PERMANENCE FOR A DISCRETE COMPETITIVE SYSTEM WITH FEEDBACK CONTROLS

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Abstract. A nonautonomous discrete competitive system with nonlinear inter-inhibition terms and feedback controls is studied in this paper. By using difference inequality theory, a set of conditions which guarantee the permanence of system is obtained. The results indicate that feedback control variables have no influence on the persistent property of the system. Our results not only supplement but also improve some existing ones. Numerical simulations show the feasibility of our results.

Keywords: Discrete; Permanence; Competitive; Feedback Controls.

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1. Introduction

For any bounded sequence \( \{a(n)\} \), \( a^L = \inf_{n \in \mathbb{N}} \{a(n)\} \), \( a^U = \sup_{n \in \mathbb{N}} \{a(n)\} \). Recently, many authors pay attention to the following competitive system with nonlinear inter-inhibition terms (see [1-5]):

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t)\left\{r_1(t) - a_1(t)x_1(t) - \frac{c_2(t)x_2(t)}{1 + x_2(t)} \right\}, \\
\dot{x}_2(t) &= x_2(t)\left\{r_2(t) - a_2(t)x_2(t) - \frac{c_1(t)x_1(t)}{1 + x_1(t)} \right\},
\end{align*}
\]

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where \( x_i \) \((i = 1, 2)\) are the population densities of two competing species; \( r_i \) \((i = 1, 2)\) are the intrinsic growth rates of species; \( a_i \) \((i = 1, 2)\) are the rates of intraspecific competition of the first and second species, respectively; \( c_i \) \((i = 1, 2)\) are the rates of interspecific competition of the first and second species, respectively. For more ecological sense of model (1), one can see [1] and the references cited therein. By using differential inequality, the module containment theorem and the Lyapunov function, the existence and global asymptotic stability of positive almost periodic solutions of system (1) is obtained by Wang et al. [2].

As we all know that continuous models can excellently show the dynamic behaviors of those populations who have a long life cycle, overlapping generations, and large quantity; Also, the discrete-time models governed by difference equations are more appropriate than the continuous ones when populations have a short life expectancy, nonoverlapping generations in the real word. Considering discrete-time models can provide efficient computational models of continuous models for numerical simulations, Qin et al. [3] study the following system which is the discrete analogue of system (1):

\[
\begin{align*}
  x_1(n+1) &= x_1(n) \exp \left\{ r_1(n) - a_1(n)x_1(n) - \frac{c_2(n)x_2(n)}{1 + x_2(n)} \right\}, \\
  x_2(n+1) &= x_2(n) \exp \left\{ r_2(n) - a_2(n)x_2(n) - \frac{c_1(n)x_1(n)}{1 + x_1(n)} \right\},
\end{align*}
\]

they investigated the permanence and global asymptotic stability of positive periodic solutions of system (2). When all coefficients in system (2) are bounded nonnegative almost periodic sequences, Wang and Liu [4] further investigate the existence, uniqueness and uniformly asymptotic stability of positive almost periodic solution of the above almost periodic system. Qin et al. [3] obtained the following result about permanence of system (2).

**Theorem A** (see [3]). *Suppose that system (2) satisfies the following assumptions:

\[
r_1^L - c_2^U > 0, \quad r_2^L - c_1^U > 0.
\]

Then system (2) is permanent i.e. any positive solution \((x_1(n), x_2(n))^T\) of system (2) satisfies

\[
0 < x_i^* \leq \liminf_{n \to +\infty} x_i(n) \leq \limsup_{n \to +\infty} x_i(n) \leq x_i^* < +\infty.
\]
Noting that ecosystems in the real world are often disturbed by outside continuous forces, Wang et al. [5] incorporate feedback controls into model (2) and consider the following model:

\[
\begin{align*}
    x_1(n+1) &= x_1(n) \exp \left\{ r_1(n) - a_1(n)x_1(n) - \frac{c_2(n)x_2(n)}{1+x_2(n)} - e_1(n)u_1(n) \right\}, \\
    x_2(n+1) &= x_2(n) \exp \left\{ r_2(n) - a_2(n)x_2(n) - \frac{c_1(n)x_1(n)}{1+x_1(n)} - e_2(n)u_2(n) \right\}, \\
    \Delta u_1(n) &= -b_1(n)u_1(n) + d_1(n)x_1(n), \quad \Delta u_2(n) = -b_2(n)u_2(n) + d_2(n)x_2(n),
\end{align*}
\]

where \( x_i(n) \) stand for the densities of species \( x_i \) (\( i = 1, 2 \)) at the \( n \)th generation, respectively, for \( i = 1, 2 \), \( \{a_i(n)\}, \{b_i(n)\}, \{c_i(n)\}, \{d_i(n)\}, \{e_i(n)\} \) and \( \{r_i(n)\} \) are all bounded nonnegative sequences such that

\[
\begin{align*}
0 < a_i^L &\leq a_i(n) \leq a_i^U, & 0 < c_i^L &\leq c_i(n) \leq c_i^U, & 0 < d_i^L &\leq d_i(n) \leq d_i^U, \\
0 < e_i^L &\leq e(n) \leq e_i^U, & 0 < r_i^L &\leq r_i(n) \leq r_i^U, & 0 < b_i^L &\leq b_i(n) \leq b_i^U \leq 1.
\end{align*}
\]

By using Lyapunov function and some preliminary lemmas, the existence and uniformly asymptotic stability of unique positive almost periodic solution of the system (3) are investigated by Wang et al. [5]. More specifically, as for permanence, Wang et al. [5] obtained the following result.

**Theorem B** (see [5]). *If the following inequalities

\[
\begin{align*}
    r_1^L - c_2^U - e_1^U u_1^* > 0, \quad r_2^L - c_1^U - e_2^U u_2^* > 0
\end{align*}
\]

hold, then system (3) is permanent i.e. any positive solution \((x_1(n), x_2(n), u_1(n), u_2(n))^T\) of system (3) satisfies

\[
\begin{align*}
    0 \leq x_{i, \ast} &\leq \liminf_{n \to +\infty} x_i(n) \leq \limsup_{n \to +\infty} x_i(n) \leq x_{i, \ast}^* < +\infty, \\
    0 \leq u_{i, \ast} &\leq \liminf_{n \to +\infty} u_i(n) \leq \limsup_{n \to +\infty} u_i(n) \leq u_{i, \ast}^* < +\infty,
\end{align*}
\]

where \( x_{i, \ast}^* = \exp(\frac{r_i^U - 1}{a_i^L}) \) and \( u_{i, \ast}^* = \frac{x_{i, \ast}^* d_i^U}{b_i^L} \), for \( i = 1, 2 \).

Comparing with Theorem A, Theorem B shows that feedback control variables play important roles on the persistent property of the system (3). But the question is whether or not the feedback control variables influence on the permanence of the system. On the other hand, as was pointed out by Fan and Wang [6], "if we use the method of comparison theorem, then the additional condition, in some extent, is necessary. But for the system itself, this condition may
not necessary.” In [6], by establishing a new difference inequality, Fan and Wang showed that feedback control has no influence on the permanence of a single species discrete model. Their success motivated us to consider the persistent property of system (3). Indeed, in this paper, we will apply the analysis technique of Fan and Wang [6] to establish sufficient conditions, which is independent of feedback control variables, to ensure the permanence of the system. We finally obtain the following main results:

**Theorem C.** Assume that

\[ r^L_1 - c^U_2 > 0, \quad r^U_2 - c^L_1 > 0 \quad (A_3) \]

hold, then system (3) is permanent.

Comparing with Theorem B, it is easy to see that (A_3) in Theorem C are weaker than (A_2) in Theorem B and feedback control variables have no influence on the permanent property of system (3), so our results improve the main results in [5]. For more works on this direction, one could refer to [7-18] and the references cited therein.

By the biological meaning, we will focus our discussion on the positive solutions of system (3). So, we consider (3) together with the following initial conditions:

\[ x_i(0) > 0, \; u_i(0) > 0, \; i = 1, 2. \quad (5) \]

It is not difficult to see that the solutions of (3)-(5) are well defined and satisfy

\[ x_i(n) > 0, \; u_i(n) > 0, \; i = 1, 2, \; \text{for} \; n \in \mathbb{N}. \quad (6) \]

The remaining part of this paper is organized as follows. In Section 2, we will introduce several lemmas. The permanence of system (3) is then studied in Section 3. In Section 4, a suitable example together with its numerical simulations shows the feasibility of our results.

**2. Preliminaries**

In this section, we will introduce several useful lemmas.

**Lemma 2.1** (see [19]). Assume that \( \{x(n)\} \) satisfies

\[ x(n + 1) \geq x(n) \exp\{a(n) - b(n)x(n)\}, \; n \geq N_0, \]
\( \limsup_{n \to +\infty} x(n) \leq x^* \) and \( x(N_0) > 0 \), where \( a(n) \) and \( b(n) \) are non-negative sequences bounded above and below by positive constants and \( N_0 \in \mathbb{N} \). Then
\[
\liminf_{n \to +\infty} x(n) \geq \min \{ \frac{a^L}{b^U} \exp \{ a^L - b^U x^* \}, \frac{a^L}{b^U} \}.
\]

**Lemma 2.2** (see [6]). Assume that \( A > 0 \) and \( y(0) > 0 \). Suppose that
\[
y(n + 1) \leq Ay(n) + B(n), \quad n = 1, 2, \ldots.
\]
Then for any integer \( k \leq n \),
\[
y(n) \leq A^k y(n - k) + \sum_{i=0}^{k-1} A^i B(n - i - 1).
\]
Especially, if \( A < 1 \) and \( B \) is bounded above with respect to \( M \), then
\[
\limsup_{n \to +\infty} y(n) \leq \frac{M}{1 - A}.
\]

**Lemma 2.3** (see [6]). Assume that \( A > 0 \) and \( y(0) > 0 \). Suppose that
\[
y(n + 1) \geq Ay(n) + B(n), \quad n = 1, 2, \ldots.
\]
Then for any integer \( k \leq n \),
\[
y(n) \geq A^k y(n - k) + \sum_{i=0}^{k-1} A^i B(n - i - 1).
\]
Especially, if \( A < 1 \) and \( B \) is bounded above with respect to \( m^* \), then
\[
\liminf_{n \to +\infty} y(n) \geq \frac{m^*}{1 - A}.
\]

### 3. Permanence

In this section, we detail the proof of our main result by several lemmas.

**Lemma 3.1** (see [5]). Any positive solution \((x_1(n), x_2(n), u_1(n), u_2(n))^T\) of system (3) satisfies
\[
(7) \quad \limsup_{n \to +\infty} x_i(n) \leq x^*_i \quad \limsup_{n \to +\infty} u_i(n) \leq u^*_i,
\]
where $x_i^*$ and $u_i^*$ ($i = 1, 2$) are defined in Theorem B.

**Lemma 3.2** Assume

$$r_1^l - c_2^U > 0$$

(A31)

holds, then there exist two positive constants $x_{1*}$ and $u_{1*}$ such that

$$\liminf_{n \to +\infty} x_{1}(n) \geq x_{1*}, \quad \liminf_{n \to +\infty} u_{1}(n) \geq u_{1*},$$

where $x_{1*}$ and $u_{1*}$ are defined in the proof.

**Proof.** According to Lemma 3.1, for any $\varepsilon > 0$ small enough, there exists enough large $N_1 > 0$, such that for $n \geq N_1$,

$$x_{1}(n) \leq x_{1}^* + \varepsilon, \quad u_{1}(n) \leq u_{1}^* + \varepsilon. \tag{8}$$

Thus, it follows from (8) and the first equation of system (3) that

$$x_{1}(n+1) \geq x_{1}(n)\exp\left\{r_1^l - d_1^U(x_{1}^* + \varepsilon) - c_2^U - e_1^U(u_{1}^* + \varepsilon)\right\} \tag{9}$$

$$\geq x_{1}(n)\exp\left\{-d_1^U(x_{1}^* + \varepsilon) - c_2^U - e_1^U(u_{1}^* + \varepsilon)\right\} \triangleq x_{1}(n)\exp\{D_{1}\varepsilon\}$$

for $n \geq N_1$, where $D_{1}\varepsilon = -d_1^U(x_{1}^* + \varepsilon) - c_2^U - e_1^U(u_{1}^* + \varepsilon) < 0$. For any integer $k \leq n$, it follows from (9) that

$$\prod_{j=n-k}^{n-1} x_{1}(j+1) / x_{1}(j) \geq \prod_{j=n-k}^{n-1} \exp\{D_{1}\varepsilon\} = \exp\{D_{1}\varepsilon \cdot k\}.$$ 

Thus

$$x_{1}(n-k) \leq x_{1}(n)\exp\{-D_{1}\varepsilon \cdot k\} \tag{10}$$

From the third equation of system (3), we have

$$u_{1}(n+1) = (1 - b_1(n))u_{1}(n) + d_1(n)x_{1}(n) \tag{11}$$

$$\leq (1 - b_1^l)u_{1}(n) + d_1^l x_{1}(n) \triangleq A_1u_{1}(n) + B_1(n),$$
where \( A_1 = 1 - b_1^f \) and \( B_1(n) = d_1^U x_1(n) \). Then, for any \( k \leq n \), according to Lemma 2.2, (10) and (11) that

\[
\begin{align*}
  u_1(n) &\leq A_1^k u_1(n-k) + \sum_{i=0}^{k-1} A_1^i B_1(n-i-1) \\
&= A_1^k u_1(n-k) + \sum_{i=0}^{k-1} A_1^i d_1^U x_1(n-i-1) \\
&\leq A_1^k u_1(n-k) + d_1^U x_1(n) \sum_{i=0}^{k-1} A_1^i \exp\{-D_1 \varepsilon (i+1)\}. 
\end{align*}
\]

(12)

Note that \( 0 < b_1^f < 1 \), hence \( 0 < A_1 < 1 \). Therefore,

\[
0 \leq A_1^k u_1(n-k) \leq A_1^k (u_1^* + \varepsilon) \to 0, \text{ as } k \to \infty. 
\]

(13)

Then, there exists a positive integer \( N_2 > N_1 \) such that for any positive solution of system (3),

\[
e_1^U A_1^{N_2} (u_1^* + \varepsilon) < \frac{1}{2} \left( r_1^f - c_2^U \right) \quad \text{for all } n \geq N_2. 
\]

In fact, we could choose \( N_2 = \max\{1, \frac{\ln P_1}{\ln A_1} + 1\} \), where \( P_1 = \frac{r_1^f - c_2^U}{2e_1^U (u_1^* + \varepsilon)} \). Fix \( N_2 \), for \( n \geq N_1 + N_2 \), we get

\[
\begin{align*}
  u_1(n) &\leq A_1^{N_2} u_1(n-N_2) + d_1^U x_1(n) \sum_{i=0}^{N_2-1} A_1^i \exp\{-D_1 \varepsilon (i+1)\} \\
&\leq A_1^{N_2} (u_1^* + \varepsilon) + d_1^U x_1(n) \sum_{i=0}^{N_2-1} A_1^i \exp\{-D_1 \varepsilon (i+1)\} \\
&\triangleq A_1^{N_2} (u_1^* + \varepsilon) + G_{1\varepsilon} x_1(n),
\end{align*}
\]

(14)

where \( G_{1\varepsilon} = d_1^U \sum_{i=0}^{N_2-1} A_1^i \exp\{-D_1 \varepsilon (i+1)\} \). Substituting (14) into the first equation of system (3), we can get

\[
\begin{align*}
x_1(n+1) &\geq x_1(n) \exp\left\{ r_1^f - d_1^U x_1(n) - c_2^U - e_1^U u_1(n) \right\} \\
&\geq x_1(n) \exp\left\{ r_1^f - d_1^U x_1(n) - c_2^U - e_1^U (A_1^{N_2} (u_1^* + \varepsilon) + G_{1\varepsilon} x_1(n)) \right\} \\
&= x_1(n) \exp\left\{ r_1^f - c_2^U - e_1^U A_1^{N_2} (u_1^* + \varepsilon) - (d_1^U + e_1^U G_{1\varepsilon}) x_1(n) \right\} \\
&\geq x_1(n) \exp\left\{ -\frac{1}{2} (r_1^f - c_2^U) - (d_1^U + e_1^U G_{1\varepsilon}) x_1(n) \right\} \\
&\triangleq x(n) \exp\left\{ E_1 - E_{2\varepsilon} x_1(n) \right\},
\end{align*}
\]

(15)

where \( E_1 = \frac{1}{2} (r_1^f - c_2^U) \) and \( E_{2\varepsilon} = d_1^U + e_1^U G_{1\varepsilon} \). By applying Lemma 2.1 to (15), it immediately follows that

\[
\liminf_{n \to +\infty} x_1(n) \geq \min\left\{ \frac{E_1}{E_{2\varepsilon}} \exp\{E_1 - E_{2\varepsilon} x_1^*\}, \frac{E_1}{E_{2\varepsilon}} \right\}. 
\]
Setting $\varepsilon \to 0$ in the above inequality, we obtain

$$\liminf_{n \to +\infty} x_1(n) \geq \min\left\{ \frac{E_1}{E_2} \exp\left\{ E_1 - E_2 x_1^* \right\}, \frac{E_1}{E_2} \right\} = x_1^*.$$  

(16)

It follows from (16) that there exists large enough $N_3 \geq N_1 + N_2$ such that

$$x_1(n) \geq \frac{x_1^*}{2}, \text{ for all } n \geq N_3.$$  

(17)

This together with the third equation of system (3) leads to

$$\Delta u_1(n) \geq -b_1(n)u_1(n) + \frac{x_1^* d_1(n)}{2}, \text{ for all } n \geq N_3.$$  

Hence,

$$u_1(n + 1) \geq (1 - b_1^U)u_1(n) + \frac{x_1^* d_1^L}{2}, \text{ for all } n \geq N_3.$$  

(18)

By applying Lemma 2.3, it follows from (18) that

$$\liminf_{n \to +\infty} u_1(n) \geq \frac{d_1^L x_1^*}{2b_1^U} \geq u_1^*.$$  

This completes the proof of Lemma 3.2.

**Lemma 3.3** Assume

$$r_2^L - c_1^U > 0 \quad (A_{32})$$

holds, then there exist two positive constants $x_2^*$ and $u_2^*$ such that

$$\liminf_{n \to +\infty} x_2(n) \geq x_2^*, \liminf_{n \to +\infty} u_2(n) \geq u_2^*,$$

where $x_2^*$ and $u_2^*$ are defined in the proof.

**Proof.** The proof of Lemma 3.3 is similar to that of Lemma 3.2. So we omit the detail here.

Lemmas 3.1-3.3 show that the conclusion of Theorem C holds.

4. Example and numeric simulation
In this section, we give the following example to verify the feasibilities of Theorem C:

\[
\begin{cases}
  x_1(n+1) = x_1(n) \exp \left\{ 2.5 + 0.5 \sin(\sqrt{7}n) - (1.3 + 0.2 \cos n)x_1(n) \ight. \\
  \left. - \frac{(0.75 + 0.25 \sin(\sqrt{11}n))x_2(n)}{1 + x_2(n)} - (0.9 + 0.1 \cos(\sqrt{3}n))u_1(n) \right\}, \\
  x_2(n+1) = x_2(n) \exp \left\{ 2.8 - (2.2 + 0.2 \sin n)x_2(n) \ight. \\
  \left. - \frac{(0.5 + 0.25 \cos(\sqrt{13}n))x_1(n)}{1 + x_1(n)} - (1 + 0.5 \sin n)u_1(n) \right\}, \\
  \Delta u_1(n) = -(0.08 + 0.02 \sin(\sqrt{2}n))u_1(n) + (0.6 + 0.4 \cos(\sqrt{7}n))x_1(n), \\
  \Delta u_2(n) = -(0.73 + 0.03 \cos(\sqrt{5}n))u_2(n) + (0.8 + 0.2 \sin(n))x_2(n),
\end{cases}
\]

In this case, we have

\[
(20) \quad r_1^L - c_2^U = 1 > 0, \quad r_2^L - c_1^U = 2.05 > 0
\]

(4.2) shows that \((A_3)\) holds, so the system (19) is permanent according to Theorem C. Our numerical simulation supports our result (see Fig. 1). However,

\[
(21) \quad r_1^L - c_2^U - e_1^U u_1^* \approx -110.9554 < 0, \quad r_2^L - c_1^U - e_2^U u_2^* \approx -2.7518 < 0,
\]

that is to say \((A_2)\) does not hold and we could not obtain the result of the permanence from Theorem B. Thus our results improve the main results in [5].

**Conflict of Interests**

The author declares that there is no conflict of interests.
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REFERENCES


