note that for each \(A, B \in CB(X)\), \(a \in A\) and \(\gamma > 0\) with \(\mathcal{H}(A, B) < \gamma\), there exists \(b \in B\) such that \(d(a, b) < \gamma\).

Let \(x_0 \in X\) and take \(x_1 \in Sx_0\), then there exists \(x_2 \in Tx_1\) such that
\[
d(x_1, x_2) \leq \frac{1}{2}d(x_0, x_1) + \phi(D(x_0, Tx_1) + D(x_1, Sx_0)) - \varphi(D(x_0, Tx_1), D(x_1, Sx_0));
\]

For \(x_2 \in Tx_1\), there exists \(x_3 \in Sx_2\) such that
\[
d(x_3, x_2) \leq \frac{1}{2}d(x_2, x_1) + \phi(D(x_2, Tx_1) + D(x_1, Sx_2)) - \varphi(D(x_2, Tx_1), D(x_1, Sx_2)).
\]

Generally, for \(x_{2n+1} \in Sx_{2n}\), there exists \(x_{2n+2} \in Tx_{2n+1}\) such that
\[
d(x_{2n+1}, x_{2n+2}) \\
\leq \frac{1}{2}d(x_{2n}, x_{2n+1}) + \phi(D(x_{2n}, Tx_{2n+1}) + D(x_{2n+1}, Sx_{2n})) - \varphi(D(x_{2n}, Tx_{2n+1}), D(x_{2n+1}, Sx_{2n}))
\]

and for \(x_{2n+2} \in Tx_{2n+1}\), there exists \(x_{2n+3} \in Sx_{2n+2}\) such that
\[
d(x_{2n+3}, x_{2n+2}) \\
\leq \frac{1}{2}d(x_{2n+2}, x_{2n+1}) + \phi(D(x_{2n+2}, Tx_{2n+1}) + D(x_{2n+1}, Sx_{2n+2})) - \varphi(D(x_{2n+2}, Tx_{2n+1}), D(x_{2n+1}, Sx_{2n+2})).
\]

For any \(n \in \mathbb{N}\),
\[
d(x_{2n+1}, x_{2n+2}) \\
\leq \frac{1}{2}d(x_{2n}, x_{2n+1}) + \phi(d(x_{2n}, x_{2n+2})) - \varphi(d(x_{2n}, x_{2n+2}), 0) \\
\leq \frac{1}{2}d(x_{2n}, x_{2n+1}) + \frac{1}{4}d(x_{2n}, x_{2n+2}) - \varphi(d(x_{2n}, x_{2n+2}), 0) \\
\leq \frac{1}{2}d(x_{2n}, x_{2n+1}) + \frac{1}{4}d(x_{2n}, x_{2n+2}) \\
\leq \frac{1}{2}d(x_{2n}, x_{2n+1}) + \frac{1}{4}d(x_{2n}, x_{2n+2}) \\
\leq \frac{1}{2}d(x_{2n}, x_{2n+1}) + \frac{1}{4}[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})].
\]

hence we obtain that
\[ d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}). \] (7)

Similarly, since
\[
\begin{align*}
d(x_{2n+3}, x_{2n+2}) & \leq \frac{1}{2}d(x_{2n+2}, x_{2n+1}) + \phi(d(x_{2n+1}, x_{2n+3})) - \phi(0, d(x_{2n+1}, x_{2n+3})) \\
& \leq \frac{1}{2}d(x_{2n+2}, x_{2n+1}) + \frac{1}{4}d(x_{2n+1}, x_{2n+3}) - \phi(0, d(x_{2n+1}, x_{2n+4})) \\
& \leq \frac{1}{2}d(x_{2n+2}, x_{2n+1}) + \frac{1}{4}[d(x_{2n+2}, x_{2n+1}) + d(x_{2n+3}, x_{2n+2})],
\end{align*}
\]
so
\[ d(x_{2n+2}, x_{2n+3}) \leq d(x_{2n+1}, x_{2n+2}). \] (9)

Combining (7) and (9), we have that for all \( n \in \mathbb{N}, \)
\[ d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}). \]

Let \( c_m = d(x_m, x_{m+1}) \) for all \( m \in \mathbb{N}, \) then \( \{c_m\} \) is a non-increasing sequence and bounded below, so there exists \( \xi \geq 0 \) such that \( \lim_{m \to \infty} c_m = \xi. \)

In view of (6),
\[
\begin{align*}
\xi & \leq c_{2n+1} = d(x_{2n+1}, x_{2n+2}) \\
& \leq \frac{1}{2}d(x_{2n}, x_{2n+1}) + \frac{1}{4}d(x_{2n}, x_{2n+2}) \\
& \leq \frac{1}{2}d(x_{2n}, x_{2n+1}) + \frac{1}{4}[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})].
\end{align*}
\]
Let \( n \to \infty \) on the above, then \( \xi \leq \frac{1}{2}\xi + \frac{1}{4}\lim_{n \to \infty} d(x_{2n}, x_{2n+2}) \leq \frac{1}{2}\xi + \frac{1}{4}[\xi + \xi] = \xi, \) hence we have that
\[ \lim_{n \to \infty} d(x_{2n}, x_{2n+2}) = 2\xi. \]

By (6) again,
\[
\xi \leq d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{2}d(x_{2n}, x_{2n+1}) + \frac{1}{4}d(x_{2n}, x_{2n+2}) - \varphi(d(x_{2n}, x_{2n+2}), 0).
\]
Let \( n \to \infty \) on the above, then
\[ \xi \leq \frac{1}{2}\xi + \frac{1}{4} \times 2\xi - \varphi(2\xi, 0) \leq \xi, \]
\[ \varphi(2\xi, 0) = 0, \]

hence

\[ \xi = 0. \]

This proves that \( \lim_{n \to \infty} d(x_n, x_{n+1}) = \lim_{n \to \infty} c_n = 0. \)

If \( \{x_n\} \) is not a Cauchy sequence, then there exists \( \gamma > 0 \) such that for all \( k \in \mathbb{N} \), there exist \( m(k), n(k) \in \mathbb{N} \) with \( m(k) > n(k) \geq k \) satisfying (i),(ii) and (iii) in Theorem 2.1 and (4) holds.

We have

\[ \gamma \leq d(x_{m(k)}, x_{n(k)}) \]
\[ \leq \mathcal{H}(Tx_{m(k)-1}, Sx_{n(k)-1}) \]
\[ = \mathcal{H}(Sx_{n(k)-1}, Tx_{m(k)-1}) \]
\[ \leq \frac{1}{2} d(x_{m(k)-1}, x_{n(k)-1}) + \phi(D(x_{m(k)-1}, Tx_{m(k)-1}) + D(x_{m(k)-1}, Sx_{n(k)-1})) \]
\[ - \phi(D(x_{m(k)-1}, Sx_{n(k)-1})) \]
\[ \leq \frac{1}{2} [c_{n(k)-1} + d(x_{m(k)}, x_{n(k)}) + c_{m(k)-1}] + \phi([c_{n(k)-1} + d(x_{m(k)}, x_{n(k)}) + d(x_{m(k)-1}, x_{n(k)})]) \]
\[ - \phi([c_{m(k)-1} + d(x_{m(k)}, c_{m(k)-1})] [d(x_{m(k)}, x_{n(k)}) + c_{m(k)-1})]. \]

Let \( k \to \infty \) on the above, then we obtain that

\[ \gamma \leq \frac{1}{2} \gamma + \phi(2\gamma) - \varphi(\gamma, \gamma) \leq \frac{1}{2} \gamma + \frac{1}{4} (2\gamma) - \varphi(\gamma, \gamma) \leq \gamma, \]

hence \( \varphi(\gamma, \gamma) = 0, \) so \( \gamma = 0, \) which is a contradiction. Hence \( \{x_n\} \) is a Cauchy sequence.

Since \( S(X) \) or \( T(X) \) is complete, there exists \( z \in S(X) \) or \( z \in T(X) \) such that \( x_n \to z \) as \( n \to \infty \) (for detail, see the proof process of Theorem 2.1).
Finally, by Lemma 1.2 and (5),

\[ D(z, Tz) = \lim_{n \to \infty} D(x_{2n+1}, Tz) \]

\[ \leq \lim_{n \to \infty} \mathcal{H}(Sx_{2n}, Tz) \]

\[ \leq \lim_{n \to \infty} \left[ \frac{1}{2} d(x_{2n}, z) + \phi (D(x_{2n}, Tz) + D(z, Sx_{2n})) - \phi(D(x_{2n}, Tz), D(z, Sx_{2n})) \right] \]

\[ \leq \lim_{n \to \infty} \left[ \frac{1}{2} d(x_{2n}, z) + \phi (D(x_{2n}, Tz) + d(z, x_{2n+1})) - \phi(D(x_{2n}, Tz), d(z, x_{2n+1})) \right] \]

\[ = \phi(D(z, Tz)) - \phi(D(z, Tz), 0) \]

\[ \leq \frac{1}{4} D(z, Tz), \]

hence \( D(z, Tz) = 0 \), therefore \( z \in Tz \) since \( Tz \) is closed.

Similarly, by Lemma 1.2 and (5),

\[ D(z, Sz) = \lim_{n \to \infty} D(x_{2n+2}, Sz) \]

\[ \leq \lim_{n \to \infty} \mathcal{H}(Tx_{2n+1}, Sz) \]

\[ = \lim_{n \to \infty} \mathcal{H}(Sz, Tx_{2n+1}) \]

\[ \leq \lim_{n \to \infty} \left[ \frac{1}{2} d(z, x_{2n+1}) + \phi (D(z, Tx_{2n+1}) + D(x_{2n+1}, Sz)) - \phi(D(z, Tx_{2n+1}), D(x_{2n+1}, Sz)) \right] \]

\[ \leq \lim_{n \to \infty} \left[ \frac{1}{2} d(z, x_{2n+1}) + \phi (d(z, x_{2n+2}) + D(x_{2n+1}, Sz)) - \phi(d(z, x_{2n+2}), D(x_{2n+1}, Sz)) \right] \]

\[ = \phi(D(z, Sz)) - \phi(0, D(z, Sz)) \]

\[ \leq \frac{1}{4} D(z, Sz), \]

hence \( D(z, Sz) = 0 \), therefore \( z \in Sz \) since \( Sz \) is closed. This complete that \( z \) is the common fixed point of \( S \) and \( T \).

Conflict of Interests
The authors declare that there is no conflict of interests.

References


