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A GENERALIZED SYSTEM OF VARIATIONAL INEQUALITIES IN A BANACH SPACE

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Abstract. In this paper, we investigate a generalized system for nonlinear variational inequalities based on a two-step algorithm. Convergence theorems of solutions are established in the framework of Banach spaces.

Keywords: variational inequality; Banach space; nonexpansive mapping; solution.

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1. Introduction-Preliminaries

Systems of variational inequalities have significant applications in various fields of mathematics, physics, economics, and engineering sciences. The solvability of systems of variational inequalities based on iterative methods has been extensively investigated; see [1-17] and the references therein.

Let B be a Banach space and let B^* be the dual space of B. Let $\langle \cdot, \cdot \rangle$ denote the duality pairing of B^* and B. We consider the following the variational inequality: Find $x \in K$ such that

$$\langle Tx, y-x \rangle \geq 0, \quad \forall y \in K, \tag{1.1}$$

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where K is a nonempty, closed and convex subset of B and $T : K \to B^*$. A point $x_0 \in K$ is called a solution of the variational inequality (1.1) if, for every $y \in K$, $\langle Tx_0, y - x_0 \rangle \geq 0$. The variational inequality (1.1) has been intensively considered due to its various applications in operations research, economic equilibrium and engineering design. When T has some monotonicity, many iterative methods for solving the variational inequality (1.1) have been developed. Recently, applying the generalized projection operator in uniformly convex and uniformly smooth Banach spaces, Li [1] established the convergence of a Mann type iterative scheme for variational inequalities in compact subsets of Banach spaces. Fan [2] also studied the same problem in noncompact subsets of Banach spaces and extended Li's results to some extent.

In this paper, we consider, based on the generalized projection methods, the approximate solvability of a system of nonlinear variational inequalities in the framework of Banach spaces. Our results obtained in this paper main generalize the results of Li [1], Fan [2], and Verma [3].

Let X, Y be Banach spaces, $T : D(T) \subset X \to Y$, the operator T is said to be compact if it is continuous and maps the bounded subsets of $D(T)$ onto the relatively compact subsets of Y; the operator T is said to be weak to norm continuous if it is continuous from the weak topology of X to the strong topology of Y . We denote by $J : B \to 2^{B^*}$ the normalized duality mapping from B to 2^{B^*}, defined by

$$J(x) := \{y \in B^* : \langle y, x \rangle = \|y\|^2 = \|x\|^2\}, \quad \forall x \in B.$$

It is well known that J is single-valued and norm to weak* continuous if B is smooth. In [4], Alber introduced the functional $V : B^* \times B \to \mathbb{R}$ defined by

$$V(\varphi, x) = \|\varphi\|^2 - 2\langle \varphi, x \rangle + \|x\|^2,$$

where $\varphi \in B^*$ and $x \in B$. It is easy to see that

$$(\|\varphi\| - \|x\|)^2 \leq V(\varphi, x) \leq (\|\varphi\| + \|x\|)^2.$$

Thus the functional $V : B^* \times B \to R^+$ is nonnegative.

Definition 1.1 [5]. If B be a uniformly convex and uniformly smooth Banach space, the generalized projection $\pi_K : B^* \to K$ is a mapping that assigns an arbitrary point $\varphi \in B^*$ to the minimum point of the functional $V(\varphi, x)$, i.e., a solution to the minimization problem

$$V(\varphi, \pi_K \varphi) = \inf_{y \in K} V(\varphi, y). \tag{1.2}$$

It was proved that the generalized projection operator $\pi_K : B^* \to K$ is continuous, if B is a reflexive, strictly convex and smooth Banach space. In what follows, we let B be a uniformly convex and uniformly smooth Banach space, unless otherwise specified. Then B is reflexive and strictly convex. Thus the generalized projection operator $\pi_K : B^* \to K$ is continuous. The functional $V_2 : B \times B \to \mathbb{R}$ is defined by

$$V_2(x, y) = V(Jx, y), \forall x, y \in B.$$

The following properties of the operators π_K, V are useful for our paper. (See, for example, [1,2,11].)

(1) $V : B^* \times B \to \mathbb{R}$ is continuous.

(2) $V(\varphi, x) = 0$ if and only if $\varphi = J(x)$.

(3) $V(J\pi_K \varphi, x) \leq V(\varphi, x)$ for all $\varphi \in B^*$ and $x \in B$.

(4) The operator π_K is J fixed in each point $x \in K$, i.e., $\pi_K(Jx) = x$.

(5) If the Banach space B is uniformly smooth, then for all $\varphi_1, \varphi_2 \in B^*$, we have

$$\|\pi_K \varphi_1 - \pi_K \varphi_2\| \leq 2R_1 g_B^{-1}(\|\varphi_1 - \varphi_2\|/R_1),$$

where $R_1 = (\|\pi_K \varphi_1\|^2 + \|\pi_K \varphi_2\|^2)^{1/2}$, and g_B^{-1} is the inverse function to g_B that is defined by the modulus of smoothness for an uniformly smooth Banach space.

(6) If B is smooth, then operator $\pi_K : B^* \to K$ is single valued and for any given $\varphi \in B^*$, $\langle \varphi - J\widetilde{\varphi}, \widetilde{\varphi} - x \rangle \geq 0, \ \forall x \in K \Leftrightarrow \widetilde{\varphi} = \pi_K \varphi$.

In 1996, Alber proved the following theorem.

Theorem A [6] *Let B be a reflexive, strictly convex and smooth Banach space with dual B^*. Let T be an arbitrary operator from Banach space B to B^*, α an arbitrary fixed positive number. Then the point $x \in K \subset B$ is a solution of variational inequality (1.1) if and only if x is a solution of the operator equation in B*

$$x = \pi_K(Jx - \alpha T x).$$

In this paper, we investigate the following system of nonlinear variational inequality (SNVI) problem:

Find $x^*, y^* \in K$ such that

$$\langle sT_1y^* + Jx^* - Jy^*, x - x^*\rangle \geq 0, \quad \forall x \in K, s > 0, \tag{1.3}$$

$$\langle tT_2x^* + Jy^* - Jx^*, x - x^*\rangle \geq 0, \quad \forall x \in K, t > 0, \tag{1.4}$$

where $T_i : K \to B^*$ are nonlinear mappings for $i = 1, 2$ and $J : B \to B^*$ is a normalized duality mapping.

One can easily see the SNVI problems (1.3) and (1.4) are equivalent to the following projection formulas:

$$x^* = \pi_K[Jy^* - sT_1y^*], \quad s > 0, \tag{1.5}$$

$$y^* = \pi_K[Jx^* - tT_2x^*], \quad t > 0, \tag{1.6}$$

Next, we consider some special classes of the SNVI problems (1.3) and (1.4) as follows:

(I) If $t = 0$ then the SNVI problems (1.3) and (1.4) collapse to the following NVI:

Find $x^* \in K$ such that

$$\langle sTx^*, x - x^*\rangle \geq 0, \quad \forall x \in K, s > 0, \tag{1.7}$$

which considered by Fan [9].

(II) If $T_1 = T_2 = T$, then the SNVI problems (1.3) and (1.4) reduce to the following nonlinear system of variational inequality problem (SNVI):

Find $x^*, y^* \in K$ such that

$$\langle sTy^* + Jx^* - Jy^*, x - x^*\rangle \geq 0, \quad \forall x \in K, s > 0, \tag{1.8}$$

$$\langle tTx^* + Jy^* - Jx^*, x - x^*\rangle \geq 0, \quad \forall x \in K, t > 0, \tag{1.9}$$

where $T : K \to B^*$ be a nonlinear mapping.

(III) If K is a closed convex cone of B, then the SNVI problems (1.3) and (1.4) are equivalent to the following system (SNC) of nonlinear complementarity problems:

Find $x^*, y^* \in K$ such that

$$T_1y^* \in K^*, \quad T_2x^* \in K^*,$$

$$\langle sT_1y^* + Jx^* - Jy^*, x^* \rangle = 0, \quad s > 0, \tag{1.10}$$

$$\langle tT_2x^* + Jy^* - Jx^*, x^* \rangle = 0, \quad t > 0, \tag{1.11}$$

where K^* is the polar cone to C defined by

$$K^* = \{f \in B : \langle f, x \rangle \geq 0, \quad \forall x \in K\}.$$

(IV) If $B = H$, a Hilbert space then the SNVI problems (1.3) and (1.4) reduce to the following nonlinear system of variational inequality problem (SNVI):

Find $x^*, y^* \in K$ such that

$$\langle sT_1y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in K, s > 0, \tag{1.12}$$

$$\langle tT_2x^* + y^* - x^*, x - x^* \rangle \geq 0, \quad \forall x \in K, t > 0, \tag{1.13}$$

where $T_i : K \to B^*$ are nonlinear mappings for $i = 1, 2$.

(V) If $B = H$ and $T_1 = T_2$, a Hilbert space then the SNVI problems (1.3) and (1.4) reduce to the following nonlinear system of variational inequality problem (SNVI):

Find $x^*, y^* \in K$ such that

$$\langle sTy^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in K, s > 0, \tag{1.14}$$

$$\langle tTx^* + y^* - x^*, x - x^* \rangle \geq 0, \quad \forall x \in K, t > 0, \tag{1.15}$$

where $T : K \to B^*$ is a nonlinear mapping.

2. Algorithms

Algorithm 2.1. For any $x_0, y_0 \in K$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative process:

$$\begin{cases} y_n = \pi_K[Jx_n - tT_2x_n], \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \pi_K[Jy_n - sT_1y_n], \end{cases} \tag{2.1}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ for all $n \geq 0$.

(I) If $t = 0$ in Algorithm 2.1, then we have the following:

Algorithm 2.2. For any $x_0 \in K$, compute the sequences $\{x_n\}$ by the iterative process:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n \pi_K[Jx_n - sTx_n], \tag{2.2}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ for all $n \geq 0$.

(II) If $T_1 = T_2 = T$ in Algorithm 2.1, then we have the following:

Algorithm 2.3. For any $x_0, y_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by the iterative process:

$$\begin{cases} y_n = \pi_K[Jx_n - tTx_n], \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \pi_K[Jy_n - sTy_n], \end{cases} \tag{2.3}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ for all $n \geq 0$.

(III) If $J = I$, the identity mapping in Algorithm 2.1, then we have the following:

Algorithm 2.4. For any $x_0, y_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by the iterative process:

$$\begin{cases} y_n = P_K[x_n - tT_2x_n], \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_K[y_n - sT_2y_n], \end{cases} \tag{2.4}$$

where P_k is a metric projection from a Hilbert space to its closed convex subset K and $\{\alpha_n\}$ is a sequence in $[0,1]$ for all $n \geq 0$.

(IV) If $J = I$ and $T_1 = T_2$, the identity mapping in Algorithm 2.1, then we have the following:

Algorithm 2.5. For any $x_0, y_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by the iterative process:

$$\begin{cases} y_n = P_K[x_n - tTx_n], \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_K[y_n - sTy_n], \end{cases} \tag{2.5}$$

where P_k is a metric projection from a Hilbert space to its closed convex subset K and $\{\alpha_n\}$ is a sequence in $[0,1]$ for all $n \geq 0$.

In order to prove our main results, we need the following lemmas and definitions.

Lemma 2.3 [7]. *Let B be a uniformly convex Banach space. Then for arbitrary $r > 0$, there exists a continuous, strictly increasing convex function $g : R^+ \to R^+, g(0) = 0$, such that for all*

x_1, x_2 and $y \in B_r(0) := \{x \in B : \|x\| \leq r\}$ and for any $\alpha \in [0,1]$, the following inequality holds:

$$V_2(\alpha x_1 + (1-\alpha)x_2, y) \leq \alpha V_2(x_1, y) + (1-\alpha)V_2(x_2, y) - \alpha(1-\alpha)g(\|x_1 - x_2\|).$$

Lemma 2.4 [8]. *Let B be a real Banach space and $J : B \to B^*$ be the normalized duality mapping, then for any $x, y \in B$ the following holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J^{-1}(x+y)\rangle, \quad \forall x, y \in B^*.$$

3. Convergence Theorems

Theorem 3.1. *Let B be an uniformly convex and uniformly smooth Banach space and let K be a closed and convex subset of B. Let T_1 and $T_2 : K \to B^*$ be mappings on K such that $J - sT_1$ and $J - tT_2$ are compact and*

$$\langle T_1 x, J^*[Jx - sT_1 x]\rangle \geq 0 \text{ and } \langle T_2 x, J^*[Jx - tT_2 x]\rangle \geq 0, \tag{3.1}$$

for all $x \in K$, where $J^* = J^{-1}$ is the normalized duality mapping on B^*. Suppose that $(x^*, y^*) \in K \times K$ is a solution to the SNVI problem (1.3)-(1.4). Assume that $\{x_n\}$ and $\{y_n\}$ are the sequences generated by Algorithm 2.1. If $\{\alpha_n\} \subset [0,1]$ satisfies condition $0 < a \leq \alpha_n \leq b < 1$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to x^* and y^*, respectively.

Proof. In view of Lemma 2.3, we have

$$\begin{aligned} V_2(x_{n+1}, u) &\leq (1-\alpha_n)V_2(x_n, u) + \alpha_n V_2(\pi_K[Jy_n - sT_1 y_n], u) \\ &\quad - \alpha_n(1-\alpha_n)g(\|\pi_K[Jy_n - sT_1 y_n] - x_n\|), \forall u \in B \end{aligned} \tag{3.2}$$

By the definition of V and V_2, we have

$$\begin{aligned} & V_2(\pi_K[Jy_n - sT_1y_n], u) \\ & \le V(Jy_n - sT_1y_n, u) \\ & = \|Jy_n - sT_1y_n\|^2 + \|u\|^2 - 2\langle Jy_n - sT_1y_n, u\rangle \\ & \le \|Jy_n\|^2 - 2s\langle T_1y_n, J^*(Jy_n - sT_1y_n)\rangle + \|u\|^2 - 2\langle Jy_n - sT_1y_n, u\rangle \\ & = V(Jy_n, u) - 2s\langle T_1y_n, J^*(Jy_n - sT_1y_n) - u\rangle \\ & = V_2(y_n, u) - 2s\langle T_1y_n, J^*(Jy_n - sT_1y_n) - u\rangle. \end{aligned} \tag{3.3}$$

In the same way, we obtain that

$$\begin{aligned} V_2(y_n, u) & = V(J\pi_K[Jx_n - tT_2x_n], u) \\ & \le V(Jx_n - tT_2x_n, u) \\ & = \|Jx_n - tT_2x_n\|^2 + \|u\|^2 - 2\langle Jx_n - tT_2x_n, u\rangle \\ & \le \|Jx_n\|^2 - 2t\langle T_2x_n, J^*(Jx_n - tT_2x_n)\rangle + \|u\|^2 - 2\langle Jx_n - tT_2x_n, u\rangle \\ & = V(Jx_n, u) - 2t\langle T_2x_n, J^*(Jx_n - tT_2x_n) - u\rangle \\ & = V_2(x_n, u) - 2t\langle T_2x_n, J^*(Jx_n - tT_2x_n) - u\rangle. \end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4) yields that

$$\begin{aligned} & V_2(\pi_K[Jy_n - sT_1y_n], u) \\ & \le V_2(x_n, u) - 2t\langle T_2x_n, J^*(Jx_n - tT_2x_n) - u\rangle - 2s\langle T_1y_n, J^*(Jy_n - sT_1y_n) - u\rangle. \end{aligned} \tag{3.5}$$

Substitute (3.5) into (3.2) yields that

$$\begin{aligned} V_2(x_{n+1}, u) \le V_2(x_n, u) & - 2\alpha_n t\langle T_2x_n, J^*(Jx_n - tT_2x_n) - u\rangle \\ & - 2\alpha_n s\langle T_1y_n, J^*(Jy_n - sT_1y_n) - u\rangle \\ & - \alpha_n(1-\alpha_n)g(\|\pi_K[Jy_n - sT_1y_n] - x_n\|). \end{aligned} \tag{3.6}$$

Since $u \in B$ is arbitrary, we, without loss any generality, choose $u = \theta$, the zero element in B. It follows from (3.6) and condition (3.1) that

$$\alpha_n(1-\alpha_n)g(\|\pi_K[Jy_n - sT_1y_n] - x_n\|) \le V_2(x_n, \theta) - V_2(x_{n+1}, \theta). \tag{3.7}$$

Noticing that $0 < a \leq \alpha_n \leq b < 1$ and taking the sum for $i = 1, 2, \ldots, n$, we arrive at

$$a(1-b)\sum_{i=1}^{n} g(\|\pi_K[Jy_i - sT_1y_i] - x_i\|) \leq V_2(x_0, \theta) - V_2(x_{n+1}, \theta).$$

It follows from $V_2 : B \times B \to R^+$ is nonnegative and $V_2(x_0, \theta) < \infty$ that $\sum_{i=1}^{\infty} g(\|\pi_K[Jy_i - sT_1y_i] - x_i\|) < \infty$. Therefore, we have $\lim_{n\to\infty} g(\|\pi_K[Jy_n - sT_1y_n] - x_n\|) = 0$. Notice the properties of g implies that

$$\lim_{n\to\infty} \|\pi_K[Jy_n - sT_1y_n] - x_n\| = 0. \tag{3.8}$$

Noticing that $V_2(x_{n+1}, \theta) \leq V_2(x_n, \theta)$, we have $\|x_{n+1}\| \leq \|x_n\|$, which implies that $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$. Since the sequence $\{y_n\}$ is bounded and $J - sT_1$ is compact on K, then the sequence $\{Jy_n - sT_1y_n\}$ must have a subsequence $\{Jy_{n_i} - sT_1y_{n_i}\}$, which converges to a point $f_1 \in B^*$. By the continuity of the projection operator π_k, we have

$$\lim_{i\to\infty} \pi_K[Jy_{n_i} - sT_1y_{n_i}] = \pi_K(f_1). \tag{3.9}$$

Let $x^* =: \pi_K(f_1)$. On the other hand, we have

$$\|x_{n_i} - x^*\| \leq \|x_{n_i} - \pi_K[Jy_{n_i} - sT_1y_{n_i}]\| + \|\pi_K[Jy_{n_i} - sT_1y_{n_i}] - x^*\|, \tag{3.10}$$

which combines with (3.8) and (3.9) yields that $\lim_{i\to\infty} x_{n_i} = x^*$. In virtue of arbitrary subsequence of $\{x_n\}$ having above property, we have

$$\lim_{n\to\infty} x_n = x^*. \tag{3.11}$$

It follows from (3.8), (3.11) and the continuity properties of the operators π_K and $J - sT_1$ that

$$x^* = \pi_K[Jy^* - sT_1y^*]. \tag{3.12}$$

Since the sequence $\{x_n\}$ is bounded and $J - tT_2$ is compact on K, then the sequence $\{Jx_n - tT_2x_n\}$ must have a subsequence $\{Jx_{n_i} - tT_2x_{n_i}\}$, which converges to a point $f_2 \in B^*$. By the continuity of the projection operator π_k, we have

$$\lim_{i\to\infty} \pi_K[Jx_{n_i} - tT_2x_{n_i}] = \pi_K(f_2). \tag{3.13}$$

Let $y^* =: \pi_K(f_2)$. On the other hand, we have $\|y_{n_i} - y^*\| \leq \|y_{n_i} - \pi_K[Jx_{n_i} - tT_2x_{n_i}]\| + \|\pi_K[Jx_{n_i} - tT_2x_{n_i}] - y^*\|$, which combines with (2.1) and (3.13) yields that $\lim_{i\to\infty} y_{n_i} = y^*$. Since $\{y_{n_i}\}$ is arbitrary, we have $\lim_{n\to\infty} y_n = y^*$. It follows from (2.1) and the continuity properties of the operators π_K and $J - tT_2$ that $y^* = \pi_K[Jx^* - tT_2x^*]$. This completes the proof.

Conflict of Interests

The author declares that there is no conflict of interests.

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REFERENCES

[1] J. Li, On the existence of solutions of variational inequalities in Banach spaces, J. Math. Anal. Appl. 295 (2004) 115-126.

[2] J.H. Fan, A Mann type iterative scheme for variational inequalities in noncompact subsets of Banach spaces, J. Math. Anal. Appl. 337 (2008) 1041-1047.

[3] R.U. Verma, Projection methods, algorithms and a new System of Nonlinear Variational Inequalities, Comput. Math. Appl. 41 (2001) 1025-1031.

[4] Ya. Alber, S. Reich, An iterative method for solving a class of nonlinear operator equations in Banach spaces, Panamer. Math. J. 4 (1994) 39-54.

[5] Ya. Alber, A. Notik, On some estimates for projection operator in Banach space, Comm. Appl. Nonlinear Anal. 2 (1995) 47-56.

[6] Ya. Alber, Metric and generalized projection operators in Banach spaces: properties and applications, in: A. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Monotonic and Accretive Type, Marcel Dekker, New York, 1996, pp. 15-50.

[7] C.E. Chidume, J.L. Li, Projection methods for approximating fixed points of Lipschitz suppressive operators, Panamer. Math. J. 15 (2005) 29-40.

[8] S.S. Chang, On Chidumes open questions and approximate solutions of multivalued strongly accretive mapping in Banach spaces, J. Math. Anal. Appl. 216 (1997) 94-111.

[9] A.Y. Al-Bayati, R.Z. Al-Kawaz, A new hybrid WC-FR conjugate gradient-algorithm with modified secant condition for unconstrained optimization, J. Math. Comput. Sci. 2 (2012) 937-966.

[10] S. Plubtieng, R. Punpaeng, A new iterative method for equilibrium problems and fixed point problems of nonexpansive mappings and monotone mappings, Appl. Math. Comput. 197 (2008) 548-558.

[11] J. Shen, L.P. Pang, An approximate bundle method for solving variational inequalities, Commn. Optim. Theory, 1 (2012) 1-18.

[12] H.S. Abdel-Salam, K. Al-Khaled, Variational iteration method for solving optimization problems, J. Math. Comput. Sci. 2 (2012) 1475-1497.

[13] H. Zegeye, N. Shahzad, Strong convergence theorem for a common point of solution of variational inequality and fixed point problem, Adv. Fixed Point Theory, 2 (2012) 374-397.

[14] B.O. Osu, O.U. Solomon, A stochastic algorithm for the valuation of financial derivatives using the hyperbolic distributional variates, Math. Fianc. Lett. 1 (2012) 43-56.

[15] Z.M. Wang, W.D. Lou, A new iterative algorithm of common solutions to quasi-variational inclusion and fixed point problems, J. Math. Comput. Sci. 3 (2013) 57-72.

[16] V.A. Khan, K. Ebadullah, I-convergent difference sequence spaces defined by a sequence of moduli, J. Math. Comput. Sci. 2 (2012) 265-273.

[17] J. Ye, J. Huang, Strong convergence theorems for fixed point problems and generalized equilibrium problems of three relatively quasi-nonexpansive mappings in Banach spaces, J. Math. Comput. Sci. 1 (2011) 1-18.