Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2017, 2017:10
ISSN: 2052-2541

SPATIAL COMPETITION MATHEMATICAL MODEL ANALYSIS FOR THE INVASION, REMOVAL OF KAPPAPHYCUS ALGAE IN GULF OF MANNAR WITH PROPAGATION DELAYS

D. PANDIARAJA[1,2], N. ARUN NAGENDRAN[1,3], D. MURUGESWARI[2], VISHNU NARAYAN MISHRA[4,*]

[1]National Centre of Excellence, Statistical and Mathematical Modelling on Bio-Resources Management, Thiagarajar College, Madurai, Tamil Nadu, India

[2]Department of Mathematics, Thiagarajar College, Madurai, Tamil Nadu, India

[3]Department of Zoology, Thiagarajar College, Madurai, Tamil Nadu, India

[4]Department of Applied Mathematics and Humanities, Sardar Vallabhai National Institute of Technology, Ichchhanath Mahadev Road, Surat-395 007, Gujarat, India

Communicated by Y. Pan

Abstract. The aim of this paper is to study the impact of manual removal of Kappaphycus Alvarezii (KA) from the corals in the region of Gulf of Mannar (GoM) and the spatial competition between KA and Native Algae (NA) on coral with propagation delays of KA and NA. A non-linear system of ordinary differential equation (ODE) model has been developed to study the linear stability and the impact of delay terms in the stability at an equilibrium point.

Keywords: delay differential equation; stability analysis.

2010 AMS Subject Classification: 34K20, 92D40.

---

*Corresponding author

E-mail address: vishnunarayanmishra@gmail.com

Received January 24, 2017

1. Introduction

The community structure of sedentary organisms is largely controlled by the outcome of direct competition for space. Reefs are one such areas of intense competition between sessile benthic organisms [25]. Sufficient access to space and light is crucial for survival on the reef. The ability to establish, maintain and extend territory (i.e., to outcompete fellow benthic organisms) can affect the composition, size, and distribution of organisms on the benthos [11]. In reefs, competition between corals and benthic algae is important in determining the fundamental structure and during coral-algae phase shifts [13]. The appearance of corals on substrate previously occupied by corals is often interpreted as evidence that algae actively compete with corals for space. Along with native algae, the exotic algae such as KA also compete for space. The faster growth rate makes it able to easily overgrow and outcompete NA and coral species. Understanding the factors defining competitive outcomes among neighbors is thus critical for predicting large-scale changes, such as transitions to alternate states within coral reefs.

Lotka-Volterra systems of differential equations have played a significant role in the development of mathematical ecology. Lotka-Volterra models lack the complexity and realism of more recently developed population models, but their generality makes them a convenient starting point for analyzing ecological systems. Lotka-Volterra systems have assisted as a basis for the development of more realistic models that entangle ratio-dependent functional responses.

Mathematical models have been developed to analyze the three way compettition between sponges, macroalgae and coral to determine the reproduction of benthic competitors [23], parrotfish exploitation [3], sufficiently small levels of fishing mortality [4], grazing intensity [2] and the role of sponge competition on coral reef alternative steady states [24].

We reported the shifting of algal dominated reef ecosystem due to the invasion of KA in Gulf of Mannar [5]. Subsequently, the dominance of KA over NA and corals in competing for space has also been reported. To simulate the three way competition among corals, KA and NA, we propose a model as a system of non-linear ODE's. Let x, y and z represent the percent cover of coral, KA and NA respectively and it is assumed that $x+y+z = 1$,

$$\frac{dx}{dt} = rx(1-(x+y+z)) - a_1xy - a_2xz \tag{1}$$

$$\frac{dy}{dt} = a_1yx + a_3yz + \nu y(1-(x+y+z)) \tag{2}$$

$$\frac{dz}{dt} = a_2zx + hz(1-(x+y+z)) - a_3zy \tag{3}$$

where x - Coral growth with respect to time (t),

y - KA growth with respect to time (t),

z - NA growth with respect to time (t),

1-(x+y+z) - Turf growth with respect to time (t),

r - coral growth on turf $(cm^2/cm^2/15d)$,

a_1 - KA growth on coral $(cm^2/cm^2/15d)$,

a_2 - NA growth on coral $(cm^2/cm^2/15d)$,

a_3 - KA growth on NA $(cm^2/cm^2/15d)$,

ν - KA growth on turf $(cm^2/cm^2/15d)$,

h - NA growth on turf $(cm^2/cm^2/15d)$.

The manual removal of KA from the coral reefs has been reported [5], we have incorporated the manual removal rate term d of KA into the above model.

$$\frac{dx}{dt} = rx - rx^2 - rxy - rxz - a_1xy - a_2xz + dy \tag{4}$$

$$\frac{dy}{dt} = a_1yx + a_3yz + \nu y - \nu yx - \nu y^2 - \nu yz - dy \tag{5}$$

$$\frac{dz}{dt} = a_2zx + hz - hxz - hzy - hz^2 - a_3zy \tag{6}$$

where d is the manual removal rate of KA.

KA sexual reproduction by spores in the Gulf of Mannar Marine Biosphere Reserve (GoM) in future, when environmental conditions unanimously favor this alga has been deliberated [6]. NA can produce sexually and asexually by forming flagellate and sometimes non-flagellate spores. The vegetative propagation is achieved through fragmentation has been reported [12]. To analyze the impact of delays in propagation of KA and NA through vegetation/spores on coral-algae interactions, we propose the following delay model:

$$\frac{dx}{dt} = rx - rx^2 - rxy - rxz - a_1xy - a_2xz + dy \tag{7}$$

$$\frac{dy}{dt} = a_1yx + a_3yz + \nu y(t-\tau_1) - \nu yx - \nu y^2 - \nu yz - dy \tag{8}$$

$$\frac{dz}{dt} = a_2zx + hz(t-\tau_2) - hxz - hzy - hz^2 - a_3zy \tag{9}$$

where τ_1 - time delay of propagation of KA through vegetation/spores, τ_2 - time delay of propagation of NA through vegetation/spores.

Stability analysis of the model for both non-delay and delay systems is investigated by Sandip Banarjee et.al., [9]. Some explicit formulae for determining the stability and direction of Hopf bifurcation periodic solution bifurcating from Hopf bifurcation are obtained [10]. The local stability of models involving delay dependent parameters has been detailed [15]. Periodic oscillations in leukopoiesis models with two delays has been reported [17]. Stability analysis of delay differential equation (DDE) model with two discrete delays has been investigated [19, 20]. Analysis of the bifurcation due to the introduction of the delay term can be reduced to finding whether a related polynomial equation has positive real roots has been discussed [21]. A decomposition technique to investigate the stability of some exponential polynomials, that is, to find conditions under which all zeros of the exponential polynomials have negative real parts has been reported [22]. Stability analysis of a multiteam prey-predator model and two-prey one-predator model has been discussed [1,16]. A delay system is absolutely stable if it is asymptotically stable for all values of the delays has been reported [27]. Several studies on stability analysis of HIV model with delay terms have been documented by many authors [7,8,14,18,26].

In section 2, we examined about equilibria of the system and the existence of the equilibria. Section 3 shows that the existence of stability analysis at different equilibria and also we have proved the existence and uniqueness theorem for the equilibria which capture stability. In section 4, stability analysis of the system of Eqs.(7)-(9) have been discussed at $E_3(0,y^*,0)$. In section 5, numerical simulations have been executed. The paper ends with a conclusion.

2. Equilibria of the system

In this section we discussed about the existence of equilibria of the system of Eqs.(4)-(6). The system has the following equilibria:

(i) the trivial equilibrium E_0(0,0,0)

(ii) the coral and KA free equilibrium E_1(0,0,1)

(iii) the KA and NA free equilibrium E_2(1,0,0)

(iv) the coral and NA free equilibrium $E_3(0,y^*,0)$

(v) the coral free equilibrium $E_4(0,y_1,z_1)$

(vi) the NA free equilibrium $E_5(x_2,y_2,0)$

Existence of equilibria:

1. For the equilibrium $E_3(0,y^*,0)$, y^* is given by $y^*=\frac{\upsilon-d}{\upsilon}$. If $d<\upsilon$, then $E_3(0,y^*,0)$ exists.

2. For $E_4(0,y_1,z_1)$, y_1 and z_1 are given by $y_1=\frac{h(a_3-d)}{\upsilon h+(h+a_3)(a_3-\upsilon)}$,
$z_1=1-\frac{(h+a_3)(a_3-d)}{\upsilon h+(h+a_3)(a_3-\upsilon)}$. If $d<a_3$ and $\frac{(h+a_3)(a_3-d)}{\upsilon h+(h+a_3)(a_3-\upsilon)}<1$, then $E_4(0,y_1,z_1)$ exists.

3. For $E_5(x_2,y_2,0)$, x_2 and y_2 is given by

$$x_2=\frac{[-d(a_1-\upsilon)+\upsilon a_1-dr-da_1]\pm\sqrt{[d(a_1-\upsilon)-\upsilon a_1+dr+da_1]^2+4[r\upsilon+r(a_1-\upsilon)+a_1(a_1-\upsilon)](\upsilon d-d^2)}}{2[-r\upsilon-r(a_1-\upsilon)-a_1(a_1-\upsilon)]},$$

$y_2=\frac{1}{\upsilon}[(a_1-\upsilon)\{\frac{-d(a_1-\upsilon)+\upsilon a_1-dr-da_1\pm\sqrt{[d(a_1-\upsilon)-\upsilon a_1+dr+da_1]^2+4[r\upsilon+r(a_1-\upsilon)+a_1(a_1-\upsilon)](\upsilon d-d^2)}}{2[-r\upsilon-r(a_1-\upsilon)-a_1(a_1-\upsilon)]}\}+\upsilon-d]$. E_5 exists if x_2 and y_2 values are positive.

3. Existence of stability analysis at different equilibrium points

Theorem 3.1. *The system of Eqs. (4)-(6) is always unstable around the trivial equilibrium E_0(0,0,0).*

Proof. The Jacobian matrix of the system of Eqs. (4)-(6) is given by,

$$\begin{vmatrix} r-2rx-ry-rz-a_1y-a_2z & -rx-a_1x+d & -rx-a_2x \\ a_1y-\nu y & a_1x+a_3z+\nu e^{-\lambda\tau_1}-\nu x-2\nu y-\nu z-d & a_3y-\nu y \\ a_2z-hz & -hz-a_3z & a_2x+he^{-\lambda\tau_2}-hx-hy-2hz-a_3y \end{vmatrix}$$

The required characteristic equation at E_0(0,0,0) is given by $(r-\lambda)(\nu-d-\lambda)(h-\lambda)$=0. The roots of the characteristic equation is r, $\upsilon-d$ and h which has two positive eigen values. So, the system is unstable around the trivial equilibrium E_0(0,0,0).

Theorem 3.2. *The system of Eqs. (4)-(6) is unstable around the equilibrium E_2(1,0,0).*

Proof. The roots of the characteristic equation of the system of Eqs. (4)-(6) at E_2(1,0,0) are -r, $a_1 - d$ and a_2 which has one positive eigen value. So, the system is unstable around the equilibrium E_2(1,0,0).

Theorem 3.3. (Existence and Uniqueness) *The system of Eqs.(4)-(6) is stable around the equilibria E_1, E_4 and E_5 and it has a unique solution under the conditions given in the proof.*

Proof. The proof of the theorem is from the following lemma.

Lemma 3.3.1. The system of Eqs. (4)-(6) is locally asymptotically stable around the equilibrium E_1(0,0,1) if $d > a_3$.

Proof. The roots of the characteristic equation of the system of Eqs. (4)-(6) at E_1(0,0,1) are $-a_2$, $a_3 - d$ and -h. All the eigen values are negative if $d > a_3$. So, the system is stable around the equilibrium E_1(0,0,1).

Lemma 3.3.2. *The system of Eqs. (4)-(6) is locally asymptotically stable around the equilibrium $E_4(0, y_1, z_1)$ if $A_1 > 0$, $C_1 > 0$ and $A_1B_1 - C_1 > 0$.*

Proof. The roots of the characteristic equation of the system of Eqs. (4)-(6) at $E_4(0, y_1, z_1)$ are given by, $\lambda^3 + A_1\lambda^2 + B_1\lambda + C_1 = 0$ where

$A_1 = -[(h - hy_1 - 2hz_1 - a_3y_1) + (a_3z_1 + \upsilon - 2\upsilon y_1 - \upsilon z_1 - d) + (r - ry_1 - rz_1 - a_1y_1 - a_2z_1)]$,

$B_1 = (h - hy_1 - 2hz_1 - a_3y_1)(r - ry_1 - rz_1 - a_1y_1 - a_2z_1) + (a_3z_1 + \upsilon - 2\upsilon y_1 - \upsilon z_1 - d)[(h - hy_1 - 2hz_1 - a_3y_1) + (r - ry_1 - rz_1 - a_1y_1 - a_2z_1)] - (a_3y_1 - \upsilon y_1)(-hz_1 - a_3z_1) - d(a_1y_1 - \upsilon y_1)$

$C_1 = (r - ry_1 - rz_1 - a_1y_1 - a_2z_1)(a_3z_1 + \upsilon - 2\upsilon y_1 - \upsilon z_1 - d)(h - hy_1 - 2hz_1 - a_3y_1) - (a_3y_1 - \upsilon y_1)(-hz_1 - a_3z_1)(r - ry_1 - rz_1 - a_1y_1 - a_2z_1) - d[(a_1y_1 - \upsilon y_1)(h - hy_1 - 2hz_1 - a_3y_1) - (a_3y_1 - \upsilon y_1)(a_2z_1 - hz_1)]$

So, the system of Eqs.(4)-(6) will be asymptotically stable in the neighbourhood of $E_4(0, y_1, z_1)$ if $A_1 > 0$, $C_1 > 0$ and $A_1B_1 - C_1 > 0$.

Lemma 3.3.3 *The system of Eqs. (4)-(6) is locally asymptotically stable around the equilibrium $E_5(x_2, y_2, 0)$ if $A_2 > 0$, $C_2 > 0$ and $A_2B_2 - C_2 > 0$.*

Proof. The roots of the characteristic equation of the system of Eqs. (4)-(6) at $E_5(x_2, y_2, 0)$ are given by, $\lambda^3 + A_2\lambda^2 + B_2\lambda + C_2 = 0$ where

$A_2 = -[a_2x_2 + h - hx_2 - hy_2 - a_3y_2 + a_1x_2 + \upsilon - \upsilon x_2 - 2\upsilon y_2 - d + r - 2rx_2 - ry_2 - a_1y_2]$,

$B_2 = (a_2x_2 + h - hx_2 - hy_2 - a_3y_2)[(a_1x_2 + \upsilon - \upsilon x_2 - 2\upsilon y_2 - d) + (r - 2rx_2 - ry_2 - a_1y_2)] + (r - 2rx_2 - ry_2 - a_1y_2)(a_1x_2 + \upsilon - \upsilon x_2 - 2\upsilon y_2 - d) + (-rx_2 - a_1x_2 + d)(a_1y_2 - \upsilon y_2)$,

$C_2 = (a_2x_2 + h - hx_2 - hy_2 - a_3y_2)[(r - 2rx_2 - ry_2 - a_1y_2)(a_1x_2 + \upsilon - \upsilon x_2 - 2\upsilon y_2 - d) - (-rx_2 - a_1x_2 + d)(a_1y_2 - \upsilon y_2)]$

So, the system of Eqs.(4)-(6) will be asymptotically stable in the neighbourhood of $E_5(x_2, y_2, 0)$ if $A_2 > 0$, $C_2 > 0$ and $A_2B_2 - C_2 > 0$.

Hence the required result is proved from the above Lemma 3.3.1, 3.3.2 and 3.3.3.

4. Stability analysis at $E_3(0, y^*, 0)$

Lemma 4.1. *Solutions of system of Eqs.(7)-(9) with initial conditions x(t), y(t) and z(t) > 0 for* $t \geq 0$ *are positive.*

We now focus on the stability of the system of Eqs.(7)-(9).

The Jacobian matrix is given by,

$$\begin{vmatrix} r-2rx-ry-rz-a_1y-a_2z & -rx-a_1x+d & -rx-a_2x \\ a_1y-\nu y & a_1x+a_3z+\nu e^{-\lambda\tau_1}-\nu x-2\nu y-\nu z-d & a_3y-\nu y \\ a_2z-hz & -hz-a_3z & a_2x+he^{-\lambda\tau_2}-hx-hy-2hz-a_3y \end{vmatrix}$$

The required characteristic equation at the equilibrium $E_3(0, y^*, 0)$is,

$$\begin{vmatrix} r-ry^*-a_1y^*-\lambda & d & 0 \\ a_1y^*-\nu y^* & \nu e^{-\lambda\tau_1}-2\nu y^*-d-\lambda & a_3y^*-\nu y^* \\ 0 & 0 & he^{-\lambda\tau_2}-hy^*-a_3y^*-\lambda \end{vmatrix} = 0$$

which gives,

$$\lambda^3 + A\lambda^2 + B\lambda + C + e^{-\lambda\tau_1}[D_1\lambda^2 + E_1\lambda + F_1] + e^{-\lambda\tau_2}[D_2\lambda^2 + E_2\lambda + F_2] + e^{-\lambda(\tau_1+\tau_2)}[G\lambda + H] = 0 \tag{10}$$

where A = $2\nu y^* + d + hy^* + a_3y^* - r + ry^* + a_1y^*$,

B = $2\nu hy^{*2} + 2\nu a_3y^{*2} + dhy^* + da_3y^* - da_1y^* + d\nu y^* - [(r - ry^* - a_1y^*)(2\nu y^* + d + hy^* + a_3y^*)]$,

C = $(ry^* + a_1y^* - r)(2\nu hy^{*2} + 2\nu a_3y^{*2} + dhy^* + da_3y^*) - da_1hy^{*2} - da_1a_3y^{*2} + d\nu hy^{*2} + d\nu a_3y^{*2}$,

$D_1 = -\nu$,

$E_1 = \nu(r - ry^* - a_1y^*) - \nu hy^* - a_3\nu y^*$,

$F_1 = (r - ry^* - a_1y^*)(\nu hy^* + a_3y^*\nu)$,

$D_2 = -h, \quad E_2 = hr - hry^* - ha_1y^* - 2\nu hy^* - dh$,

$F_2 = (r - ry^* - a_1y^*)(2\nu hy^* + dh) + d\nu hy^* - da_1hy^*$,

$G = \nu h, \quad H = (ry^* + a_1y^* - r)\nu h.$

Case(i): $\tau_1 = 0, \ \tau_2 = 0$

In this case the characteristic equation (10) reduces to,

$$\lambda^3 + \lambda^2(A + D_1 + D_2) + \lambda(B + E_1 + E_2 + G) + (C + F_1 + F_2 + H) = 0 \tag{11}$$

Theorem 4.1. *Assume that (as1)* $(B + E_1 + E_2 + G) > 0, (C + F_1 + F_2 + H) > 0$ *and* $(A + D_1 + D_2)(B + E_1 + E_2 + G) - (C + F_1 + F_2 + H) > 0$, *then the system (7-9) without delay will be locally asymptotically stable around* $E_3(0, y^*, 0)$.

Case(ii): $\tau_1 = 0, \ \tau_2 > 0$

In this case the characteristic equation (10) reduces to,

$$\begin{aligned} &\lambda^3 + \lambda^2(A + D_1) + \lambda(B + E_1) + (C + F_1) + e^{-\lambda\tau_2}(D_2\lambda^2 + (E_2 + G)\lambda \\ &\qquad + (F_2 + H)) = 0 \end{aligned} \tag{12}$$

Let $i\omega(\omega > 0)$ be a root of Eq.(12).

$$\begin{aligned} &-i\omega^3 - \omega^2(A + D_1) + i\omega(B + E_1) + (C + F_1) + (-D_2\omega^2 + (E_2 + G)i\omega \\ &\qquad + (F_2 + H))(\cos\ \omega\tau_2 - i\sin\ \omega\tau_2) = 0 \end{aligned} \tag{13}$$

Separating the real and imaginary parts we get,

$$(E_2 + G)\omega\sin\ \omega\tau_2 + (F_2 + H - D_2\omega^2)\cos\ \omega\tau_2 = \omega^2(A + D_1) - (C + F_1) \tag{14}$$

$$(E_2 + G)\omega\cos\ \omega\tau_2 - (F_2 + H - D_2\omega^2)\sin\ \omega\tau_2 = \omega^3 - \omega(B + E_1) \tag{15}$$

Squaring and adding (14) and (15),

$$\begin{aligned} &\omega^6 + \omega^4[(A + D_1)^2 - 2(B + E_1) - D_2^2] + \omega^2[(B + E_1)^2 - 2(A + D_1)(C + F_1) \\ &\qquad + 2(F_2 + H)D_2 - (E_2 + G)^2] + (C + F_1)^2 - (F_2 + H)^2 = 0 \end{aligned} \tag{16}$$

It follows that if, $(A+D_1)^2-2(B+E_1)-D_2^2>0$, $(B+E_1)^2-2(A+D_1)(C+F_1)+2(F_2+H)D_2-(E_2+G)^2>0$ and (as2) $(C+F_1)^2-(F_2+H)^2>0$ hold then Eq.(16) has no positive roots.

Hence all roots of Eq.(12) have negative real parts when $\tau_2\in[0,\infty)$ under (as1) and (as2).

If (as1) and (as3) $(C+F_1)^2-(F_2+H)^2<0$ hold, then Eq.(16) has a unique positive root ω_0^2. Substitute ω_0^2 in Eq.(14) and Eq.(15) we have,

$$\begin{aligned}\tau_{2n} &= \frac{1}{\omega_0}cos^{-1}\{\frac{\omega^4(E_2+G)-\omega^2(B+E_1)(E_2+G)}{\omega^2(E_2+G)^2+(F_2+H-D_2\omega^2)^2}\\ &+ \frac{(F_2+H-D_2\omega^2)[\omega^2(A+D_1)-(C+F_1)]}{\omega^2(E_2+G)^2+(F_2+H-D_2\omega^2)^2}\}+\frac{2n\pi}{\omega_0},\ n=0,1,2...\end{aligned}$$

If (as1) and (as4) $(A+D_1)^2-2(B+E_1)-D_2^2>0$, $(B+E_1)^2-2(A+D_1)(C+F_1)+2(F_2+H)D_2-(E_2+G)^2>0$, $(C+F_1)^2-(F_2+H)^2>0$ and $[(B+E_1)^2-2(A+D_1)(C+F_1)+2(F_2+H)D_2-(E_2+G)^2]^2>4[(C+F_1)^2-(F_2+H)^2]$ hold then Eq.(16) has two positive roots ω_+^2,ω_-^2. Substitute $\omega_\pm^2$ into Eq.(12) gives,

$$\begin{aligned}\tau_{2k}^{\pm} &= \frac{1}{\omega_\pm}cos^{-1}\{\frac{\omega_\pm^4(E_2+G)-\omega_\pm^2(B+E_1)(E_2+G)}{\omega_\pm^2(E_2+G)^2+(F_2+H-D_2\omega_\pm^2)^2}\\ &+ \frac{(F_2+H-D_2\omega_\pm^2)[\omega_\pm^2(A+D_1)-(C+F_1)]}{\omega_\pm^2(E_2+G)^2+(F_2+H-D_2\omega_\pm^2)^2}\}+\frac{2k\pi}{\omega_\pm},\ k=0,1,2...\end{aligned}$$

Let $\lambda(\tau_2)$ be the root of Eq.(12) satisfying $Re\lambda(\tau_{2n})=0$ $(rep.Re\lambda(\tau_{2k}^{\pm})=0)$ and $Im\lambda(\tau_{2n})=\omega_0$ $(rep.Im\lambda(\tau_{2k}^{\pm})=\omega_\pm)$.

We can obtain that,

$$\left[\frac{d}{d\tau_2}Re(\lambda)\right]_{\tau_2=\tau_{2_0},\omega=\omega_0}>0,\left[\frac{d}{d\tau_2}Re(\lambda)\right]_{\tau_2=\tau_{2k}^{+},\omega=\omega_+}>0,\left[\frac{d}{d\tau_2}Re(\lambda)\right]_{\tau_2=\tau_{2k}^{-},\omega=\omega_-}<0$$

Lemma 4.2. *For $\tau_1=0$, assume that (as1) is satisfied. Then, the following holds.*

(i) If (as2) holds, then the equilibrium E(0, y^*,0) is asymptotically stable for all $\tau_2\geq 0$.

(ii) If (as3) holds, then the equilibrium E(0, y^*,0) is asymptotically stable for $\tau_2<\tau_{2_0}$ and unstable for $\tau_2>\tau_{2_0}$. Furthermore system undergoes a Hopf bifurcation at $E_3(0,y^*,0)$ when $\tau_2=\tau_{2_0}$.

Case(iii): $\tau_1>0$, $\tau_2=0$

In this case the characteristic equation will be,

$$\lambda^3+\lambda^2(A+D_2)+\lambda(B+E_2)+(C+F_2)+e^{-\lambda\tau_1}[D_1\lambda^2+\lambda(E_1+G) \\ +(F_1+H)]=0 \tag{17}$$

Let $i\omega(\omega>0)$ be a root of Eq.(17). Then, separating the real and imaginary parts,

$$(E_1+G)\omega\sin\ \omega\tau_1+(F_1+H-D_1\omega^2)\omega\cos\ \omega\tau_1 = \omega^2(A+D_2)-(C+F_2) \tag{18}$$

$$(E_1+G)\omega\cos\ \omega\tau_1-(F_1+H-D_1\omega^2)sin\ \omega\tau_1 = \omega^3-\omega(B+E_2) \tag{19}$$

which implies,

$$\begin{aligned}\cos\ \omega\tau_1 &= \frac{\omega^4[E_1+G-D_1(A+D_2)]+\omega^2[CD_1+F_2D_1+AF_1+AH+D_2F_1]}{\omega^4D_1^2+\omega^2[(E_1+G)^2-2D_1(F_1+H)]+(F_1+H)^2} \\ &+ \frac{\omega^2[D_2H-(B+E_2)(E_1+G)]-(C+F_2)(F_1+H)}{\omega^4D_1^2+\omega^2[(E_1+G)^2-2D_1(F_1+H)]+(F_1+H)^2}\end{aligned} \tag{20}$$

$$\begin{aligned}\sin\ \omega\tau_1 &= \frac{\omega^7D_1^2+\omega^5[D_1(A+D_2)(E_1+G)-2D_1(F_1+H)-D_1^2(B+E_2)]+\omega^3[(F_1+H)^2-(B+E_2)(E_1+G)^2]}{\omega^6D_1^3+\omega^4[D_1(E_1+G)^2-3D_1(F_1+H)]+\omega^2[3D_1(F_1+H)^2-(E_1+G)^2(F_1+H)]-(F_1+H)^3} \\ &+ \frac{\omega^3[(B+E_2)2D_1(F_1+H)-(E_1+G)[CD_1+F_2D_1+AF_1+AH+D_2F_1+D_2H-(B+E_2)(E_1+G)]]}{\omega^6D_1^3+\omega^4[D_1(E_1+G)^2-3D_1(F_1+H)]+\omega^2[3D_1(F_1+H)^2-(E_1+G)^2(F_1+H)]-(F_1+H)^3} \\ &+ \frac{\omega[(E_1+G)(C+F_2)(F_1+H)-(B+E_2)(F_1+H)^2]}{\omega^6D_1^3+\omega^4[D_1(E_1+G)^2-3D_1(F_1+H)]+\omega^2[3D_1(F_1+H)^2-(E_1+G)^2(F_1+H)]-(F_1+H)^3}\end{aligned} \tag{21}$$

Squaring and adding (20) and (21),

$$\omega^6+\omega^4[(A+D_2)^2-2(B+E_2)-D_1^2]+\omega^2[(B+E_2)^2-2(A+D_2)(C+F_2) \\ +2(F_1+H)D_1-(E_1+G)^2]+(C+F_2)^2-(F_1+H)^2=0 \tag{22}$$

Let $\psi(W)=W^3+W^2[(A+D_2)^2-2(B+E_2)-D_1^2]+W[(B+E_2)^2-2(A+D_2)(C+F_2)+2(F_1+H)D_1-(E_1+G)^2]+(C+F_2)^2-(F_1+H)^2=0$ where $W=\omega^2$.

The function ψ has positive roots iff $(C+F_2)^2-(F_1+H)^2<0$.

Without loss of generality, let W_p be the positive roots of $\psi=0$ and let $\omega_p=\sqrt{W_p}$. We note that unique solution of $\theta\in[0,2\pi]$ of Eq.(20) and Eq.(21) is,

$$\begin{aligned}\theta &= \cos^{-1}\{\frac{\omega^4[E_1+G-D_1(A+D_2)]+\omega^2[CD_1+F_2D_1+AF_1+AH+D_2F_1]}{\omega^4D_1^2+\omega^2[(E_1+G)^2-2D_1(F_1+H)]+(F_1+H)^2} \\ &+ \frac{\omega^2[D_2H-(B+E_2)(E_1+G)]-(C+F_2)(F_1+H)}{\omega^4D_1^2+\omega^2[(E_1+G)^2-2D_1(F_1+H)]+(F_1+H)^2}\}\end{aligned} \tag{23}$$

if $\sin(\theta) > 0$, i.e., $\omega^6 D_1^2 + \omega^4[D_1(A+D_2)(E_1+G) - 2D_1(F_1+H) - D_1^2(B+E_2)] + \omega^2[(F_1+H)^2 - (B+E_2)[(E_1+G)^2 - 2D_1(F_1+H)] - (E_1+G)[CD_1+F_2D_1+AF_1+AH+D_2F_1+D_2H-(B+E_2)(E_1+G)]] + (E_1+G)(C+F_2)(F_1+H) - (B+E_2)(F_1+H)^2 > 0$ and

$$\begin{aligned}\theta &= 2\pi - \cos^{-1}\{\frac{\omega^4[E_1+G-D_1(A+D_2)] + \omega^2[CD_1+F_2D_1+AF_1+AH]}{\omega^4 D_1^2 + \omega^2[(E_1+G)^2 - 2D_1(F_1+H)] + (F_1+H)^2} \\ &+ \frac{\omega^2[D_2F_1+D_2H-(B+E_2)(E_1+G)] - (C+F_2)(F_1+H)}{\omega^4 D_1^2 + \omega^2[(E_1+G)^2 - 2D_1(F_1+H)] + (F_1+H)^2}\} \end{aligned} \tag{24}$$

if $\omega^6 D_1^2 + \omega^4[D_1(A+D_2)(E_1+G) - 2D_1(F_1+H) - D_1^2(B+E_2)] + \omega^2[(F_1+H)^2 - (B+E_2)[(E_1+G)^2 - 2D_1(F_1+H)] - (E_1+G)[CD_1+F_2D_1+AF_1+AH+D_2F_1+D_2H-(B+E_2)(E_1+G)]] + (E_1+G)(C+F_2)(F_1+H) - (B+E_2)(F_1+H)^2 \leq 0$.

We now define two sequences,

$$\begin{aligned}\tau_{1,p}^{1,i} &= \frac{1}{\omega_p}\{\cos^{-1}\{\frac{\omega^4[E_1+G-D_1(A+D_2)] + \omega^2[CD_1+F_2D_1+AF_1+AH]}{\omega^4 D_1^2 + \omega^2[(E_1+G)^2 - 2D_1(F_1+H)] + (F_1+H)^2} \\ &+ \frac{\omega^2[D_2F_1+D_2H-(B+E_2)(E_1+G)] - (C+F_2)(F_1+H)}{\omega^4 D_1^2 + \omega^2[(E_1+G)^2 - 2D_1(F_1+H)] + (F_1+H)^2}\} + 2i\pi\} \end{aligned} \tag{25}$$

$$\begin{aligned}\tau_{1,p}^{2,i} &= \frac{1}{\omega_p}\{2\pi - \cos^{-1}\{\frac{\omega^4[E_1+G-D_1(A+D_2)] + \omega^2[CD_1+F_2D_1+AF_1]}{\omega^4 D_1^2 + \omega^2[(E_1+G)^2 - 2D_1(F_1+H)] + (F_1+H)^2} \\ &+ \frac{\omega^2[AH+D_2F_1+D_2H-(B+E_2)(E_1+G)] - (C+F_2)(F_1+H)}{\omega^4 D_1^2 + \omega^2[(E_1+G)^2 - 2D_1(F_1+H)] + (F_1+H)^2}\} + 2i\pi\} \end{aligned} \tag{26}$$

Theorem 4.2. *Let $\tau_{1,p}^* = \tau_{1,p}^{1,i}$ or $\tau_{1,p}^* = \tau_{1,p}^{2,i}$ that is $\tau_{1,p}^*$ represents an element either of the sequence $\tau_{1,p}^{1,i}$ or $\tau_{1,p}^{2,i}$ associated with ω_p. Then the equation, $\lambda^3 + \lambda^2(A+D_2) + \lambda(B+E_2) + (C+F_2) + e^{-\lambda\tau_1}[D_1\lambda^2 + \lambda(E_1+G) + (F_1+H)] = 0$ has a pair of simple conjugate roots $\pm i\omega_p$ for $\tau_2 = \tau_{1,p}^*$ which satisfies, $sign\{\frac{dRe\lambda}{d\tau_1}|_{\tau=\tau_{1,p}^*}\} = sign\ \psi'(\omega_p^2)$. Denoting $\tau_1^* = min_{i\in N}\{\tau_{1,p}^{1,i}, \tau_{1,p}^{2,i}\}$, it is concluded that the steady state $(0,y^*,0)$ is locally asymptotically stable if $\tau_1 < \tau_1^*$ and a Hopf bifurcation occurs at $(0,y^*,0)$ when $\tau_1 = \tau_1^*$ iff $\psi'(\omega_p^2) > 0$.*

Proof. We prove the theorem by contradiction. Let $\pm i\omega_p$ be a pair of purely imaginary roots of Eq.(17) and let $\lambda(\tau_1) = \psi(\tau_1) + i\omega(\tau_1)$ be the branch of roots of Eq.(17) with $\psi(\tau_{1,p}^*) = 0$ and $\omega(\tau_{1,p}^*) = \omega_p$.

We assume that $\lambda(\tau_{1,p}^*)$ is not a simple root of Eq.(17), then both Eq.(17) and derivative of Eq.(17) share the same root, which implies,

$$\lambda^3+\lambda^2(A+D_2)+\lambda(B+E_2)+(C+F_2)+e^{-\lambda\tau_1}[D_1\lambda^2+\lambda(E_1+G) +(F_1+H)] = 0 \tag{27}$$

$$\frac{d\lambda}{d\tau_1}\{3\lambda^2+2\lambda(A+D_2)+(B+E_2)+e^{-\lambda\tau_1}(-D_1\tau_1\lambda^2+2D_1\lambda-\lambda\tau_1(E_1+G) +(E_1+G)-\tau_1(F_1+H))\}-\lambda(D_1\lambda^2+\lambda(E_1+G)+(F_1+H))e^{-\lambda\tau_1} = 0 \tag{28}$$

at $\lambda=\lambda(\tau^*_{1,p})$. Substitute $\lambda=\lambda(\tau^*_{1,p})=\omega(\tau^*_{1,p})=\omega_p$ in Eq.(27) and Eq.(28). Separating the real and imaginary parts,

$$(E_1+G)\omega_p\sin\ \omega_p\tau^*_{1,p}-(D_1\omega_p^2-(F_1+H))\cos\ \omega_p\tau^*_{1,p}=0 \tag{29}$$

$$(E_1+G)\omega_p\cos\ \omega_p\tau^*_{1,p}+(D_1\omega_p^2-(F_1+H))\sin\ \omega_p\tau^*_{1,p}=0 \tag{30}$$

and

$$(E_1+G)\omega_p\cos\ \omega_p\tau^*_{1,p}+(D_1\omega_p^2-(F_1+H))\sin\ \omega_p\tau^*_{1,p} = \omega_p^3-\omega_p(B+E_2) \tag{31}$$

$$(E_1+G)\omega_p\sin\ \omega_p\tau^*_{1,p}-(D_1\omega_p^2-(F_1+H))\cos\ \omega_p\tau^*_{1,p} = \omega_p^2(A+D_2)-(C+F_2) \tag{32}$$

Considering the fact that $\omega_p>0$ and using Eqs.(29),(30),(31) and (32), we obtain $\omega_p^2=\frac{C+F_2}{A+D_2}$ and $\omega_p^2=B+E_2$. Since $\frac{C+F_2}{A+D_2}\neq B+E_2$, we arrive at a contracdiction.

Hence we conclude that $\pm i\omega_p$ are simple roots of Eq.(17). From Eqs.(27) and (28), we get

$$e^{\lambda\tau_1} = -\frac{[\lambda^2D_1+\lambda(E_1+G)+(F_1+H)]}{\lambda^3+\lambda^2(A+D_2)+\lambda(B+E_2)+C+F_2} \tag{33}$$

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1} = -\frac{[3\lambda^2+2\lambda(A+D_2)+(B+E_2)]}{\lambda(\lambda^3+\lambda^2(A+D_2)+\lambda(B+E_2)+C+F_2)} + \frac{2D_1\lambda+E_1+G}{\lambda(\lambda^2D_1+\lambda(E_1+G)+(F_1+H))}-\frac{\tau_1}{\lambda} \tag{34}$$

Eliminating $e^{\lambda\tau_1}$ we get,

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1}_{\tau_1=\tau^*_{1,p}} = \frac{-[3(-\omega_p^2)+2i\omega_p(A+D_2)+B+E_2]}{i\omega_p[-i\omega_p^3-\omega_p^2(A+D_2)+i\omega_p(B+E_2)+C+F_2]} + \frac{2D_1i\omega_p+E_1+G}{i\omega_p[-D_1\omega_p^2+i\omega_p(E_1+G)+F_1+H]}-\frac{\tau^*_{1,p}}{i\omega_p} \tag{35}$$

Consequently,

$$\begin{aligned} Re\left(\frac{d\lambda}{d\tau_1}\right)^{-1}_{\tau_1=\tau^*_{1,p}} &= \frac{3\omega_p^4+\omega_p^2[2(A+D_2)^2-4(B+E_2)]+(B+E_2)^2}{\omega_p^2(\omega_p^2-(B+E_2))^2+[C+F_2-\omega_p^2(A+D_2)]^2} \\ &- \frac{2(A+D_2)(C+F_2)}{\omega_p^2(\omega_p^2-(B+E_2))^2+[C+F_2-\omega_p^2(A+D_2)]^2} \\ &+ \frac{2D_1[F_1+H-D_1\omega_p^2]-(E_1+G)^2}{\omega_p^2(E_1+G)^2+[F_1+H-D_1\omega_p^2]^2} \end{aligned} \tag{36}$$

Now, $\omega_p^2(E_1+G)^2+[F_1+H-D_1\omega_p^2]^2 = \omega_p^6+\omega_p^4[(A+D_2)^2-2(B+E_2)]+\omega_p^2[(B+E_2)^2-2(A+D_2)(C+F_2)]+(C+F_2)^2 = \omega_p^2(\omega_p^2-(B+E_2))^2-\omega_p^2(A+D_2)+(C+F_2)$

which gives,

$$\begin{aligned} Re\left(\frac{d\lambda}{d\tau_1}\right)^{-1}_{\tau_1=\tau^*_{1,p}} &= \frac{3\omega_p^4+\omega_p^2[2(A+D_2)^2-4(B+E_2)]+(B+E_2)^2}{\omega_p^2(\omega_p^2-(B+E_2))^2+[C+F_2-\omega_p^2(A+D_2)]^2} \\ &- \frac{2(A+D_2)(C+F_2)}{\omega_p^2(\omega_p^2-(B+E_2))^2+[C+F_2-\omega_p^2(A+D_2)]^2} \\ &+ \frac{2D_1[F_1+H-D_1\omega_p^2]-(E_1+G)^2}{\omega_p^2(\omega_p^2-(B+E_2))^2+[C+F_2-\omega_p^2(A+D_2)]^2} \\ Re\left(\frac{d\lambda}{d\tau_1}\right)^{-1}_{\tau_1=\tau^*_{1,p}} &= \frac{\psi'(\omega_p^2)}{\omega_p^2(\omega_p^2-(B+E_2))^2+[C+F_2-\omega_p^2(A+D_2)]^2} \end{aligned}$$

Since

$$sign\{Re\left(\frac{d\lambda}{d\tau_1}\right)^{-1}_{\tau_1=\tau^*_{1,p}}\} = sign\{\frac{dRe(\lambda)}{d\tau_1}\mid_{\tau_1=\tau^*_{1,p}}\}$$

we get,

$$sign\{\frac{dRe(\lambda)}{d\tau_1}\mid_{\tau_1=\tau^*_{1,p}}\} = sign\{\psi'(\omega_p^2)\}$$

If $\psi'(\omega_p^2)>0$, then $sign\{\frac{dRe(\lambda)}{d\tau_1}\mid_{\tau_1=\tau^*_{1,p}}\}>0$. So the system will be locally asymptotically stable when $\tau_1<\tau^*_{1,p}$ and a Hopf bifurcation occurs at $(0,y^*,0)$ at $\tau_1=\tau^*_{2,p}$ iff $\psi'(\omega_p^2)>0$.

Case(iv): $\tau_1>0$, $\tau_2>0$

Let us now state a result according to the sign of the real parts of the roots of Eq.(10) in order to study the local stability of the positive steady state $(0,y^*,0)$ of the system (7)-(9).

Proposition 4.1. If all the roots of the Eq.(10) have negative real parts for some $\tau_1>0$, then there exists a $\tau_2^*(\tau_1) > 0$ such that all the roots of Eq.(10) (i.e., with $\tau_2 > 0$) have negative real parts when $\tau_2 < \tau_2^*(\tau_1)$.

Considering the above proposition we can now state the following theorem.

Theorem 4.3. If we assume that the hypothesis (Proposition 2.5) hold, then for any $\tau_1 \in [0, \tau_1^*)$ there exists a $\tau_2^*(\tau_1) > 0$ such that the positive equilibrium $(0, y^*, 0)$ of the system is locally asymptotically stable when $\tau_1 \in [0, \tau_1^*)$.

Proof. Using the above theorem, we can say that all the roots of Eq.(10) have negative real parts when $\tau_1 \in [0, \tau_1^*)$ and by proposition we can conclude that there exists a $\tau_2^*(\tau_1) > 0$ such that all roots of Eq.(10) have negative real parts when $\tau_2 < \tau_2^*(\tau_1)$. Hence the equilibrium $(0, y^*, 0)$ of system (7)-(9) is locally asymptotically stable when $\tau_1 \in [0, \tau_2^*(\tau_1))$.

5. Numerical Simulation

In this section we present numerical results of the system (7)-(9) to verify the analytical predictions obtained in previous section. Let us consider the system with the parameter values r=0.47, a_1=0.45, a_2=0.01, a_3=0.39, v=0.76, h=0.07. So the system (7)-(9) has a positive equilibrium E(0, 0.8684, 0). When $\tau_1 = 0$ and $\tau_2 = 0$ the equilibrium E is asymptotically stable. Fig 1 shows that the impact of the coral, KA and NA without delays after the manual removal rate of KA. Fig 2 shows that growth rate of KA when $\tau_1 = 4$ and there is no time delay of propagation of NA through vegetation/spores.

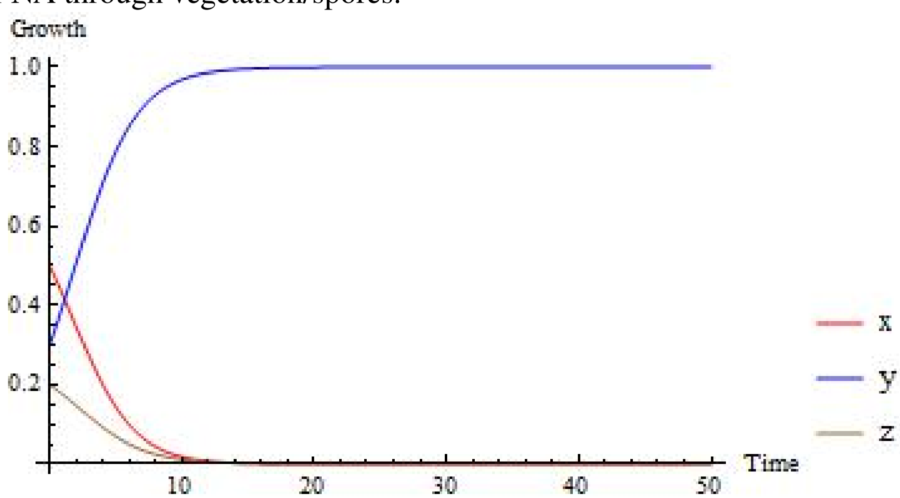

Figure 1: $\tau_1 = 0$,$\tau_2 = 0$ and d = 0.

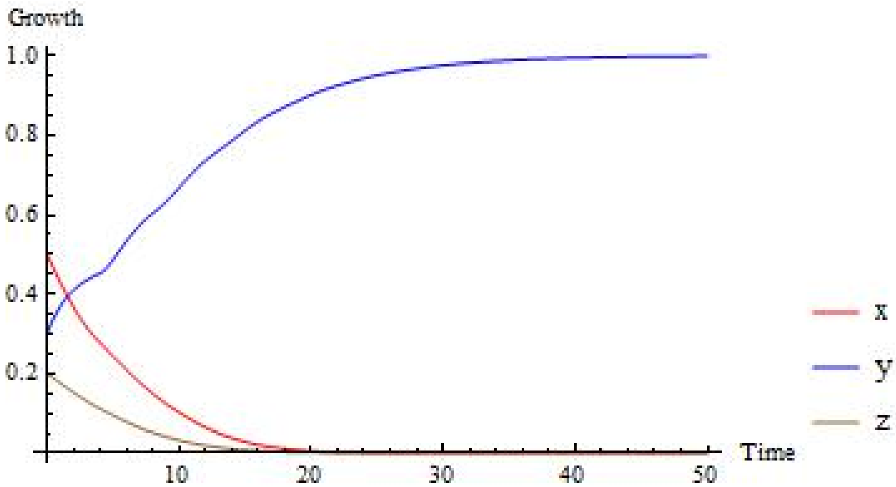

Figure 2: $\tau_1 = 4$, $\tau_2 = 0$ and d = 0

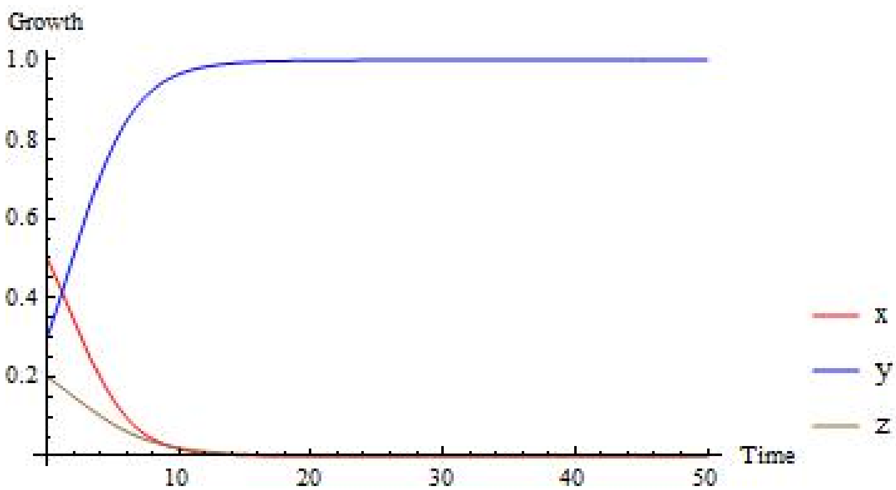

Figure 3: $\tau_1 = 0$, $\tau_2 = 3$ and d = 0

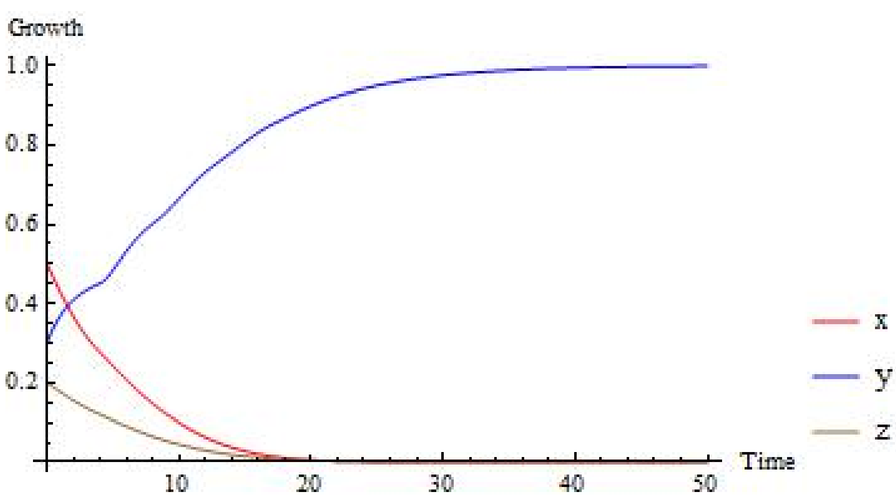

Figure 4: $\tau_1 = 4$, $\tau_2 = 3$ and d = 0

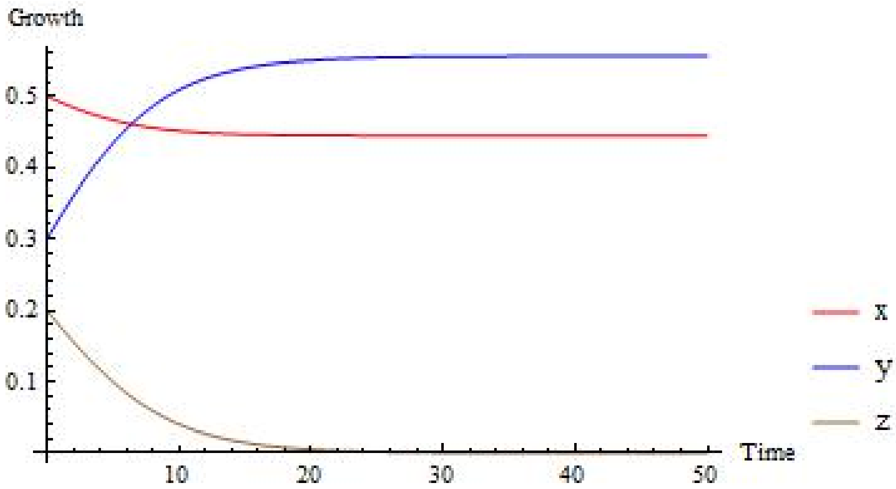

Figure 5: $\tau_1 = 0$,$\tau_2 = 0$ and d = 0.2

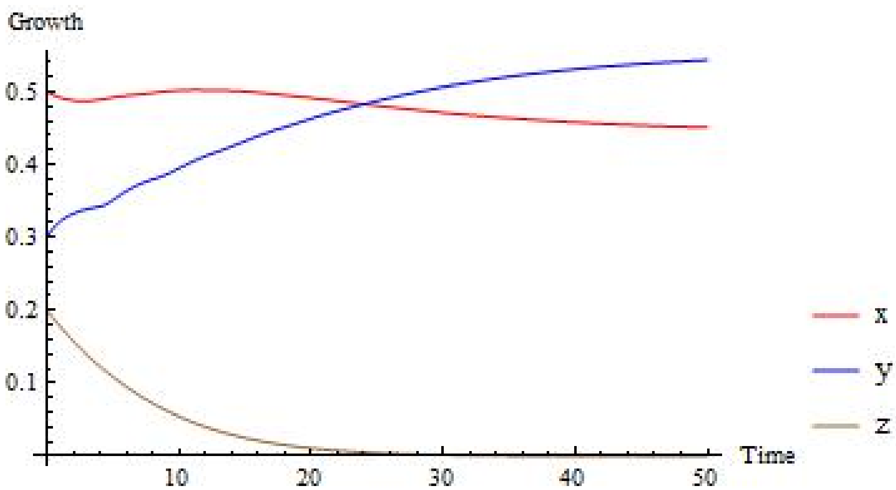

Figure 6: $\tau_1 = 4$,$\tau_2 = 0$ and d = 0.2

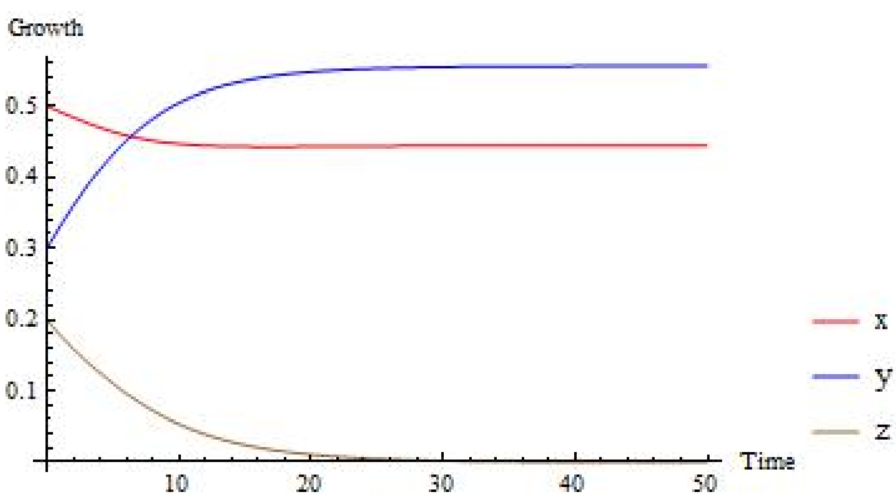

Figure 7: $\tau_1 = 0$,$\tau_2 = 3$ and d = 0.2

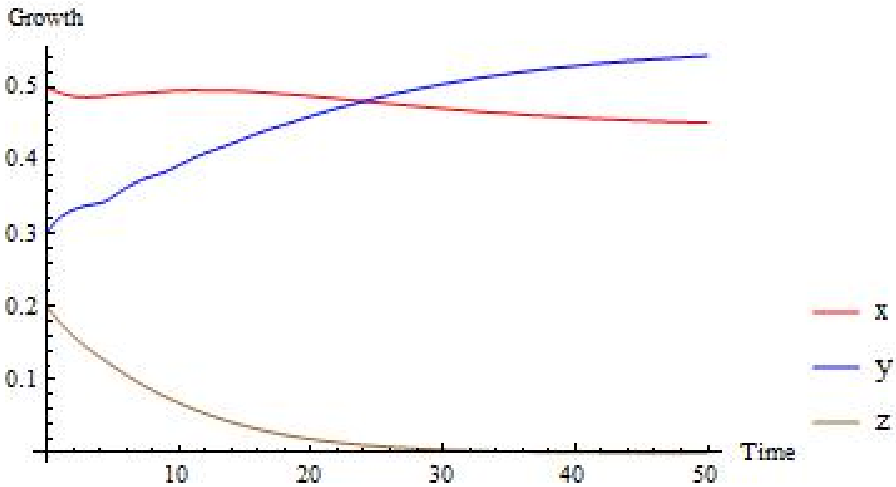

Figure 8: $\tau_1 = 4$,$\tau_2 = 3$ and d = 0.2

Fig 3 gives the time evolution of three populations when time delay of propagation of NA through vegetation/spores is taken into account, where time delay of propagation of KA through vegetation/spores is zero. The positive equilibrium E is asymptotically stable for τ_1=0 and τ_2=3. Fig 4 shows that the growth rate of KA and NA. The steady state is asymptotically stable when $\tau_1 = 4$ and $\tau_2 = 3$.

6. Conclusion

We have analyzed the effect of two time delays on the coral reef system with the manual removal rate of KA. The actual manual removal rate of KA is very much lesser than our vision in the field. For non-delay case the growth rate of coral increases after the manual removal rate of KA. Then, we have considered the delay case. In the presence of the time delay of propagation of KA through vegetation/spores, the growth rate of KA increases. In the absence of the time delay of propagation of KA through vegetation/spores, the time evolution of three populations are asymptotiocally stable. In the presence of both the time delays (τ_1, τ_2), the system is asymptotically stable. Finally, the impact of manual removal rate of KA triggers its growth.

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgement

We are grateful to Ministry of Human Resource Development, Government of India, New Delhi and The Management, Thiagarajar College, Madurai, Tamilnadu for rendering financial and other facilities.

REFERENCES

[1] Abhijit Ghorai, Soovoojeet Jana, Srabani Guria and T.K.Kar., Global dynamics of a predator, weaker prey and stronger prey system, J.Appl.Math.Comp. 250 (2015), 235-248.

[2] Alan Hastings, Hao Wang, Xiong Li and Zheng Zhang., Mathematical analysis of coral reef models, J. Math. Anal. Appl. 416 (2014), 352-373.

[3] Alan Hastings, Helen J. Edwards and Peter. J. Mumby., Thresholds and the resilience of Carribean coral reefs, Nature. 450 (2007), 98-101.

[4] Alan Hastings, Julie C. Blackwood and Peter. J. Mumby., The effect of fishing on hysteresis in Carribean coral reefs, Theor Ecol. 5 (2012), 105-114.

[5] N. Arun Nagendran, S. Chandrasekaran, J. Joyson Joe Jeevamani, B. Kamalakannan and D. Pandiaraja., Impact of removal of invasive species Kappaphycus alvarezii from coral reef ecosystem in Gulf of Mannar, India, Current Sci. 106 (10) (2014), 1401-1408.

[6] N. Arun Nagendran, S. Chandrasekaran, B. Kamalakannan, N. Krishnankutty and D. Pandiaraja., Bioinvasion of Kappaphycus alvarezii on corals in the Gulf of Mannar, India, Current Sci. 94 (9) (2008), 1167-1172.

[7] Ahmed Elaiw, Ismail Hassanien and Shimaa Azoz., Global stability of HIV infection models with intracellular delays, J. Korean Math. Soc., 49 (4) (2012), 779-794.

[8] Baoazhu Guo, Liming Cai, Mini Ghosh and Xuezhi Li., Stability analysis of an HIV/AIDS epidemic model with treatment. J.Comput.Appl.Math. 229 (2009), 313-323.

[9] R. Bhattacharya, B. Mukhopadhyay and Sandip Banerjee., Effect of maturation and gestation delays in a stage structure predator prey model, J.Appl.Math and Informatics, 28 (5-6) (2010), 1379-1393.

[10] Changjin Xu, Maoxin Liao and Xiaoffi He., Stability and Hopf bifurcation analysis for Lotka-Volterra predator-prey model with two delays, Int.J.Appl.Math.Comput.Sci., 21 (1) (2011), 97-107.

[11] Chornesky EA and Lang JC, Competition between scleractinian reef corals: a review of mechanisms and effects. In: Dubinsky Z (ed), Ecosystems of the world, 25. Coral reefs. Elsevier Science, Amsterdam, (1990), pp. 209-252.

[12] V. K. Dhargalkar., Seaweeds - a field manual, National Institute of Oceanography, Goa, (2004).

[13] G. Diaz-Pulido, J. Jompa and L. J. McCook , Competition between corals and algae on coral reefs: a review of evidence and mechanisms, Coral reefs, 19 (2001), 400 - 417.

[14] Dileep Sharma and Ram Naresh., An HIV/AIDS model with vertical transmission and time delay. World.J.Modelling and Simul, 7 (3) (2011), 230-240.

[15] Edoardo Beretta and Yang Kuang., Geometric stability switch criteria in delay differential systems with delay dependent parameters. SIAM.J.Math.Anal, 33 (5) (2002), 1144-1165.

[16] M.F. Elettreby., Two-prey one predator model, Chaos, Solitons and Fractals, 39 (5) (2009), 2018-2027.

[17] Fabien Crauste, Mostafa Adimy and Shigui Ruan., Periodic oscillations in leukopoiesis models with two delays, J.Theoret.Biol, 242 (2006), 288-299.

[18] Farida Mosally, Mona Alsolami and Sarah Al-Sheikh., Properties of stability and Hopf bifurcation for an HIV/AIDS epidemic model with screening and time delay, Int.J.Differ.Equ.Appl. 11 (2) (2012), 87-99.

[19] Guang Ping and XiaoLing Li., Stability and bifurcation in a Network Model with two delays, International Mathematical Forum, 6 (35) (2011), 1725-1731.

[20] Hao Wang and Xihui Lin., Stability analysis of delay differential equations with two discrete delays, Can.Appl.Math.Q. 20 (4) (2012), 519-533.

[21] Jonathan Forde and Patrick Nelson., Applications of Sturm sequences to bifurcation analysis of delay differential equation models, J.Math.Anal.Appl. 300 (2004), 273-284.

[22] Junjie Wei and Shigui Ruan., On the zeros of transcendental functions with applications to stability of delay differential equations with two delays, Dyn.Contin.Discrete Impuls.Syst.Ser.A Math.Anal. 10 (2003), 863-874.

[23] Kathleen M. Morrow and Nanette E. Chadwick., Competition Among Sessile Organisms on Coral Reefs, Coral Reefs: An Ecosystem in Transition, (2011).

[24] Laith Yakob, Manuel Gonzalez-Rivero and Peter. J. Mumby., The role of sponge competition on coral reef alternative steady states, Ecol. Model, 222 (2011), 1847-1853.

[25] Mc Namara D.E. and S. A. Sandin, Spatial dynamics of benthic competition on coral reefs, Oecologia, 168 (4) (2012), 1079-1090.

[26] Rujira Ouncharoen, Thongchai Dumrongpokaphan, Yongwimon Lenbury and Yuesheng Xu., An Intracellular delay-differential equation model of the HIV infection and immune control, Math. Modelling of the Natural Phenomena. 2 (1) (2007), 75-99.

[27] Shigui Ruan., Absolute stability, conditional stability and bifurcation in Kolomogorov-Type predator-prey systems with discrete delays, Quart.Appl.Math. 6 (1) (2001), 159-173.