DIRECTION AND STABILITY OF HOPF BIFURCATION

D. PANDIARAJA\textsuperscript{1}, N. ARUN NAGENDRAN\textsuperscript{2}, D. MURUGESWARI\textsuperscript{3}, VISHNU NARAYAN MISHRA\textsuperscript{4,5,}\textsuperscript{*}

\textsuperscript{1}National Centre of Excellence, Statistical and Mathematical Modelling on Bio-Resources Management, Thiagarajar College, Madurai, India
\textsuperscript{2}Department of Mathematics, Thiagarajar College, Madurai, India
\textsuperscript{3}Department of Zoology, Thiagarajar College, Madurai, India
\textsuperscript{4}L. 1627 Awadh Puri Colony Beniganj, Phase - III, Opposite-Industrial Training Institute (I.T.I.), Faizabad-224 001, Uttar Pradesh, India
\textsuperscript{5}Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak-484887, Madhya Pradesh, India

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Abstract. The dynamics of the spatial competition mathematical model for the invasion, removal of Kappaphycus Algae (KA) in Gulf of Mannar (GoM) with propagation delays is investigated by applying the normal form theory and the center manifold theorem.

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1. Introduction

\textsuperscript{*}Corresponding author.

E-mail address: vishnunarayamishra@gmail.com

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In recent years we have witnessed an increasing interest in dynamical systems with time delays, especially in applied mathematics. Stability and direction of the Hopf bifurcation for the predator-prey system have been discussed by using normal form theory and center manifold theory [5,6,10,13,15,16,17]. Direction and stability of the equilibrium for a neural network model with two delays have been investigated [7,12]. Bifurcation analysis of the predator-prey model has been detailed [8]. Direction and stability of the equilibrium involving various fields have been discussed [4,9,11,14]. We reported the shifting of algal dominated reef ecosystem due to the invasion of KA in Gulf of Mannar [1]. Subsequently, the dominance of KA over NA and corals in competing for space has also been reported. KA sexual reproduction by spores in the Gulf of Mannar Marine Biosphere Reserve (GoM) in future, when environmental conditions unanimously favor this alga has been deliberated [2]. To simulate the three way competition among corals, KA and NA, we proposed the following system of non-linear ODE’s [3].

\[
\begin{align*}
\frac{dx}{dt} &= rx - r x^2 - r x y^2 - r x z - a_1 y x y - a_2 x y + dy \\
\frac{dy}{dt} &= a_1 y x + a_3 y z + v y (t - \tau_1) - v y x - v y^2 - v y z - dy \\
\frac{dz}{dt} &= a_2 z x + h z (t - \tau_2) - h x z - h z y - h z^2 - a_3 z y
\end{align*}
\]

(1.1)

2. Direction and Stability of Hopf bifurcation

We assume that the system undergoes a Hopf bifurcation at the positive equilibrium \( E(0,y^*,0) \) for \( \tau_1 = \tau_1^* \) and then \( \pm i \omega \) denotes the corresponding purely imaginary roots of the characteristic equation at the positive equilibrium \( E(0,y^*,0) \).

Without loss of generality, we assume that \( \tau_2^* < \tau_1^* \) where \( \tau_2^* \in (0, \tau_2^0) \) and \( \tau_1 = \tau_1^* + \mu \). Let \( x_{1 i} = x - x_i^* \), \( x_{2 i} = y - y_i^* \), \( x_{3 i} = y - y_i^* \), \( x_{i1} = \mu_i(\tau_i), i=1,2,3,... \) Here \( \mu = 0 \) is the bifurcation parameter and dropping the bars, the system becomes a functional differential equation in \( C = C([-1,0],R^3) \) as

\[
\frac{dX}{dt} = L_\mu(X_t) + f(\mu,X_t)
\]

(2.1)
where \( x(t) = (x_{11}, x_{21}, x_{31}) \in \mathbb{R}^3 \) and \( L_\mu : C \to \mathbb{R}^3, f : R \times C \to \mathbb{R}^3 \) are respectively given by

\[
L_\mu(\phi) = (\tau^*_1 + \mu)B \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix} + (\tau^*_1 + \mu)C \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \end{pmatrix}
\]

and

\[
f(\mu, \phi) = (\tau^*_1 + \mu)Q
\]

where \( Q = \begin{pmatrix} (r - r \phi_2(0) - a_1 \phi_1(0)) \phi_1(0) + d \phi_2(0) \\ (a_1 \phi_2(0) - v \phi_2(0)) \phi_1(0) - 2v \phi_2^2(0) - d \phi_2(0) + (a_3 \phi_2(0) - v \phi_2(0)) \phi_3(0) + ve^{-\lambda \tau} \phi_2(-1) \\ (-h \phi_2(0) - a_3 \phi_2(0)) \phi_3(0) + he^{-\lambda \tau} \phi_3(-1) \end{pmatrix} \)

respectively where \( \phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta))^T \in C, \)

\[
B = \begin{pmatrix} r - ry^* - a_1 y^* & d & 0 \\ a_1 y^* - vy^* & -2vy^* - d & a_3 y^* - vy^* \\ 0 & 0 & -hy^* - a_3 y^* \end{pmatrix},
\]

\[
C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & he^{-\lambda \tau} \end{pmatrix},
\]

\[
D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & ve^{-\lambda \tau} & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

By the Riesz representation theorem, we claim about the existence of a function \( \eta(\theta, \mu) \) of bounded variation for \( \theta \in [-1, 0) \) such that

\[
L_\mu(\phi) = \int_{-1}^{0} d \eta(\theta, \mu) \phi(\theta) \text{ for } \phi \in C
\]

Now let us choose,

\[
\eta(\theta, \mu) = \begin{cases} (\tau^*_1 + \mu)(B + C + D), & \theta = 0 \\ (\tau^*_1 + \mu)(C + D), & \theta \in [-\frac{\tau^*_2}{\tau_1}, 0) \\ (\tau^*_1 + \mu)(D), & \theta \in (-1, -\frac{\tau^*_1}{\tau_1}) \\ 0, & \theta = -1. \end{cases}
\]
For $\phi \in C([-1, 0], R^3)$, we define

$$A(\mu)\phi = \begin{cases} 
\frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0) \\
\int_{-1}^{0} d\eta(s, \mu)\phi(s), & \theta = 0 
\end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 
0, & \theta \in [-1, 0) \\
f(\mu, \phi), & \theta = 0 
\end{cases}$$

Then the system is equivalent to

$$\frac{dX}{dt} = A(\mu)X_t + R(\mu)X_t, \quad (2.5)$$

where $X_t(\theta) = X(t + \theta)$ for $\theta \in [-1, 0]$.

Now for $\psi \in C([-1, 0], (R^3)^*)$, we define

$$A^*\psi(s) = \begin{cases} 
\frac{-d\psi(s)}{ds}, & s \in (0, 1) \\
\int_{-1}^{0} d\eta^T(t, 0)\psi(-t), & s = 0 
\end{cases}$$

Further we define a bilinear inner product

$$<\psi(s), \phi(0)> = \bar{\psi}(0)\phi(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi. \quad (2.6)$$

where $\eta(\theta) = \eta(\theta, 0)$. Clearly here $A$ and $A^*$ are adjoint operators and $\pm i\omega^*\tau_0^*$ are eigen values of $A(0)$ and so they are also eigen values $A^*$. Let $q(\theta) = (1 \alpha \beta)^Te^{i\omega^*\tau_0^*\theta}$ be the eigen vector of $A(0)$ corresponding to $i\omega^*\tau_0^*$ where

$$\alpha = \frac{-(r-r^*a_1y^*+iw)}{a_1y^*-vy^*}, \quad \beta = \frac{(r-r^*a_1y^*+iw)(ve^{-iw_0}-2vy^*-d-iv)-d(a_1y^*-vy^*)}{d(a_1y^*-vy^*)}$$

Similarly if $q^*(s) = M(1 \alpha^* \beta^*)e^{i\omega^*\tau_0^{-}s}$ be the eigen vector of $A^*$ where

$$\alpha^* = \frac{-(r-r^*a_1y^*+iw)}{a_1y^*-vy^*}, \quad \beta^* = \frac{(a_3y^*-vy^*)(r-r^*a_1y^*+iw)}{(a_1y^*-vy^*)(he^{-iw_0}-hy^*-a_1y^*-iw)}$$
Then we have to determine $M$ from $\langle q^*(s), q(\theta) \rangle = 1$.

Thus we can take

$$\bar{M} = \frac{1}{1 + \alpha \alpha^* + \beta \beta^* + \tau_1 e^{i\alpha_0^* \tau_0^*} (\alpha \alpha^* + \beta \beta^* \bar{h})}$$

(2.7)

We first compute the coordinate to describe the center manifold $C_0$ at $\mu = 0$. Let $X_t$ be the solution of the system (2.5) when $\mu = 0$. Define $z(t) = \langle q^*, X_t \rangle$

$$W(t, \theta) = X_t(\theta) - 2Rez(t)q(\theta)$$

(2.8)

On the center manifold $C_0$, we have

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta)$$

where

$$W(z, \bar{z}, \theta) = W_{20}(\theta) \frac{z^2}{2} + W_{11}(\theta) z \bar{z} + W_{02}(\theta) \frac{\bar{z}^2}{2} + ....$$

(2.9)

and $z$ and $\bar{z}$ are local coordinates for center manifold $C_0$ in the direction of $q^*$ and $\bar{q}^*$.

Note that $W$ is real if $X_t$ is real. We consider only real solutions. For solution $X_t \in C_0$ of Eq. (2.1), since $\mu = 0$ we have

$$\dot{z}(t) = i\omega^* \tau_0^* z + \langle \bar{q}^*(0) f(0, W(z, \bar{z}, 0) + 2Rezq(\theta) \rangle$$

$$\cong i\omega^* \tau_0^* z + \bar{q}^*(0) f_0(z, \bar{z})$$

$$= i\omega^* \tau_0^* z + g(z, \bar{z})$$

(2.10)

where

$$g(z, \bar{z}) = \bar{q}^*(0) f_0(z, \bar{z})$$

$$= g_{20} \frac{z^2}{2} + g_{11} z \bar{z} + g_{02} \frac{\bar{z}^2}{2} + g_{21} \frac{z^2 \bar{z}}{2} + ....$$

(2.11)

From (2.8) and (2.9), we get

$$X_t(\theta) = W(t, \theta) + 2Rez(t)q(\theta)$$

$$= W_{20}(\theta) \frac{z^2}{2} + W_{11}(\theta) z \bar{z} + W_{02}(\theta) \frac{\bar{z}^2}{2} + zq + \bar{z}q + ....$$

$$= W_{20}(\theta) \frac{z^2}{2} + W_{11}(\theta) z \bar{z} + W_{02}(\theta) \frac{\bar{z}^2}{2} + (1 \alpha \beta)^T e^{i\omega^* \tau_0^*} + (1 \alpha \bar{\beta})^T e^{i\omega^* \tau_0^*} \bar{z} + ....$$

(2.12)
Hence we have
\[ g(z, \bar{z}) = \bar{q}^*(0)f_0(z, \bar{z}) = \bar{q}^*(0)f(0, X_t) = \tau_0^*\bar{M}(1 \bar{\alpha}^* \bar{\beta}^*)T = \tau_0^*\bar{M}(p_1z^2 + 2p_2z\bar{z} + p_3\bar{z}^2 + p_4z^2\bar{z}) + H.O.T \]

where \( T = \begin{pmatrix} (r - x_2(0) - a_1x_2(0)x_1(0) + dx_2(0)) \\ (a_1x_2(0) - v_x2(0)x_1(0) - 2v_x^2(0) - dx_2 + (a_3x_2 - v_x2)x_1(0) + ve^{-\kappa_2}x_2(-1)) \\ (-hx_2(0) - a_1x_2(0) + e^{-\kappa_2}x_2(-1)) \end{pmatrix} \)

\( p_1, p_2, p_3 \) and \( p_4 \) values can be calculated by using the formula.

Comparing (2.11) and (2.13)
\[ g_{20} = 2\tau_0^*\bar{M}p_1 \]
\[ g_{11} = 2\tau_0^*\bar{M}p_2 \]
\[ g_{02} = 2\tau_0^*\bar{M}p_3 \]
\[ g_{21} = 2\tau_0^*\bar{M}p_4 \]

For unknown \( W^{(i)}_{20}(\theta), W^{(i)}_{11}(\theta), i=1,2 \) in \( g_{21} \), we still have to compute them. From (2.5) and (2.8)
\[ \dot{W} = \dot{X}_t - \dot{z}q - \dot{z}\bar{q} \]

\[ \dot{W} = AW - 2Re\{q^*(0)f_0q(\theta)\}, \quad -1 \leq \theta \leq 0, \]
\[ \dot{W} = AW - 2Re\{q^*(0)f_0q(\theta)\} + f_0, \quad \theta = 0, \]

where
\[ H(z, \bar{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{z^2}{2} + \ldots \]

From (2.14) and (2.15)
\[ [A(0) - 2i\omega^*\tau_0^*I]W_{20}(\theta) = -H_{20}(\theta) \]
\[ A(0)W_{11}(\theta) = -H_{11}(\theta) \]
From (2.14) we have for \( \theta \in [-1, 0] \)

\[
H(z, \bar{z}, \theta) = -g(z, \bar{z})q(\theta) - \bar{g}(z, \bar{z})\bar{q}(\theta)
\]

(2.18)

Comparing (2.15) and (2.18)

\[
H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta)
\]

(2.19)

\[
H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta)
\]

(2.20)

By the definition of \( A(\theta) \) and from the above equations

\[
W_{20}(\theta) = \frac{i g_{20}}{\omega^* \tau_0^2} q(0) e^{i \omega^* \tau_0^2} + \frac{i \bar{g}_{02}}{3 \omega^* \tau_0^2} \bar{q}(0) e^{-i \omega^* \tau_0^2} + E_1 e^{2i \omega^* \tau_0^2}.
\]

(2.21)

and

\[
W_{11}(\theta) = \frac{-i g_{11}}{\omega^* \tau_0^2} q(0) e^{i \omega^* \tau_0^2} + \frac{i \bar{g}_{11}}{\omega^* \tau_0^2} \bar{q}(0) e^{-i \omega^* \tau_0^2} + E_2.
\]

(2.22)

where \( q(\theta) = (1, \alpha, \beta)^T e^{i \omega^* \tau_0^2} \), \( E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)}) \in \mathbb{R}^3 \) and \( E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)}) \in \mathbb{R}^3 \) are constant vectors. From (2.14) and (2.15)

\[
H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\tau_0^*(c_1 c_2 c_3)^T
\]

\[
H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + 2\tau_0^*(d_1 d_2 d_3)^T
\]

(2.23)

where \((c_1 c_2 c_3)^T = C_1, (d_1 d_2 d_3)^T = D_1\) are respective coefficients of \( z^2 \) and \( \bar{z} \bar{z} \) of \( f_0(z, \bar{z}) \) and they are

\[
C_1 = \begin{pmatrix}
    c_1 \\
    c_2 \\
    c_3
\end{pmatrix}
= \begin{pmatrix}
    -(a_1 + r)\alpha \\
    (a_1 - \nu)\alpha - 2\nu\alpha^2 - \frac{dW_{12}^{(2)}(0)}{2} + (a_3 - \nu)\alpha\beta + \frac{\nu e^{-\lambda_2}}{2} W_{20}^{(2)}(-1) \\
    (-h - a_3)\alpha\beta + \frac{he^{-\lambda_2}}{2} W_{20}^{(3)}(-1)
\end{pmatrix}
\]

and

\[
D_1 = \begin{pmatrix}
    d_1 \\
    d_2 \\
    d_3
\end{pmatrix}
= 2 \begin{pmatrix}
    (a_1 - \nu)Re(\alpha) - 4\nu\alpha - \frac{dW_{12}^{(2)}(0)}{2} + (a_3 - \nu)Re(\bar{\alpha}\beta) + \frac{\nu e^{-\lambda_2}}{2} W_{11}^{(2)}(-1) \\
    (-h - a_3)Re(\bar{\alpha}\beta) + \frac{he^{-\lambda_2}}{2} W_{11}^{(3)}(-1)
\end{pmatrix}
\]
Finally we have \((2i\omega^*\tau_0^* I - \int_0^1 e^{2i\omega^*\tau_0^* \theta} d\eta(\theta))E_1 = 2\tau_0^* C_1\) or \(C^* E_1 = 2C_1\) where

\[
C^* = 
\begin{pmatrix}
2rx + ry + rz + a_1y + a_2z - r + 2\omega & rx + a_1x - d & rx + a_2x \\
v y - a_1y & vx + 2vy + vz - a_1x - a_2z - v & vy - a_3y \\
 h z - a_2z & h z + a_3z & -a_2x - h e^{-2i\omega \tau^*_1} + h x + hy + 2hz + a_3y + 2i\omega \\
\end{pmatrix}
\]

Thus \(E_1^* = \frac{2\Delta_i}{\Delta}\) where \(\Delta = \text{Det}(C^*)\) and \(\Delta_i\) be the value of the determinant \(U_i\), where \(U_i\) formed by replacing \(i^{th}\) column vector of \(C^*\) by another column vector \((c_1 c_2 c_3)^T, i = 1, 2, 3\). Similarly \(D^*E_2 = 2D_1\), where

\[
D^* = 
\begin{pmatrix}
2rx + ry + rz + a_1y + a_2z - r & rx + a_1x - d & rx + a_2x \\
v y - a_1y & vx + 2vy + vz - a_1x - a_2z - v & vy - a_3y \\
 h z - a_2z & h z + a_3z & -a_2x - h e^{-2i\omega \tau^*_1} + h x + hy + 2hz + a_3y \\
\end{pmatrix}
\]

Thus \(E_2^* = \frac{2\bar{\Delta}_i}{\bar{\Delta}}\) where \(\bar{\Delta} = \text{Det}(D^*)\) and \(\bar{\Delta}_i\) be the value of the determinant \(V_i\), where \(V_i\) formed by replacing \(i^{th}\) column vector of \(D^*\) by another column vector \((d_1 d_2 d_3)^T, i = 1, 2, 3\). Thus we can determine \(W_{20}(\theta)\) and \(W_{11}(\theta)\) from (2.12) and (2.13). Furthermore using them we can compute \(g_{21}\) and derive the following values.

\[
C_1(0) = \frac{i}{2\omega^*\tau_0^*}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{21}|^2}{3}) + \frac{g_{21}}{2}
\]

\[
\mu_2 = \frac{-\text{Re}\{C_1(0)\}}{\text{Re}\{\frac{d\lambda(\tau_0^*)}{dt}\}}
\]

\[
\beta_2 = 2\text{Re}\{C_1(0)\}
\]

\[
T_2 = \frac{-\text{Im}\{C_1(0) + \mu_2\text{Im}\{\frac{d\lambda(\tau_0^*)}{dt}\}\}}{\omega^*\tau_0^*}
\]

These formulae give a description of the Hopf bifurcation periodic solutions of system (1.1) at \(\tau = \tau_0^*\) on the center manifold. Hence we have the following result.

**Theorem 2.1.** The periodic solutions is supercritical (resp. subcritical) if \(\mu_2 > 0\) (resp. \(\mu_2 < 0\)). The bifurcating periodic solutions are orbitally asymptotically stable with an asymptotical
phase (resp.unstable) if $\beta_2 < 0$ (resp. $\beta_2 > 0$). The period of bifurcating periodic solutions increases (resp. decreases) if $T_2 > 0$ (resp. $T_2 < 0$).

4. Conclusion

We have derived the bifurcating periodic solutions areorbitally asymptotically stable with an asymptotical phase if $\beta_2 < 0$ and unstable if $\beta_2 > 0$ and the period of bifurcating periodic solutions increases if $T_2 > 0$ and decreases if $T_2 < 0$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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