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VISCOSITY SOLUTIONS FOR HAMILTONIAN EQUATIONS WITH YOUNG MEASURES

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Abstract. The paper is concerned with a type value functions which occurs in the control problems subject to evolution inclusions involving time-dependent subdifferential operators where the controls are Young measures. We study their link with the viscosity solution of the associated Hamilton-Jacobi-Bellman equation.

Keywords: evolution inclusions; subdifferential operator; Young measures; control; value function; viscosity solution.

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1. INTRODUCTION

The aim of the present paper is to produce some viscosity results related to evolution problems governed by time-dependent subdifferential operators. This work is mainly motivated by several papers in viscosity theory and the results in Saïdi et al [21] (see also [22] for more recent results), concerning the study of a class of evolution problems in Hilbert spaces and its application to optimal control.

In this article, we are interested in the value function to the problem of maximizing minimizing "sup inf" an integral functional of the trajectories. These trajectories are the absolutely

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continuous solutions to differential inclusions, driven by the subdifferential of a convex function involving a Lipschitz perturbation that contains two controls. The existence and uniqueness result for such problems, has been recently proved in [21] (see also [22]). The two control spaces here, are compact metric, the control measure belongs to the space of Young measures and the cost function appearing in the value function, is an integrand. On the one hand, we show in the finite dimensional setting, that under suitable conditions imposed on the cost functional and the dynamics, the associated value function, is a viscosity subsolution of the corresponding Hamilton-Jacobi-Bellman equation. On the other hand, we prove that, under some extra conditions on the cost functional, the dynamics, and the first space of Young measure controls, the value function under consideration is a viscosity supersolution of the associated Hamilton-Jacobi-Bellman equation.

In a recent paper, the authors Saïdi et al [22], have presented a kind of value functions involving, in the quest of viscosity subsolutions of some Hamilton-Jacobi-Bellman equation, one relaxed control. For results of this kind, see also [9].

The theory of viscosity and the resolution of Hamilton-Jacobi-Bellman equations have interested many authors, in the finite dimensional setting. The paper [4], is concerned with non-convex sweeping processes and m -accretive operators. Dealing with problems when the dynamic is governed by the subdifferential of an integral functional, we refer to [9] (see also [7]). In the case of the subdifferential of a primal lower nice function depending only on the state variable, related results were found in [6]. In all these papers, two Young measures (controls), were considered. The authors in [18] and [14, 15, 16, 17] investigated such problems arising from differential games theory (with two players). They involved measurable mappings as controls. A large literature on viscosity solutions applied to optimal control of evolution equations, sweeping process, some classes of evolution inclusions of second order and related results including papers and some books, see, e.g. [2, 3, 5, 8, 10, 11, 12, 19, 20].

The paper is organized as follows. In section 2, we give the notations and definitions used through the paper. We introduce in the next section, the formulation of the model and the description of the value function under consideration. In section 4, we investigate in the finite dimensional setting, the existence of both viscosity subsolutions and viscosity supersolutions of

Hamilton-Jacobi-Bellman equations related to control problems subject to evolution inclusions involving time-dependent subdifferential operators. The last section contains some concluding remarks on the presented work.

2. NOTATIONS AND DEFINITIONS

We provide here, notations and definitions which will be needed in the development of the paper. Throughout the paper $I := [0, T]$ ($T > 0$) is an interval of \mathbb{R} . In the real Hilbert space \mathbb{R}^d , the inner product is denoted by $\langle \cdot, \cdot \rangle$ and the associated norm by $\|\cdot\|$. We denote by λ the Lebesgue measure, by $B[x, r]$ the closed ball of center x and radius r on \mathbb{R}^d and by $\mathcal{L}(I)$ (resp. $\mathcal{B}(\mathbb{R}^d)$) the σ -algebra of measurable sets of I (resp. Borel σ -algebra of measurable sets of \mathbb{R}^d). By $L^p_{\mathbb{R}^d}(I)$ for $p \in [1, +\infty[$ (resp. $p = +\infty$), we denote the space of measurable maps $x : I \rightarrow \mathbb{R}^d$ such that $\int_I \|x(t)\|^p dt < +\infty$ (resp. which are essentially bounded) endowed with the usual norm $\|x\|_{L^p_{\mathbb{R}^d}(I)} = (\int_I \|x(t)\|^p dt)^{\frac{1}{p}}$, $1 \leq p < +\infty$ (resp. endowed with the usual essential supremum norm $\|\cdot\|$). On the space $\mathcal{C}_{\mathbb{R}^d}(I)$ of continuous maps $x : I \rightarrow \mathbb{R}^d$, we consider the norm of uniform convergence defined by $\|x\|_{\infty} = \sup_{t \in I} \|x(t)\|$. Let X be a metric space, we denote by $\mathcal{C}(X)$ the set of all continuous functions from X into \mathbb{R} . When X is compact, the topological dual space of $(\mathcal{C}(X), \|\cdot\|_{\infty})$ corresponds to the space $\mathcal{M}(X)$ of all Radon measures on X . For any subset S of \mathbb{R}^d , $\delta^*(S, \cdot)$ represents the support function of S , that is, for any $y \in \mathbb{R}^d$, $\delta^*(S, y) := \sup_{x \in S} \langle y, x \rangle$.

Let φ be a lower semi-continuous (lsc) convex function from \mathbb{R}^d into $\mathbb{R} \cup \{+\infty\}$ which is proper in the sense that its effective domain $\text{dom } \varphi$ defined by $\text{dom } \varphi := \{x \in \mathbb{R}^d : \varphi(x) < +\infty\}$ is nonempty and, as usual, its Fenchel conjugate is defined by $\varphi^*(v) := \sup_{x \in \mathbb{R}^d} [\langle v, x \rangle - \varphi(x)]$. The subdifferential $\partial\varphi(x)$ of φ at $x \in \text{dom } \varphi$ is

$$\partial\varphi(x) = \{v \in \mathbb{R}^d : \varphi(y) \geq \langle v, y - x \rangle + \varphi(x) \quad \forall y \in \text{dom } \varphi\}.$$

3. FORMULATION OF THE MODEL

The class of evolution problems under consideration is governed by the subdifferential of a function φ from $I \times \mathbb{R}^d$ to $[0, +\infty]$. Assume that

(H₁) The function $\varphi(\cdot, \cdot)$ is convex and globally Lipschitz on $I \times \mathbb{R}^d$.

(H₂) There exist a non-negative ρ -Lipschitz function $k : \mathbb{R}^d \rightarrow \mathbb{R}_+$ and an absolutely continuous function $a : I \rightarrow \mathbb{R}$, with a non-negative derivative $\dot{a} \in L^2_{\mathbb{R}}(I)$, such that for any $(t, s, x) \in I \times I \times \mathbb{R}^d$

$$\varphi^*(t, x) \leq \varphi^*(s, x) + k(x)|a(t) - a(s)|,$$

where $\varphi^*(t, \cdot)$ is the conjugate function of $\varphi(t, \cdot)$.

(H₃) The perturbation $g : I \times \mathbb{R}^d \times Y \times Z \rightarrow \mathbb{R}^d$ is bounded, continuous, uniformly Lipschitz continuous with respect to its second variable.

(H₄) The cost function $J : I \times \mathbb{R}^d \times Y \times Z \rightarrow \mathbb{R}$ is bounded and continuous.

Then, the associated value function V_J on $I \times \mathbb{R}^d$ is defined by

$$V_J(\tau, x) := \sup_{v \in \mathcal{L}} \inf_{\mu \in \mathcal{Y}} \left\{ \int_{\tau}^T \left[\int_Z \left[\int_Y J(t, u_{x, \mu, v}(t), y, z) \mu_t(dy) \right] v_t(dz) \right] dt \right\},$$

where $u_{x, \mu, v}(\cdot)$ is the unique absolutely continuous solution of the problem

$$\begin{cases} \dot{u}_{x, \mu, v}(t) \in -\partial \varphi(t, u_{x, \mu, v}(t)) + \int_Z \left[\int_Y g(t, u_{x, \mu, v}(t), y, z) \mu_t(dy) \right] v_t(dz) \\ \text{for a.e. } t \in [\tau, T] \\ u_{x, \mu, v}(\tau) = x \in \text{dom } \varphi(\tau, \cdot) \end{cases}$$

(see [21, 22] for existence and uniqueness result). The spaces Y and Z are compact metric, $\mathcal{M}_+^1(Y)$ (resp. $\mathcal{M}_+^1(Z)$) is the compact metrizable space of all probability Radon measures on Y , (resp. Z) endowed with the vague topology $\sigma(\mathcal{C}(Y)', \mathcal{C}(Y))$ (resp. $\sigma(\mathcal{C}(Z)', \mathcal{C}(Z))$). The space of Young measures which is the set of all measurable mappings from I to $\mathcal{M}_+^1(Y)$ (resp. $\mathcal{M}_+^1(Z)$) is denoted by \mathcal{Y} (resp. \mathcal{Z}). The space \mathcal{Y} (resp. \mathcal{Z}) is compact metrizable for the stable convergence. For more details concerning Young measures and stable convergence, see, e.g. [1, 10, 13].

We will prove that the value function above is a viscosity solution to the following Hamilton-Jacobi-Bellman equation

$$\frac{\partial V}{\partial t}(t, x) + H(t, x, \nabla V(t, x)) = 0.$$

4. CONNECTION WITH VISCOSITY THEORY

To begin with, let's introduce the following theorem.

Theorem 4.1. Suppose that the assumptions above are made and $\text{dom } \varphi(\cdot, \cdot) = I \times \mathbb{R}^d$. For any $x_0 \in \mathbb{R}^d$ and for any $(\mu, \nu) \in \mathcal{Y} \times \mathcal{Z}$,

(i) the following problem has a unique absolutely continuous solution $x_{x_0, \mu, \nu}(\cdot)$

$$\begin{cases} \dot{x}_{x_0, \mu, \nu}(t) \in -\partial \varphi(t, x_{x_0, \mu, \nu}(t)) + \int_{\mathbb{Z}} \left[\int_Y g(t, x_{x_0, \mu, \nu}(t), y, z) \mu_t(dy) \right] \nu_t(dz) \\ \text{for a.e. } t \in I, \\ x_{x_0, \mu, \nu}(0) = x_0. \end{cases}$$

Moreover, there exists a constant $M > 0$ independent of (μ, ν) such that

$$\|x_{x_0, \mu, \nu}(t) - x_{x_0, \mu, \nu}(s)\| \leq (t-s)^{\frac{1}{2}} M \text{ for all } 0 \leq s \leq t \leq T.$$

(ii) If a sequence (t_n) in I converges to t_∞ , a sequence (ν^n) in \mathcal{Z} converges stably to $\nu^\infty \in \mathcal{Z}$ and x_{x_0, μ, ν^n} is the unique absolutely continuous solution of

$$\begin{cases} -\dot{x}_{x_0, \mu, \nu^n}(t) \in \partial \varphi(t, x_{x_0, \mu, \nu^n}(t)) + \int_{\mathbb{Z}} \left[\int_Y g(t, x_{x_0, \mu, \nu^n}(t), y, z) \mu_t(dy) \right] \nu_t^n(dz) \\ \text{for a.e. } t \in I, \\ x_{x_0, \mu, \nu^n}(0) = x_0, \end{cases}$$

then,

$$\lim_{n \rightarrow \infty} \|x_{x_0, \mu, \nu^n}(t_n) - x_{x_0, \mu, \nu^\infty}(t_\infty)\| = 0.$$

Proof. To prove the existence and uniqueness of a solution to the problem above and the required inequality in (i), follow the arguments like in Theorem 3.3. [22]. To get the continuous dependence of the solution on the control, see Theorem 3.4. and Corollary 3.1. [22].

Next, we establish the following lemma using arguments as in [6]. We may find other variants of this result in [4, 5, 7, 10].

Lemma 4.1. Under the assumptions above. Let $\mathcal{D} = \{(t, z) : t \in I, z \in \text{dom } \varphi(t, \cdot)\}$, and $(t_0, x_0) \in \mathcal{D}$. Suppose that $\Lambda_1 : I \times \mathbb{R}^d \times \mathcal{M}_+^1(Y) \times \mathcal{M}_+^1(Z) \rightarrow \mathbb{R}$ is a continuous mapping, $\Lambda_2 : I \times \mathbb{R}^d \times \mathcal{M}_+^1(Z) \rightarrow \mathbb{R}$ is upper semicontinuous such that, for any bounded subset B of \mathbb{R}^d ,

$\Lambda_2|_{I \times B \times \mathcal{M}_+^1(Z)}$ is bounded. Define $\Lambda := \Lambda_1 + \Lambda_2$ such that

$$\min_{\mu \in \mathcal{M}_+^1(Y)} \max_{\nu \in \mathcal{M}_+^1(Z)} \Lambda(t_0, x_0, \mu, \nu) < -\eta \text{ for some } \eta > 0.$$

Let $V : I \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous function such that V reaches a local maximum at (t_0, x_0) .

Then, there exist $\bar{\mu} \in \mathcal{M}_+^1(Y)$ and a real number $\rho > 0$ such that

$$(1) \quad \sup_{\nu \in \mathcal{L}} \int_{t_0}^{t_0+\rho} \Lambda(t, x_{x_0, \bar{\mu}, \nu}(t), \bar{\mu}, \nu_t) dt < -\frac{\rho\eta}{2},$$

where $x_{x_0, \bar{\mu}, \nu}(\cdot)$ is the unique absolutely continuous solution of the problem

$$\left\{ \begin{array}{l} -\dot{x}_{x_0, \bar{\mu}, \nu}(t) \in \partial \varphi(t, x_{x_0, \bar{\mu}, \nu}(t)) + \int_Z \left[\int_Y g(t, x_{x_0, \bar{\mu}, \nu}(t), y, z) \bar{\mu}_t(dy) \right] \nu_t(dz) \\ \text{for a.e. } t \in [t_0, T], \\ x_{x_0, \bar{\mu}, \nu}(t_0) = x_0, \end{array} \right.$$

corresponding to the controls $(\bar{\mu}, \nu) \in \mathcal{M}_+^1(Y) \times \mathcal{L}$, and such that

$$(2) \quad V(t_0, x_0) \geq V(t_0 + \rho, x_{x_0, \bar{\mu}, \nu}(t_0 + \rho))$$

for all $\nu \in \mathcal{L}$.

Proof. One has by assumption

$$\min_{\mu \in \mathcal{M}_+^1(Y)} \max_{\nu \in \mathcal{M}_+^1(Z)} \Lambda(t_0, x_0, \mu, \nu) < -\eta < 0,$$

i.e.,

$$\min_{\mu \in \mathcal{M}_+^1(Y)} \max_{\nu \in \mathcal{M}_+^1(Z)} [\Lambda_1(t_0, x_0, \mu, \nu) + \Lambda_2(t_0, x_0, \nu)] < -\eta < 0.$$

Since the function Λ_1 is continuous, so is the mapping

$$\mu \mapsto \max_{\nu \in \mathcal{M}_+^1(Z)} [\Lambda_1(t_0, x_0, \mu, \nu) + \Lambda_2(t_0, x_0, \nu)].$$

Thus, there exists $\bar{\mu} \in \mathcal{M}_+^1(Y)$ such that

$$\max_{\nu \in \mathcal{M}_+^1(Z)} \Lambda(t_0, x_0, \bar{\mu}, \nu) = \min_{\mu \in \mathcal{M}_+^1(Y)} \max_{\nu \in \mathcal{M}_+^1(Z)} \Lambda(t_0, x_0, \mu, \nu) < -\eta < 0.$$

Since the function $(t, x, \nu) \mapsto \Lambda_1(t, x, \bar{\mu}, \nu)$ is continuous and the function $(t, x, \nu) \mapsto \Lambda_2(t, x, \nu)$ is upper semicontinuous, $(t, x, \nu) \mapsto \Lambda_1(t, x, \bar{\mu}, \nu) + \Lambda_2(t, x, \nu)$ is upper semicontinuous, so is the function

$$(t, x) \mapsto \max_{\nu \in \mathcal{M}_+^1(Z)} \Lambda(t, x, \bar{\mu}, \nu).$$

Then, there exists $\xi > 0$ such that

$$\max_{\nu \in \mathcal{M}_+^1(Z)} \Lambda(t, x, \bar{\mu}, \nu) < -\frac{\eta}{2},$$

whenever $0 < t - t_0 \leq \xi$ and $\|x - x_0\| \leq \xi$. Assume that there exists some constant real number $\theta > 0$ such that

$$V(t_0, x_0) \geq V(t_0 + s, x_{x_0, \bar{\mu}, \nu}(t_0 + s))$$

for all $s \in]0, \theta]$, for all $\nu \in \mathcal{Z}$. This fact needs a subtle argument due to P. Raynaud de Fitte using both the continuity of $(t, \nu) \mapsto x_{x_0, \bar{\mu}, \nu}$ and the compactness of \mathcal{Z} . That is, as V has a local maximum at (t_0, x_0) , for δ and $r > 0$ small enough (we can decrease δ), one has

$$V(t_0, x_0) \geq V(t_0 + s, x)$$

for any $s \geq 0$ such that $s \leq \delta$ and for every $x \in \mathbb{R}^d$ such that $\|x - x_0\| \leq r$. Thanks to the continuity of $(t, \nu) \mapsto x_{x_0, \bar{\mu}, \nu}(t)$, one can find for each $\nu \in \mathcal{Z}$ an open neighborhood V_ν of ν in \mathcal{Z} and $\theta_\nu \in]0, \delta]$ such that, for all $(s, \nu') \in [0, \theta_\nu[\times V_\nu$, $\|x_{x_0, \bar{\mu}, \nu'}(t_0 + s) - x_0\| \leq r$. Since \mathcal{Z} is compact, one finds a finite family ν^1, \dots, ν^n such that $\mathcal{Z} = \cup_{j=1}^n V_{\nu^j}$. The assertion is then proved by taking $\theta = \min\{\theta_{\nu^j} : 1 \leq j \leq n\}$. Recall that

$$\|x_{x_0, \bar{\mu}, \nu}(t) - x_{x_0, \bar{\mu}, \nu}(s)\| \leq (t - s)^{\frac{1}{2}} M \text{ for all } t_0 \leq s \leq t \leq T,$$

where M is a positive real constant independent of $(\mu, \nu) \in \mathcal{Y} \times \mathcal{Z}$.

Choose $0 < \rho \leq \min\{\theta, \xi, (\frac{\xi}{M})^2\}$, then we obtain

$$\|x_{x_0, \bar{\mu}, \nu}(t) - x_{x_0, \bar{\mu}, \nu}(t_0)\| \leq \xi,$$

for all $t \in [t_0, t_0 + \rho]$, and for all $\nu \in \mathcal{Z}$, so that the estimate (1) results by integrating $t \mapsto \Lambda(t, x_{x_0, \bar{\mu}, \nu}(t), \bar{\mu}, \nu_t)$ on $[t_0, t_0 + \rho]$

$$\int_{t_0}^{t_0 + \rho} \Lambda(t, x_{x_0, \bar{\mu}, \nu}(t), \bar{\mu}, \nu_t) dt \leq \int_{t_0}^{t_0 + \rho} \left[\max_{\nu' \in \mathcal{M}_+^1(Z)} \Lambda(t, x_{x_0, \bar{\mu}, \nu}(t), \bar{\mu}, \nu') \right] dt$$

$$< \frac{-\rho\eta}{2} < 0,$$

for all $v \in \mathcal{Z}$. The estimate (2) results by the choice of ρ .

We address now, the dynamic programming theorem.

Theorem 4.2. Let $(\tau, x) \in \mathcal{D}$ (defined above) and let $\rho > 0$ be such that $\tau + \rho < T$. Then,

$$V_J(\tau, x) = \sup_{v \in \mathcal{Z}} \inf_{\mu \in \mathcal{Y}} \left\{ \int_{\tau}^{\tau+\rho} \left[\int_Z \left[\int_Y J(t, u_{x,\mu,v}(t), y, z) \mu_t(dy) \right] v_t(dz) \right] dt \right. \\ \left. + V_J(\tau + \rho, u_{x,\mu,v}(\tau + \rho)) \right\},$$

where

$$V_J(\tau + \rho, u_{x,\mu,v}(\tau + \rho)) := \sup_{\gamma \in \mathcal{Z}} \inf_{\beta \in \mathcal{Y}} \int_{\tau+\rho}^T \int_Z \int_Y J(t, v_{x,\beta,\gamma}(t), y, z) \beta_t(dy) \gamma_t(dz) dt,$$

the map $v_{x,\beta,\gamma}(\cdot)$ denotes the solution on $[\tau + \rho, T]$ of the evolution inclusion

$$-\dot{v}_{x,\beta,\gamma}(t) \in \partial \varphi(t, v_{x,\beta,\gamma}(t)) + \int_Z \left[\int_Y g(t, v_{x,\beta,\gamma}(t), y, z) \beta_t(dy) \right] \gamma_t(dz)$$

where the controls $(\beta, \gamma) \in \mathcal{Y} \times \mathcal{Z}$ and the starting from $v_{x,\beta,\gamma}(\tau + \rho) = u_{x,\mu,v}(\tau + \rho)$.

We are going to prove the existence of viscosity subsolutions.

Theorem 4.3. Under the assumptions above, let $V_J : I \times \mathbb{R}^d \rightarrow \mathbb{R}$ be the value function defined

by

$$V_J(\tau, x) := \sup_{v \in \mathcal{Z}} \inf_{\mu \in \mathcal{Y}} \left\{ \int_{\tau}^T \left[\int_Z \left[\int_Y J(t, u_{x,\mu,v}(t), y, z) \mu_t(dy) \right] v_t(dz) \right] dt \right\},$$

where $u_{x,\mu,v}(\cdot)$ is the unique absolutely continuous solution of the problem

$$\begin{cases} -\dot{u}_{x,\mu,v}(t) \in \partial \varphi(t, u_{x,\mu,v}(t)) + \int_Z \left[\int_Y g(t, u_{x,\mu,v}(t), y, z) \mu_t(dy) \right] v_t(dz) \\ \text{a.e. in } [\tau, T], \\ u_{x,\mu,v}(\tau) = x \in \text{dom } \varphi(\tau, \cdot), \end{cases}$$

Let $H(\cdot, \cdot, \cdot)$ be the Hamiltonian on $I \times \mathbb{R}^d \times \mathbb{R}^d$ given by

$$H(t, x, \xi) = \inf_{\mu \in \mathcal{M}_+^1(Y)} \sup_{v \in \mathcal{M}_+^1(Z)} \left\{ \langle \xi, \int_Z \left[\int_Y g(t, x, y, z) \mu_t(dy) \right] v_t(dz) \rangle \right. \\ \left. + \int_Z \left[\int_Y J(t, x, y, z) \mu_t(dy) \right] v_t(dz) \right\} + \delta^*(\xi, -\partial \varphi(t, x)),$$

the function $\delta^*(\cdot, -\partial\varphi(t, z))$ is the support function of the set-valued map that associates z , $-\partial\varphi(t, z)$. Then, V_J is a viscosity subsolution of the Hamilton-Jacobi-Bellman equation

$$\frac{\partial V}{\partial t}(t, x) + H(t, x, \nabla V(t, x)) = 0,$$

i.e., for any $\phi \in \mathcal{C}^1(I \times \mathbb{R}^d)$ such that $V_J - \phi$ reaches a local maximum at $(t_0, x_0) \in I \times \mathbb{R}^d$, one obtains

$$\frac{\partial \phi}{\partial t}(t_0, x_0) + H(t_0, x_0, \nabla \phi(t_0, x_0)) \geq 0.$$

Proof. We follow arguments from [4, 5, 6, 7, 10] and originally used in [14, 18].

Assume by contradiction that there exist some $\phi \in \mathcal{C}^1(I \times \mathbb{R}^d)$ and a point $(t_0, x_0) \in \mathcal{D}$ for which

$$\frac{\partial \phi}{\partial t}(t_0, x_0) + H(t_0, x_0, \nabla \phi(t_0, x_0)) < -\eta \text{ for some } \eta > 0.$$

In view of Proposition I.17 in [23], for each $t \in I$, the set-valued mapping that associates $z \in \mathbb{R}^d$, $\partial\varphi(t, z)$ is upper semicontinuous with convex compact values in \mathbb{R}^d . As well known, the global Lipschitz property of $\varphi(\cdot, \cdot)$ also entails that the range of $\partial\varphi(\cdot, \cdot)$ is a bounded set. It follows that the function

$$(t, z) \in I \times \mathbb{R}^d \mapsto \Lambda_2(t, z) := \delta^*(\nabla \phi(t, z), -\partial\varphi(t, z))$$

is upper semicontinuous. Thanks to the continuity of $\nabla \phi(\cdot, \cdot)$ and the boundedness of the range of $\partial\varphi(\cdot, \cdot)$, for any bounded subset B of \mathbb{R}^d , $\Lambda_2|_{I \times B}$ is bounded. Under our assumptions, it is clear that the function $\Lambda_1 : I \times \mathbb{R}^d \times \mathcal{M}_+^1(Y) \times \mathcal{M}_+^1(Z) \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} \Lambda_1(t, x, \mu, \nu) &:= \int_Z \left[\int_Y J(t, x, y, z) \mu_t(dy) \right] \nu_t(dz) \\ &+ \langle \nabla \phi(t, x), \int_Z \left[\int_Y g(t, x, y, z) \mu_t(dy) \right] \nu_t(dz) \rangle + \frac{\partial \phi}{\partial t}(t, x) \end{aligned}$$

is continuous, $\mathcal{M}_+^1(Y)$ and $\mathcal{M}_+^1(Z)$ being endowed with the vague topology $\sigma(\mathcal{M}(Y), \mathcal{C}(Y))$ and $\sigma(\mathcal{M}(Z), \mathcal{C}(Z))$ respectively. Hence, applying Lemma 4.1. to $\Lambda := \Lambda_1 + \Lambda_2$ and find $\bar{\mu} \in \mathcal{M}_+^1(Y)$ and $\rho > 0$ independent of $\nu \in \mathcal{Z}$ such that

$$(3) \quad \frac{-\rho\eta}{2} \geq \sup_{\nu \in \mathcal{Z}} \left\{ \int_{t_0}^{t_0+\rho} \left[\int_Z \left[\int_Y J(t, x_{x_0, \bar{\mu}, \nu}(t), y, z) \bar{\mu}(dy) \right] \nu_t(dz) \right] dt \right.$$

$$\begin{aligned}
& + \int_{t_0}^{t_0+\rho} \left[\int_Z \left[\int_Y \langle \nabla \phi(t, x_{x_0, \bar{\mu}, \nu}(t)), g(t, x_{x_0, \bar{\mu}, \nu}(t), y, z) \rangle \bar{\mu}(dy) \right] \mathbf{v}_t(dz) \right] dt + \\
& \int_{t_0}^{t_0+\rho} \frac{\partial \phi}{\partial t}(t, x_{x_0, \bar{\mu}, \nu}(t)) dt + \int_{t_0}^{t_0+\rho} \delta^*(\nabla \phi(t, x_{x_0, \bar{\mu}, \nu}(t)), -\partial \phi(t, x_{x_0, \bar{\mu}, \nu}(t))) dt \}
\end{aligned}$$

where $x_{x_0, \bar{\mu}, \nu}(\cdot) : [\tau, T] \rightarrow \mathbb{R}^d$ denotes the unique absolutely continuous solution of the problem

$$\begin{cases} -\dot{x}_{x_0, \bar{\mu}, \nu}(t) \in \partial \phi(t, x_{x_0, \bar{\mu}, \nu}(t)) + \int_Z \left[\int_Y g(t, x_{x_0, \bar{\mu}, \nu}(t), y, z) \bar{\mu}(dy) \right] \mathbf{v}_t(dz) \\ \text{for a.e. } t \in [\tau, T], \\ x_{x_0, \bar{\mu}, \nu}(\tau) = x_0, \end{cases}$$

the controls $(\bar{\mu}, \nu)$ belong to $\mathcal{M}_+^1(Y) \times \mathcal{Z}$ and such that

$$(4) \quad V_J(t_0, x_0) - \phi(t_0, x_0) \geq V_J(t_0 + \rho, x_{x_0, \bar{\mu}, \nu}(t_0 + \rho)) - \phi(t_0 + \rho, x_{x_0, \bar{\mu}, \nu}(t_0 + \rho))$$

for all $\nu \in \mathcal{Z}$. Thanks to Theorem 4.2. of dynamic programming, we know that

$$\begin{aligned}
V_J(t_0, x_0) & \leq \sup_{\nu \in \mathcal{Z}} \left\{ \int_{t_0}^{t_0+\rho} \left[\int_Z \left[\int_Y J(t, x_{x_0, \bar{\mu}, \nu}(t), y, z) \bar{\mu}(dy) \right] \mathbf{v}_t(dz) \right] dt \right. \\
& \quad \left. + V_J(t_0 + \rho, x_{x_0, \bar{\mu}, \nu}(t_0 + \rho)) \right\}.
\end{aligned}$$

We need to an argument from Proposition 6.2 [9]. For any $n \in \mathbb{N}$, there is $\nu^n \in \mathcal{Z}$ such that

$$\begin{aligned}
V_J(t_0, x_0) & \leq \int_{t_0}^{t_0+\rho} \left[\int_Z \left[\int_Y J(t, x_{x_0, \bar{\mu}, \nu^n}(t), y, z) \bar{\mu}(dy) \right] \mathbf{v}_t^n(dz) \right] dt \\
& \quad + V_J(t_0 + \rho, x_{x_0, \bar{\mu}, \nu^n}(t_0 + \rho)) + \frac{1}{n}.
\end{aligned}$$

Hence from (4), it follows that

$$\begin{aligned}
& V_J(t_0 + \rho, x_{x_0, \bar{\mu}, \nu^n}(t_0 + \rho)) - \phi(t_0 + \rho, x_{x_0, \bar{\mu}, \nu^n}(t_0 + \rho)) \\
& \leq \int_{t_0}^{t_0+\rho} \left[\int_Z \left[\int_Y J(t, x_{x_0, \bar{\mu}, \nu^n}(t), y, z) \bar{\mu}(dy) \right] \mathbf{v}_t^n(dz) \right] dt + \frac{1}{n} \\
& \quad - \phi(t_0, x_0) + V_J(t_0 + \rho, x_{x_0, \bar{\mu}, \nu^n}(t_0 + \rho)).
\end{aligned}$$

As a result,

$$\begin{aligned}
0 & \leq \int_{t_0}^{t_0+\rho} \left[\int_Z \left[\int_Y J(t, x_{x_0, \bar{\mu}, \nu^n}(t), y, z) \bar{\mu}(dy) \right] \mathbf{v}_t^n(dz) \right] dt \\
& \quad + \phi(t_0 + \rho, x_{x_0, \bar{\mu}, \nu^n}(t_0 + \rho)) - \phi(t_0, x_0) + \frac{1}{n}.
\end{aligned}$$

Making use of the \mathcal{C}^1 regularity of ϕ and the absolute continuity of $x_{x_0, \bar{\mu}, v^n}(\cdot)$,

$$\begin{aligned} & \phi(t_0 + \rho, x_{x_0, \bar{\mu}, v^n}(t_0 + \rho)) - \phi(t_0, x_0) \\ & \leq \int_{t_0}^{t_0 + \rho} \left[\int_Z \left[\int_Y \langle \nabla \phi(t, x_{x_0, \bar{\mu}, v^n}(t)), g(t, x_{x_0, \bar{\mu}, v^n}(t), y, z) \rangle \bar{\mu}(dy) \right] v_t^n(dz) \right] dt \\ & \quad + \int_{t_0}^{t_0 + \rho} \delta^*(\nabla \phi(t, x_{x_0, \bar{\mu}, v^n}(t)), -\partial \varphi(t, x_{x_0, \bar{\mu}, v^n}(t))) dt \\ & \quad + \int_{t_0}^{t_0 + \rho} \frac{\partial \phi}{\partial t}(t, x_{x_0, \bar{\mu}, v^n}(t)) dt, \end{aligned}$$

one has for any n ,

$$\begin{aligned} (5) \quad & 0 \leq \int_{t_0}^{t_0 + \rho} \left[\int_Z \left[\int_Y J(t, x_{x_0, \bar{\mu}, v^n}(t), y, z) \bar{\mu}(dy) \right] v_t^n(dz) \right] dt \\ & \quad + \int_{t_0}^{t_0 + \rho} \left[\int_Z \left[\int_Y \langle \nabla \phi(t, x_{x_0, \bar{\mu}, v^n}(t)), g(t, x_{x_0, \bar{\mu}, v^n}(t), y, z) \rangle \bar{\mu}(dy) \right] v_t^n(dz) \right] dt \\ & \quad + \int_{t_0}^{t_0 + \rho} \delta^*(\nabla \phi(t, x_{x_0, \bar{\mu}, v^n}(t)), -\partial \varphi(t, x_{x_0, \bar{\mu}, v^n}(t))) dt \\ & \quad + \int_{t_0}^{t_0 + \rho} \frac{\partial \phi}{\partial t}(t, x_{x_0, \bar{\mu}, v^n}(t)) dt + \frac{1}{n}. \end{aligned}$$

The space \mathcal{Z} being compact metrizable for the stable topology, assume that (v^n) stably converges to a Young measure $\bar{v} \in \mathcal{Z}$. This entails that $x_{x_0, \bar{\mu}, v^n}$ converges uniformly to $x_{x_0, \bar{\mu}, \bar{v}}$ which is the trajectory solution of

$$\begin{cases} -\dot{x}_{x_0, \bar{\mu}, \bar{v}}(t) \in \partial \varphi(t, x_{x_0, \bar{\mu}, \bar{v}}(t)) + \int_Z \left[\int_Y g(t, x_{x_0, \bar{\mu}, \bar{v}}(t), y, z) \bar{\mu}(dy) \right] \bar{v}_t(dz) \\ \text{for a.e. } t \in [\tau, T], \\ x_{x_0, \bar{\mu}, \bar{v}}(\tau) = x_0, \end{cases}$$

where the controls $(\bar{\mu}, \bar{v})$ belong to $\mathcal{M}_+^1(Y) \times \mathcal{Z}$, and $\delta_{x_{x_0, \bar{\mu}, v^n}} \otimes v^n$ stably converges to $\delta_{x_{x_0, \bar{\mu}, \bar{v}}} \otimes \bar{v}$ (see [10]). Consequently,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{t_0}^{t_0 + \rho} \left[\int_Z \left[\int_Y J(t, x_{x_0, \bar{\mu}, v^n}(t), y, z) \bar{\mu}(dy) \right] v_t^n(dz) \right] dt \\ & = \int_{t_0}^{t_0 + \rho} \left[\int_Z \left[\int_Y J(t, x_{x_0, \bar{\mu}, \bar{v}}(t), y, z) \bar{\mu}(dy) \right] \bar{v}_t(dz) \right] dt, \end{aligned}$$

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \int_{t_0}^{t_0+\rho} \left[\int_Z \left[\int_Y \langle \nabla \phi(t, x_{x_0, \bar{\mu}, v^n}(t)), g(t, x_{x_0, \bar{\mu}, v^n}(t), y, z) \rangle \bar{\mu}(dy) \right] v_t^n(dz) \right] dt \\
&= \int_{t_0}^{t_0+\rho} \left[\int_Z \left[\int_Y \langle \nabla \phi(t, x_{x_0, \bar{\mu}, \bar{v}}(t)), g(t, x_{x_0, \bar{\mu}, \bar{v}}(t), y, z) \rangle \bar{\mu}(dy) \right] \bar{v}_t(dz) \right] dt.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \int_{t_0}^{t_0+\rho} \delta^*(\nabla \phi(t, x_{x_0, \bar{\mu}, v^n}(t)), -\partial \varphi(t, x_{x_0, \bar{\mu}, v^n}(t))) dt \\
& \leq \int_{t_0}^{t_0+\rho} \delta^*(\nabla \phi(t, x_{x_0, \bar{\mu}, \bar{v}}(t)), -\partial \varphi(t, x_{x_0, \bar{\mu}, \bar{v}}(t))) dt,
\end{aligned}$$

because

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \delta^*(\nabla \phi(t, x_{x_0, \bar{\mu}, v^n}(t)), -\partial \varphi(t, x_{x_0, \bar{\mu}, v^n}(t))) \\
& \leq \delta^*(\nabla \phi(t, x_{x_0, \bar{\mu}, \bar{v}}(t)), -\partial \varphi(t, x_{x_0, \bar{\mu}, \bar{v}}(t)))
\end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \int_{t_0}^{t_0+\rho} \frac{\partial \phi}{\partial t}(t, x_{x_0, \bar{\mu}, v^n}(t)) dt = \int_{t_0}^{t_0+\rho} \frac{\partial \phi}{\partial t}(t, x_{x_0, \bar{\mu}, \bar{v}}(t)) dt.$$

Passing to the limit in (5), when $n \rightarrow \infty$,

$$\begin{aligned}
0 & \leq \int_{t_0}^{t_0+\rho} \left[\int_Z \left[\int_Y J(t, x_{x_0, \bar{\mu}, \bar{v}}(t), y, z) \bar{\mu}(dy) \right] \bar{v}_t(dz) \right] dt \\
& + \int_{t_0}^{t_0+\rho} \left[\int_Z \left[\int_Y \langle \nabla \phi(t, x_{x_0, \bar{\mu}, \bar{v}}(t)), g(t, x_{x_0, \bar{\mu}, \bar{v}}(t), y, z) \rangle \bar{\mu}(dy) \right] \bar{v}_t(dz) \right] dt \\
& + \int_{t_0}^{t_0+\rho} \delta^*(\nabla \phi(t, x_{x_0, \bar{\mu}, \bar{v}}(t)), -\partial \varphi(t, x_{x_0, \bar{\mu}, \bar{v}}(t))) dt \\
& + \int_{t_0}^{t_0+\rho} \frac{\partial \phi}{\partial t}(t, x_{x_0, \bar{\mu}, \bar{v}}(t)) dt.
\end{aligned}$$

This is in contradiction with (3).

We check now, the superviscosity property of the value function V_J by imposing some extra conditions on φ , g , J and the first space of Young measure controls. Assume the following

(C₁) The subset of \mathcal{Y} denoted by \mathcal{H} , is compact for the convergence in probability, in particular \mathcal{H} is compact for the stable convergence (see e.g. [10]).

This condition entails that the mapping $(\mu, \nu) \mapsto x_{x_0, \mu, \nu}$ is continuous on $\mathcal{H} \times \mathcal{L}$ using the fiber product and the arguments of Theorem 5.1 [10], along with the continuous dependence of the solution of the problem on the initial position and the control.

(C₂) The functions J and g are bounded, continuous and g is uniformly Lipschitz with respect to its second variable, the family $(J(\cdot, \cdot, \mu, \nu))_{(\mu, \nu) \in \mathcal{M}_+^1(Y) \times \mathcal{M}_+^1(Z)}$ (resp. $(g(\cdot, \cdot, \mu, \nu))_{(\mu, \nu) \in \mathcal{M}_+^1(Y) \times \mathcal{M}_+^1(Z)}$) is equicontinuous on $I \times \mathbb{R}^d$.

(C₃) The function $\varphi : I \times \mathbb{R}^d \rightarrow [0, +\infty]$ satisfies (H₁)-(H₂), φ is \mathcal{C}^1 on $I \times \mathbb{R}^d$ so that $\partial\varphi(t, x) = \{\nabla\varphi(t, x)\}$, $(t, x) \in I \times \mathbb{R}^d$ (see Proposition I.18 in [23]).

Under assumptions (C₁)-(C₃), we get a variant of Lemma 4.1. which allows us to prove the superviscosity property.

Lemma 4.2. Let $(t_0, x_0) \in \mathcal{D}$ and let $\Lambda : I \times \mathbb{R}^d \times \mathcal{M}_+^1(Y) \times \mathcal{M}_+^1(Z) \rightarrow \mathbb{R}$ be a continuous mapping, and the family $(\Lambda(\cdot, \cdot, \mu, \nu))$, $(\mu, \nu) \in \mathcal{M}_+^1(Y) \times \mathcal{M}_+^1(Z)$ be equicontinuous on $I \times \mathbb{R}^d$. Assume further that

$$\min_{\mu \in \mathcal{M}_+^1(Y)} \max_{\nu \in \mathcal{M}_+^1(Z)} \Lambda(t_0, x_0, \mu, \nu) > \eta > 0 \text{ for some } \eta > 0.$$

Let $V : I \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous function such that V reaches a local minimum at (t_0, x_0) .

Then, there exists a real number $\rho > 0$ such that for any $\mu \in \mathcal{H}$, one has

$$(6) \quad \sup_{\nu \in \mathcal{L}} \int_{t_0}^{t_0+\rho} \Lambda(t, x_{x_0, \mu, \nu}(t), \mu_t, \nu_t) dt > \frac{\rho\eta}{2},$$

where $x_{x_0, \mu, \nu}(\cdot)$ is the unique absolutely continuous solution of the problem

$$\begin{cases} \dot{x}_{x_0, \mu, \nu}(t) = -\nabla\varphi(t, x_{x_0, \mu, \nu}(t)) + \int_Z \left[\int_Y g(t, x_{x_0, \mu, \nu}(t), y, z) \mu_t(dy) \right] \nu_t(dz) \\ \text{for a.e. } t \in I, \\ x_{x_0, \mu, \nu}(t_0) = x_0, \end{cases}$$

the controls (μ, ν) belong to $\mathcal{H} \times \mathcal{L}$, and such that

$$(7) \quad V(t_0, x_0) \leq V(t_0 + \rho, x_{x_0, \mu, \nu}(t_0 + \rho))$$

for all $(\mu, \nu) \in \mathcal{H} \times \mathcal{L}$.

Proof. As V has a local minimum at (t_0, x_0) , there are $\theta > 0$, $r > 0$ such that

$$V(t_0, x_0) \leq V(t, x) \text{ whenever } 0 < t - t_0 \leq \theta \text{ and } x \in B[x_0, r].$$

Thanks to the equicontinuity of the family $(\Lambda(\cdot, \cdot, \mu, \nu))_{(\mu, \nu) \in \mathcal{M}_+^1(Y) \times \mathcal{M}_+^1(Z)}$ there is ξ such that $\xi \in]0, r[$ independent of (μ, ν) such that for all $t \in [t_0, t_0 + \xi]$ and x such that $\|x - x_0\| \leq \xi$

$$\Lambda(t_0, x_0, \mu, \nu) - \frac{\eta}{2} < \Lambda(t, x, \mu, \nu)$$

for any $(\mu, \nu) \in \mathcal{M}_+^1(Y) \times \mathcal{M}_+^1(Z)$.

Consider an arbitrary element μ in \mathcal{H} . Then, there exists a λ -measurable mapping $\nu^\mu : I \rightarrow \mathcal{M}_+^1(Z)$ such that

$$\Lambda(t_0, x_0, \mu_t, \nu_t^\mu) = \max_{\nu' \in \mathcal{M}_+^1(Z)} \Lambda(t_0, x_0, \mu_t, \nu')$$

for all $t \in I$, since the nonempty compact-valued set-valued

$$t \mapsto \{\nu \in \mathcal{M}_+^1(Z) : \Lambda(t_0, x_0, \mu_t, \nu) = \max_{\nu' \in \mathcal{M}_+^1(Z)} \Lambda(t_0, x_0, \mu_t, \nu')\}$$

has its graph in $\mathcal{L}(I) \otimes \mathcal{B}(\mathcal{M}_+^1(Z))$. Recall that

$$\|x_{x_0, \mu, \nu}(t) - x_{x_0, \mu, \nu}(s)\| \leq (t - s)^{\frac{1}{2}} M \text{ for all } t_0 \leq s \leq t \leq T,$$

where M is a positive real constant independent of $(\mu, \nu) \in \mathcal{Y} \times \mathcal{Z}$. Choose $0 < \rho \leq \min\{\theta, \xi, (\frac{\xi}{M})^2\}$, one obtains

$$\|x_{x_0, \mu, \nu}(t) - x_{x_0, \mu, \nu}(t_0)\| \leq \xi,$$

for all $t \in [t_0, t_0 + \rho]$, and for all $\nu \in \mathcal{Z}$. By integration,

$$\begin{aligned} \int_{t_0}^{t_0+\rho} \Lambda(t, x_{x_0, \mu, \nu}(t), \mu_t, \nu_t^\mu) dt &\geq \int_{t_0}^{t_0+\rho} [\Lambda(t_0, x_0, \mu_t, \nu_t^\mu) - \frac{\eta}{2}] dt \\ &> \int_{t_0}^{t_0+\rho} \frac{\eta}{2} dt = \frac{\rho\eta}{2}. \end{aligned}$$

Then, (6) follows from the choice of ρ . This ends the proof.

We are going to prove the existence of viscosity supersolutions.

Theorem 4.4. Under assumptions (C_1) - (C_3) above, let $V_J : I \times \mathbb{R}^d \rightarrow \mathbb{R}$ be the value function defined by

$$V_J(\tau, x) := \sup_{\nu \in \mathcal{Z}} \inf_{\mu \in \mathcal{H}} \left\{ \int_{\tau}^T \left[\int_Z \left[\int_Y J(t, u_{x, \mu, \nu}(t), y, z) \mu_t(dy) \right] \nu_t(dz) \right] dt \right\},$$

where $u_{x,\mu,\nu}(\cdot)$ is the unique absolutely continuous solution of the inclusion

$$\begin{cases} \dot{u}_{x,\mu,\nu}(t) = -\nabla\varphi(t, u_{x,\mu,\nu}(t)) + \int_Z \left[\int_Y g(t, u_{x,\mu,\nu}(t), y, z) \mu_t(dy) \right] \nu_t(dz) \\ \text{a.e. in } [\tau, T], \\ u_{x,\mu,\nu}(\tau) = x \in \text{dom } \varphi(\tau, \cdot). \end{cases}$$

Let $H(\cdot, \cdot, \cdot)$ be the Hamiltonian on $I \times \mathbb{R}^d \times \mathbb{R}^d$ given by

$$\begin{aligned} H(t, x, \xi) &= \inf_{\mu \in \mathcal{M}_+^1(Y)} \sup_{\nu \in \mathcal{M}_+^1(Z)} \left\{ \langle \xi, \int_Z \left[\int_Y g(t, x, y, z) \mu_t(dy) \right] \nu_t(dz) \rangle \right. \\ &\quad \left. + \int_Z \left[\int_Y J(t, x, y, z) \mu_t(dy) \right] \nu_t(dz) \right\} + \langle \xi, -\nabla\varphi(t, x) \rangle. \end{aligned}$$

Then, V_J is a viscosity supersolution of the Hamilton-Jacobi-Bellman equation

$$\frac{\partial V}{\partial t}(t, x) + H(t, x, \nabla V(t, x)) = 0,$$

i.e., for any $\phi \in \mathcal{C}^1(I \times \mathbb{R}^d)$ such that $V_J - \phi$ reaches a local minimum at $(t_0, x_0) \in I \times \mathbb{R}^d$, one has

$$\frac{\partial \phi}{\partial t}(t_0, x_0) + H(t_0, x_0, \nabla \phi(t_0, x_0)) \leq 0.$$

Proof. We use the arguments of Theorem 4.3., with some modifications.

Assume by contradiction that there exist some $\phi \in \mathcal{C}^1(I \times \mathbb{R}^d)$ and a point $(t_0, x_0) \in I \times \mathbb{R}^d$ for which

$$(8) \quad \frac{\partial \phi}{\partial t}(t_0, x_0) + H(t_0, x_0, \nabla \phi(t_0, x_0)) > \eta \text{ for some } \eta > 0.$$

As $V_J - \phi$ reaches a local minimum at (t_0, x_0) , hence, applying Lemma 4.2. to $V_J - \phi$ and the integrand Λ defined on $I \times \mathbb{R}^d \times \mathcal{M}_+^1(Y) \times \mathcal{M}_+^1(Z)$ by

$$\begin{aligned} \Lambda(t, x, \mu, \nu) &= \int_Z \left[\int_Y J(t, x, y, z) \mu_t(dy) \right] \nu_t(dz) + \langle \nabla \phi(t, x), -\nabla \varphi(t, x) \rangle \\ &\quad + \langle \nabla \varphi(t, x), \int_Z \left[\int_Y g(t, x, y, z) \mu_t(dy) \right] \nu_t(dz) \rangle + \frac{\partial \phi}{\partial t}(t, x) \end{aligned}$$

$(t, x, \mu, \nu) \in I \times \mathbb{R}^d \times \mathcal{M}_+^1(Y) \times \mathcal{M}_+^1(Z)$ gives $\rho > 0$ such that

$$(9) \quad \sup_{\nu \in \mathcal{H}} \min_{\mu \in \mathcal{H}} \left\{ \int_{t_0}^{t_0+\rho} \left[\int_Z \left[\int_Y J(t, x_{t_0, \mu, \nu}(t), y, z) \mu_t(dy) \right] \nu_t(dz) \right] dt \right.$$

$$\begin{aligned}
& + \int_{t_0}^{t_0+\rho} \left[\int_Z \left[\int_Y \langle \nabla \varphi(t, x), g(t, x_{x_0, \mu, \nu}(t), y, z) \rangle \mu_t(dy) \right] \nu_t(dz) \right] dt \\
& + \int_{t_0}^{t_0+\rho} \frac{\partial \phi}{\partial t}(t, x_{x_0, \mu, \nu}(t)) dt + \int_{t_0}^{t_0+\rho} \langle \nabla \phi(t, x_{x_0, \mu, \nu}(t)), -\nabla \varphi(t, x_{x_0, \mu, \nu}(t)) \rangle dt \} \\
& \geq \frac{\rho \eta}{2},
\end{aligned}$$

where $x_{x_0, \mu, \nu}(\cdot)$ is the unique absolutely continuous solution of the problem

$$\begin{cases} \dot{x}_{x_0, \mu, \nu}(t) = -\nabla \varphi(t, x_{x_0, \mu, \nu}(t)) + \int_Z \left[\int_Y g(t, x_{x_0, \mu, \nu}(t), y, z) \mu_t(dy) \right] \nu_t(dz) \\ x_{x_0, \mu, \nu}(t_0) = x_0 \in \text{dom } \varphi(t_0, \cdot), \end{cases}$$

the controls (μ, ν) belong to $\mathcal{H} \times \mathcal{L}$ and such that

$$(10) \quad V_J(t_0, x_0) - \phi(t_0, x_0) \leq V_J(t_0 + \rho, x_{x_0, \mu, \nu}(t_0 + \rho)) - \phi(t_0 + \rho, x_{x_0, \mu, \nu}(t_0 + \rho))$$

for all $(\mu, \nu) \in \mathcal{H} \times \mathcal{L}$. Next, thanks to (10) and Theorem 4.2. of dynamic programming, we know that

$$\begin{aligned}
& \sup_{\nu \in \mathcal{L}} \min_{\mu \in \mathcal{H}} \left\{ \int_{t_0}^{t_0+\rho} \left[\int_Z \left[\int_Y J(t, x_{x_0, \mu, \nu}(t), y, z) \mu_t(dy) \right] \nu_t(dz) \right] dt \right. \\
& \quad \left. + V_J(t_0 + \rho, x_{x_0, \mu, \nu}(t_0 + \rho)) \right\} + \phi(t_0 + \rho, x_{x_0, \mu, \nu}(t_0 + \rho)) \\
(11) \quad & - \phi(t_0, x_0) - V_J(t_0 + \rho, x_{x_0, \mu, \nu}(t_0 + \rho)) \leq 0.
\end{aligned}$$

Choose $\bar{\mu} \in \mathcal{H}$ such that

$$\begin{aligned}
(12) \quad & \sup_{\nu \in \mathcal{L}} \min_{\mu \in \mathcal{H}} \left\{ \int_{t_0}^{t_0+\rho} \left[\int_Z \left[\int_Y J(t, x_{x_0, \mu, \nu}(t), y, z) \mu_t(dy) \right] \nu_t(dz) \right] dt \right. \\
& \quad \left. + V_J(t_0 + \rho, x_{x_0, \mu, \nu}(t_0 + \rho)) \right\} \\
& = \sup_{\nu \in \mathcal{L}} \left\{ \int_{t_0}^{t_0+\rho} \left[\int_Z \left[\int_Y J(t, x_{x_0, \bar{\mu}, \nu}(t), y, z) \bar{\mu}_t(dy) \right] \nu_t(dz) \right] dt \right. \\
& \quad \left. + V_J(t_0 + \rho, x_{x_0, \bar{\mu}, \nu}(t_0 + \rho)) \right\}.
\end{aligned}$$

We come back to (10) and (12), we get

$$\begin{aligned}
& \sup_{\nu \in \mathcal{L}} \left\{ \int_{t_0}^{t_0+\rho} \left[\int_Z \left[\int_Y J(t, x_{x_0, \bar{\mu}, \nu}(t), y, z) \bar{\mu}_t(dy) \right] \nu_t(dz) \right] dt + V_J(t_0 + \rho, x_{x_0, \bar{\mu}, \nu}(t_0 + \rho)) \right\} \\
& + \sup_{\nu \in \mathcal{L}} \left\{ \phi(t_0 + \rho, x_{x_0, \bar{\mu}, \nu}(t_0 + \rho)) - \phi(t_0, x_0) - V_J(t_0 + \rho, x_{x_0, \bar{\mu}, \nu}(t_0 + \rho)) \right\} \leq 0.
\end{aligned}$$

Then, it follows that

$$(13) \quad 0 \geq \sup_{v \in \mathcal{Z}} \left\{ \int_{t_0}^{t_0+\rho} \left[\int_Z \left[\int_Y J(t, x_{x_0, \bar{\mu}, v}(t), y, z) \bar{\mu}_t(dy) \right] v_t(dz) \right] dt \right. \\ \left. + \phi(t_0 + \rho, x_{x_0, \bar{\mu}, v}(t_0 + \rho)) - \phi(t_0, x_0) \right\}.$$

Making use of the \mathcal{C}^1 regularity of ϕ and the fact that $x_{x_0, \bar{\mu}, v}(\cdot)$ is the solution of the dynamic, we clearly have

$$(14) \quad \phi(t_0 + \rho, x_{x_0, \bar{\mu}, v}(t_0 + \rho)) - \phi(t_0, x_0) \\ = \int_{t_0}^{t_0+\rho} \left[\int_Z \left[\int_Y \langle \nabla \phi(t, x_{x_0, \bar{\mu}, v}(t)), g(t, x_{x_0, \bar{\mu}, v}(t), y, z) \rangle \bar{\mu}_t(dy) \right] v_t(dz) \right] dt \\ + \int_{t_0}^{t_0+\rho} \frac{\partial \phi}{\partial t}(t, x_{x_0, \bar{\mu}, v}(t)) dt + \int_{t_0}^{t_0+\rho} \langle \nabla \phi(t, x_{x_0, \bar{\mu}, v}(t)), -\nabla \varphi(t, x_{x_0, \bar{\mu}, v}(t)) \rangle dt.$$

In view of (14), we may write (13) as follows

$$\sup_{v \in \mathcal{Z}} \left\{ \int_{t_0}^{t_0+\rho} \left[\int_Z \left[\int_Y J(t, x_{z, \bar{\mu}, v}(t), y, z) \bar{\mu}_t(dy) \right] v_t(dz) \right] dt \right. \\ + \int_{t_0}^{t_0+\rho} \left[\int_Z \left[\int_Y \langle \nabla \phi(t, x_{x_0, \bar{\mu}, v}(t)), g(t, x_{x_0, \bar{\mu}, v}(t), y, z) \rangle \bar{\mu}_t(dy) \right] v_t(dz) \right] dt \\ \left. + \int_{t_0}^{t_0+\rho} \langle \nabla \phi(t, x_{x_0, \bar{\mu}, v}(t)), -\nabla \varphi(t, x_{x_0, \bar{\mu}, v}(t)) \rangle dt + \int_{t_0}^{t_0+\rho} \frac{\partial \phi}{\partial t}(t, x_{x_0, \bar{\mu}, v}(t)) dt \right\} \leq 0.$$

This latter inequality leads to a contradiction with (9).

5. CONCLUSION

The present article consists in finding a solution to an associated Hamilton-Jacobi-Bellman equation. The first result of the paper is Theorem 4.1. in Section 4, concerning the continuous dependence of the trajectories with respect to the relaxed control, in suitable topologies. The latter plays an important role in the development of the paper. It allows us to prove a dynamic programming principle (Theorem 4.2.), in the finite dimensional setting, that the value function is a subsolution of an associated Hamilton-Jacobi-Bellman equation (Theorem 4.3.). In order to show further that this value function is a supersolution, we impose extra conditions on the first space control, the dynamic and the cost functional (Theorem 4.4.). These additional conditions lead to a viscosity solution. We will continue the research started here, and investigate such

equations related to similar evolution problems when the dynamic contains a delay in another paper.

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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