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STABILITY OF A STAGE-STRUCTURED PREDATOR-PREY MODEL WITH ALLEE EFFECT AND HARVESTING

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Abstract. A stage-structured predator-prey model with Allee effect on predator species and harvesting on prey species is introduced and studied. The local stability of the positive equilibrium is discussed by the sign of eigenvalue. Furthermore, by using the iterative method, some suitable sufficient conditions for the global attractivity of the equilibria are obtained. Our results show that the Allee effect has no influence on the local stability and global attractivity of the interior equilibrium, such a result is different from the known results. Furthermore, numeric simulations show the interior equilibrium spends a much longer time to achieve the stable state due to the Allee effect.

Keywords: predator-prey; stage-structured; Allee effect; harvesting; global stability.

2010 Mathematics Subject Classification: 34C25, 92D25, 34D20, 34D40.

1. Introduction

In the classical predator-prey model, many scholars assumed that each prey species has the same fertility and survivability, but this is not realistic. As is well known, there exist numerous species whose individual have a life history that divide them into two stages: the mature and

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immature. In fact, the species exhibit different characteristic at different stages. Hence, there is much practical significance to consider the difference of species stage.

Since the pioneer work of Aiello and Freedman [1] on the single species stage-structured models, the stage-structured model have been extensively studied by many scholars see [1], [2], [3], [5], [9], [11], [16]-[20], [22], [25], [28], [29] and the references cited therein.

In [2], Song and Chen proposed the predator-prey system with stage structure and optimal harvesting.

$$\begin{aligned} \dot{x}_i(t) &= \alpha x_m(t) - \gamma x_i(t) - \alpha e^{-\gamma\tau} x_m(t-\tau), \\ \dot{x}_m(t) &= \alpha e^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t) - a_1 x_m(t) y(t) - E x_m(t), \\ \dot{y}(t) &= y(t)\big(-r + a_2 x_m(t) - b y(t)\big), \end{aligned} \tag{1.1}$$

where $x_i(t), x_m(t), y(t)$ can be described as the immature and mature prey, the predator densities at time t, $e^{-\gamma\tau}$ denotes the surviving rate of immaturity to reach maturity. E can be regarded as the harvesting effort. All parameters of system (1.1) are positive constants. Song and Chen showed that stage structure of the prey and optimal harvesting play an important role in the stability of the system (1.1).

In 1931, Allee [13] found that the population growth rate is negative or decreasing function at low population size, which is now called as the Allee effect. The Allee effect can be caused by a variety of biological and environmental factors: lack of mates, low rate of mating success, social dysfunction of reproduction, food exploitation and inbreeding depression etc. During the last decade, many scholars ([4], [6], [7], [8], [10], [12], [13], [14], [26], [27], [30]) investigated the dynamic behaviors of the population model incorporating the Alleee effect.

Merdan [6] introduced a predator-prey population dynamics system subject to Allee effect on the prey population.

$$\begin{aligned} \dot{x}(t) &= \frac{x(t)}{\beta + x(t)} r x(t)\big(1 - x(t)\big) - a x(t) y(t), \\ \dot{y}(t) &= y(t) a\big(x(t) - y(t)\big), \end{aligned}$$

where $\dfrac{x(t)}{\beta + x(t)}$ is the term for Allee effect and $\beta > 0$ can be called the Allee effect constant. His results showed that the Allee effect on the prey population decreases the population densities of both species at the stable steady-state solutions and the system spends a much longer time to achieve its stable steady-state solution as increases in the Allee effect.

Guan, Liu and Xie [7] argued that the higher the hierarchy in the food chain, the more likely it is to become extinct. They proposed the following predator prey model with predator species subject to Allee effect:

$$\frac{dx}{dt} = rx(1-x) - axy, \ \ \frac{dy}{dt} = a\frac{y}{\beta + y}y(x-y),$$

where β is positive constant, represents the Allee effect of the predator species, r, a are positive constants. Unlike the results of Merdan[6], they showed that the Allee effect has no influence on the final density of the predator and prey species.

Recently, Wu, Li and Lin[8] proposed the following two species commensal symbiosis model with Holling type functional response and Allee effect on the second species

$$\begin{aligned} \frac{dx}{dt} &= x\Big(a_1 - b_1 x + \frac{c_1 y^p}{1+y^p}\Big), \\ \frac{dy}{dt} &= y(a_2 - b_2 y)\frac{y}{u+y}, \end{aligned}$$

where $a_i, b_i, i = 1,2$ p, u and c_1 are all positive constants, $p \geq 1$. They also showed that the unique positive equilibrium is globally stable and the system always permanent, consequently, Allee effect has no influence on the final density of the species.

It bring to our attention that to this day, still no scholar investigate the influence of the Allee effect on the stage structure model, also, since the creatures of higher rank are more likely to die out or have Allee effect, due to the lack of food, it is necessary to investigated the predator prey system incorporating Allee effect to predator species.

In this paper, base on the works of Song and Chen [2], we propose the stage-structured predator-prey model with Allee effect on predator species and harvesting on prey species.

$$\begin{aligned} \dot{x}_i(t) &= \alpha x_m(t) - \gamma x_i(t) - \alpha e^{-\gamma\tau} x_m(t-\tau), \\ \dot{x}_m(t) &= \alpha e^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t) - a_1 x_m(t) y(t) - E x_m(t), \\ \dot{y}(t) &= \frac{y^2(t)}{A + y(t)}\big(-r + a_2 x_m(t) - b y(t)\big), \end{aligned} \tag{1.2}$$

the initial conditions for system (1.2) take the form

$$\begin{aligned}
&x_i(\theta) = \phi_m(\theta) \geq 0,\ x_m(\theta) = \phi_m(\theta) \geq 0,\ y(\theta) = \varphi(\theta) \geq 0,\ -\tau \leq \theta < 0,\\
&x_i(0) > 0,\ x_m(0) > 0,\ y(0) > 0,
\end{aligned}$$

where $\big(\phi_i(\theta),\, \phi_m(\theta),\, \varphi(\theta)\big) \in C\big([-\tau, 0], R_+^3\big)$, $x_i(t), x_m(t), y(t)$ can be described as the immature and mature prey, the predator densities at time t. $\alpha, \gamma, \tau, \beta, a_1, E, r, a_2, b$ are defined as in system (1.1). All the parameters of system (1.2) are positive. $B(y) = \dfrac{y}{A+y}$ is the term for Allee effect and $A > 0$ is called the Allee effect constant. According to biological fact, the function $B(y)$ satisfies the following conditions:

(1)If $y = 0$ then $B(y) = 0$, that is, there is no reproduction without partners;

(2)$\dot{B}(y) > 0$ for $y \in (0, \infty)$, that is, the Allee effect decreases as predator population increases.

(3)$\lim\limits_{y \to \infty} B(y) = 1$, that is, the Allee effect vanishes at high densities.

For the continuity of the solutions to system (1.2), in this paper, we require

$$x_i(0) = \int_{-\tau}^{0} \alpha e^{-\gamma s} \phi_m(s) ds. \tag{1.3}$$

Now integrating both sides of the first equation of system (1.2) over (0,t),we obtain that

$$x_i(t) = \int_{t-\tau}^{t} \alpha e^{-\gamma(t-s)} x_m(s) ds. \tag{1.4}$$

From (1.4), one could easily see that the dynamic behaviors of $x_i(t)$ is determined by $x_m(t)$. Hence, we only need to analyse the the following subsystem of the system (1.2).

$$\begin{aligned}
\dot{x}_m(t) &= \alpha e^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t) - a_1 x_m(t) y(t) - E x_m(t),\\
\dot{y}(t) &= \frac{y^2(t)}{A+y(t)}\big(-r + a_2 x_m(t) - b y(t)\big),\\
x_m(\theta) &= \phi_m(\theta) \geq 0, y(\theta) = \varphi(\theta) \geq 0, -\tau \leq \theta < 0,\\
x_m(0) &> 0, y(0) > 0.
\end{aligned} \tag{1.5}$$

2. Local stability

According to the equations of system (1.5), if $E < \dfrac{a_2\alpha e^{-\gamma\tau}-\beta r}{a_2}$, then system (1.5) admits three nonnegative equilibria.

$$E_0(0,0),\quad E_1(\frac{\alpha e^{-\gamma\tau}-E}{\beta},0),\quad E_2(x_m^*,y^*),$$

where $x_m^* = \dfrac{a_1 r + b(\alpha e^{-\gamma\tau}-E)}{a_1a_2+\beta b}$, $y^* = \dfrac{a_2(\alpha e^{-\gamma\tau}-E)-\beta r}{a_1a_2+\beta b}$.

Lemma 2.1. *Assume $\phi(\theta)\geq 0$ is continuous on $\theta\in[-\tau,0]$, $x_m(0)$, $y(0)>0$, then the solutions of system (1.5) with initial condition are positive for all $t>0$.*

Proof. The proof of $x_m(t)>0$ for all $t>0$ is similar to the proof of Theorem 2.1 in [25], so we omit its proof.

Now, we prove $y(t)>0$ for all t. Integrating both sides of the second equation of system (1.5) over $(0,t)$,we obtain that

$$y(t) = y(0)\exp\int_0^t \frac{y(s)\big(-r+a_2x_m(s)-by(s)\big)}{A+y(s)}ds > 0.$$

Theorem 2.1. *Assume that $E < \dfrac{a_2\alpha e^{-\gamma\tau}-\beta r}{a_2}$, then the positive equilibrium point $E_2(x_m^*,y^*)$ is locally stable.*

Proof. The variational matrix of the system (1.5) at the equilibrium point $E_2(x_m^*,y^*)$ is

$$V(E_2) = \begin{bmatrix} \alpha e^{-(\gamma+\lambda)\tau}-2\beta x_m^*-a_1y^*-E & -a_1x_m^* \\ \dfrac{a_2y^{*2}}{A+y^*} & \dfrac{-by^{*2}}{A+y^*} \end{bmatrix}.$$

The characteristic equation at the equilibrium point E_2 is

$$F(\lambda,\tau) \;=\; (\lambda-\alpha e^{-(\gamma+\lambda)\tau}+2\beta x_m^*+a_1y^*+E)(\lambda+\frac{by^{*2}}{A+y^*})+\frac{a_1a_2x_m^*y^{*2}}{A+y^*}=0.$$

It is easy to check that $\lambda = -\dfrac{by^{*2}}{A+y^*}$ does not satisfy the above equation, so we have

$$F(\lambda,\tau) = (\lambda+\frac{by^{*2}}{A+y^*})G(\lambda)=0,$$

where

$$G(\lambda)=\lambda-\alpha e^{-(\gamma+\lambda)\tau}+2\beta x_m^*+a_1y^*+E+\frac{a_1a_2x_m^*y^{*2}}{(A+y^*)(\lambda+\frac{by^{*2}}{A+y^*})},$$

which implies that all solutions of $F(\lambda,\tau)=0$ are given by

$$G(\lambda)=\lambda-\alpha e^{-(\gamma+\lambda)\tau}+2\beta x_m^*+a_1y^*+E+\frac{a_1a_2x_m^*y^{*2}}{(A+y^*)(\lambda+\frac{by^{*2}}{A+y^*})}=0.$$

We declare that $Re\lambda<0$. We will prove it by contradiction. Suppose that $Re\lambda\geq 0$, we can obtain

$$\begin{aligned}
Re\lambda &= \alpha e^{-\gamma\tau}e^{-Re(\lambda)\tau}\cos(Im\lambda)\tau-2\beta x_m^*-a_1y^*-E\\
&\quad-\frac{a_1a_2x_m^*y^{*2}\left[(A+y^*)Re\lambda+by^{*2}\right]}{\left[(A+y^*)Re\lambda+by^{*2}\right]^2+\left[Im\lambda(A+y^*)\right]^2}\\
&= \alpha e^{-\gamma\tau}e^{-Re(\lambda)\tau}\cos(Im\lambda)\tau-\beta x_m^*-\alpha e^{-\gamma\tau}\\
&\quad-\frac{a_1a_2x_m^*y^{*2}\left[(A+y^*)Re\lambda+by^{*2}\right]}{\left[(A+y^*)Re\lambda+by^{*2}\right]^2+\left[Im\lambda(A+y^*)\right]^2}\\
&\leq -\beta x_m^*-\frac{a_1a_2x_m^*y^{*2}\left[(A+y^*)Re\lambda+by^{*2}\right]}{\left[(A+y^*)Re\lambda+by^{*2}\right]^2+\left[Im\lambda(A+y^*)\right]^2}\\
&< 0.
\end{aligned}$$

It is a contradiction, hence, $Re\lambda<0$. Above analysis shows that under the conditions of Theorem 2.2 the positive equilibrium E_2 is locally asymptotically stable.

3. Global attractivity

Lemma 3.1. ([11])*Consider the following equation:*

$$\dot{x}(t)=ax(t-\tau)-bx(t)-cx^2(t), \tag{3.1}$$

where $a, c, \tau>0, b\geq 0,$ and $x(t)>0,$ $for-\tau\leq t\leq 0.$
(1)If $a>b$, then $\lim\limits_{t\to\infty}x(t)=\dfrac{(a-b)}{c}$.
(2)If $a\leq b$, then $\lim\limits_{t\to\infty}x(t)=0$.

Lemma 3.2.([23]) *Consider the following equation*

$$\dot{N}=NF(N). \tag{3.2}$$

Assume that function $F(N)$ satisfies the following conditions.

(1)There is a N^*, such that $F(N^*) = 0$.

(2)For all $N^* > N > 0$, $F(N) > 0$.

(3)For all $N > N^* > 0$, $F(N) < 0$.

Then the system (3.2) is global stability.

Lemma 3.3. *Consider the following equation*

$$\dot{y}(t) = \frac{y^2(t)}{A + y(t)}\big(-r + a_2K - by(t)\big), \tag{3.3}$$

where A, r, a_2, b, K are positive constants. Assume that $r < a_2K$, then

$$\lim_{t\to\infty} y(t) = \frac{a_2K - r}{b}.$$

Proof.

$$\begin{aligned}
\dot{y}(t) &= \frac{y^2(t)}{A + y(t)}\big(-r + a_2K - by(t)\big)\\
&= y(t)\Big(\frac{-ry(t) + a_2Ky(t) - by^2(t)}{A + y(t)}\Big)\\
&= y(t)F\big(y(t)\big).
\end{aligned}$$

Let

$$F(y) = F_1(y)F_2(y),$$

where

$$\begin{aligned}
F_1(y) &= \frac{y}{A + y},\\
F_2(y) &= -r + a_2K - by.
\end{aligned} \tag{3.4}$$

Clearly, if $y > 0$, then $F_1(y) > 0$.

According to (3.4) and the assumptions, we obtain

$$F_2(0) = -r + a_2K > 0,$$

$$\dot{F}_2(y) = -b < 0.$$

Hence, there exists a $y^* = \dfrac{a_2K - r}{b}$ such that for all $0 < y < y^*,\ F_2(y) > 0,\ $ for all $y^* < y,\ F_2(y) <$ 0.

Above analysis shows that

(1)There exists a $y^* = \dfrac{a_2K - r}{b}$ such that $F(y^*) = 0$.

(2) for all $0 < y < y^*$, $F(y) > 0$.

(3)for all $y^* < y$, $F_2(y) < 0$.

It follow from Lemma 3.2 that $\lim\limits_{t\to\infty} y(t) = \dfrac{a_2K - r}{b}$.

Lemma 3.4.([24])(Fluctuation lemma)*Let $x(t)$ be a bounded differentiable function on $[\alpha, \infty)$. Then there exist sequences $\tau_n \to \infty$ and $\sigma_n \to \infty$ such that*

(1)$\dot{x}(\tau_n) \to 0$ and $x(\tau_n) \to \limsup\limits_{t\to\infty} x(t) = \overline{x}$ as $n \to \infty$.

(2)$\dot{x}(\sigma_n) \to 0$ and $x(\sigma_n) \to \liminf\limits_{t\to\infty} x(t) = \underline{x}$ as $n \to \infty$.

Lemma 3.5. *Assume that $a_2K \le r$, then system (3.3) admits a unique nonnegative equilibrium $y = 0$, which is globally asymptotically stable.*

Proof. Let $y(t)$ be any nonnegative solution of system (3.3). According to $a_2K \le r$, we obtain $\dot{y}(t) < 0$, which implies that $y(t)$ is a bounded differentiable function. By Lemma 3.4, there exist sequences $\tau_n \to \infty$, such that $y(\tau_n) \to \limsup\limits_{t\to\infty} y(t) = \overline{y}$. From system (3.3), we obtain

$$\dot{y}(t) \mid_{t=\tau_n} = \frac{y^2(\tau_n)}{A + y(\tau_n)}\big(-r + a_2K - by(\tau_n)\big).$$

Let $n \to \infty$, we have

$$0 = \frac{\overline{y}^2}{A + \overline{y}}\big(-r + a_2K - b\overline{y}\big).$$

From the above equation, we obtain $\overline{y} = 0$ or $\overline{y} = \dfrac{a_2K - r}{b}$. It follows from $a_2K < r$ that $\overline{y} = \dfrac{a_2K - r}{b} < 0$, which is not the nonnegative equilibrium, hence, $0 \le \liminf\limits_{t\to\infty} y(t) \le \limsup\limits_{t\to\infty} y(t) = \overline{y} = 0$.

Theorem 3.1. *Assume that $\alpha e^{-\gamma\tau} \le E$, then $E_0(0,0)$ is globally attractive.*

Proof. It follows from the first equation of the system (1.5) that

$$\dot{x}_m(t) \quad \le \quad \alpha e^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t) - E x_m(t). \tag{3.5}$$

Consider the following system

$$\dot{u}(t) = \alpha e^{-\gamma\tau} u(t-\tau) - \beta u^2(t) - E u(t), t \ge 0,$$

$$u(t) = \varphi(t), -\tau \le t \le 0.$$

It follows from $\alpha e^{-\gamma\tau} \le E$ and Lemma 3.1 that $\lim\limits_{t\to\infty} u(t) = 0$. By the comparison principle, we obtain $x_m(t) \le u(t), t \ge 0$. Therefore, $\lim\limits_{t\to\infty} x_m(t) = 0$. That is, for any sufficiently small $\varepsilon > 0$, without lose of generality, we may assume that $0 < \varepsilon < r/a_2$, there exists a $T_1^* > 0$ such that

$$x_m(t) < \varepsilon,\ t \ge T_1^*. \tag{3.6}$$

From the second equation of system (1.5) and (2.6), we obtain

$$\dot{y}(t) \le y(t)\big(-r + a_2\varepsilon - by(t)\big),\ t \ge T_1^*. \tag{3.7}$$

Let $P_0 = -r + a_2\varepsilon$, then $P_0 < 0$. From Lemma 3.5, $\lim\limits_{t\to\infty} y(t) = 0$. Hence, there exists a $T_2^* > T_1^*$ such that

$$y(t) < \varepsilon,\ t \ge T_2^*. \tag{3.8}$$

Above analysis shows that if $\alpha e^{-\gamma\tau} \le E$, then $E_0(0,0)$ is globally attractive.

Theorem 3.2. *Assume* $\dfrac{a_2\alpha e^{-\gamma\tau} - \beta r}{a_2} \le E < \alpha e^{-\gamma\tau}$ *hold, then* $E_1(\dfrac{\alpha e^{-\gamma\tau} - E}{\beta}, 0)$ *is globally attractive.*

Proof. It follows from the first equation of the system (1.5) that

$$\dot{x}_m(t) \ \le\ \alpha e^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t) - E x_m(t). \tag{3.9}$$

Consider the following system

$$\begin{aligned} &\dot{u}(t) = \alpha e^{-\gamma\tau} u(t-\tau) - \beta u^2(t) - E u(t), t \ge 0,\\ &u(t) = \varphi(t), -\tau \le t \le 0. \end{aligned}$$

It follows $E < \alpha e^{-\gamma\tau}$ and Lemma 3.1 that $\lim\limits_{t\to\infty} u(t) = \dfrac{\alpha e^{-\gamma\tau} - E}{\beta}$. By the comparison principle, we obtain $x_m(t) \le u(t), t \ge 0$. Therefore, $\limsup\limits_{t\to+\infty} x_m(t) \le \dfrac{\alpha e^{-\gamma\tau} - E}{\beta}$. That is, for any sufficiently small $\varepsilon > 0$, without lose of generality, we may assume that $0 < \varepsilon < \dfrac{r\beta - a_2(\alpha e^{-\gamma\tau} - E)}{2\beta a_2}$, there exists a $\widetilde{T}_1 > 0$ such that

$$x_m(t) < \frac{\alpha e^{-\gamma\tau} - E}{\beta} + \varepsilon,\ t \ge \widetilde{T}_1. \tag{3.10}$$

From the second equation of system (1.5) and (3.10), we obtain

$$\dot{y}(t) \le y(t)\big(-r + a_2(\frac{\alpha e^{-\gamma\tau} - E}{\beta} + \varepsilon) - by(t)\big),\ t \ge \widetilde{T}_1. \tag{3.11}$$

Let $P_1 = -r + \dfrac{a_2(\alpha e^{-\gamma\tau} - E)}{\beta} + a_2\varepsilon$, then $P_1 \leq 0$. From Lemma 3.5, $\lim\limits_{t\to\infty} y(t) = 0$. Hence, there exists a $\widetilde{T}_2 > \widetilde{T}_1$ such that

$$y(t) < \varepsilon,\ t \geq \widetilde{T}_2. \tag{3.12}$$

It follows from the first equation of the system (1.5) that

$$\dot{x}_m(t) \quad \geq \quad \alpha e^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t) - a_1 x_m(t)\varepsilon - E x_m(t). \tag{3.13}$$

Consider the following system

$$\dot{u}(t) = \alpha e^{-\gamma\tau} u(t-\tau) - \beta u^2(t) - a_1 u(t)\varepsilon - E u(t),$$

$$u(t) = \varphi(t), \widetilde{T}_2 \leq t \leq \widetilde{T}_2 + \tau.$$

According to $\alpha e^{-\gamma\tau} > E$ and Lemma 3.1, we obtain that $\lim\limits_{t\to\infty} u(t) = \dfrac{\alpha e^{-\gamma\tau} - E}{\beta}$. By the comparison principle, we obtain $x_m(t) \geq u(t), t \geq \widetilde{T}_2$. Hence, for the above ε, there exists a $\widetilde{T}_3 > \widetilde{T}_2$ such that

$$x_m(t) > \frac{\alpha e^{-\gamma\tau} - E}{\beta} - \varepsilon,\ t \geq \widetilde{T}_3. \tag{3.14}$$

Above analysis shows that if $\dfrac{a_2\alpha e^{-\gamma\tau} - \beta r}{a_2} \leq E < \alpha e^{-\gamma\tau}$ holds, then $E_1(\dfrac{\alpha e^{-\gamma\tau} - E}{\beta}, 0)$ is globally attractive.

Theorem 3.3. *Assume $a_1 a_2 r < \beta b r < a_2 b(\alpha e^{-\gamma\tau} - E)$ hold, then the unique positive equilibrium of the system (1.5) is globally attractive.*

Proof. It follows from the first equation of the system (1.5) that

$$\dot{x}_m(t) \quad \leq \quad \alpha e^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t) - E x_m(t). \tag{3.15}$$

Consider the following system

$$\dot{u}(t) = \alpha e^{-\gamma\tau} u(t-\tau) - \beta u^2(t) - E u(t), t \geq 0,$$

$$u(t) = \varphi(t), -\tau \leq t \leq 0.$$

It follows Lemma 3.1 that $\lim\limits_{t\to\infty} u(t) = \dfrac{\alpha e^{-\gamma\tau} - E}{\beta}$. By the comparison principle, we obtain $x_m(t) \leq u(t), t \geq 0$. Therefore, for any sufficiently small $\varepsilon > 0$, without lose of generality, we may assume that

$$\varepsilon < \min\left\{\frac{b\beta - a_1 a_2\Big(a_2(\alpha e^{-\gamma\tau} - E) - \beta r\Big)}{\beta a_2(a_1 a_2 + a_1 b + b\beta)}, \frac{(b\beta - a_1 a_2)(\alpha e^{-\gamma\tau} - E) + a_1\beta r}{\beta a_1(a_2 + b)}\right\},$$

there exists a $T_1 > 0$ such that

$$x_m(t) < \frac{\alpha e^{-\gamma\tau} - E}{\beta} + \varepsilon = \bar{u}_1,\ t \geq T_1. \tag{3.16}$$

From the second equation of system (1.5) and (3.16), we obtain

$$\dot{y}(t) \leq y(t)\big(-r + a_2\bar{u}_1 - by(t)\big),\ t \geq T_1.$$

Let $P_2 = -r + a_2\bar{u}_1 = \dfrac{a_2\alpha e^{-\gamma\tau} - a_2 E - \beta r}{\beta} + a_2\varepsilon$, then $P_2 > 0$. From Lemma 3.3, there exists a $T_2 > T_1$ such that

$$y(t) < \frac{a_2\bar{u}_1 - r}{b} + \varepsilon = \bar{v}_1,\ t \geq T_2. \tag{3.17}$$

From (3.17) and the first equation of system (1.5), we obtain

$$\dot{x}_m(t) \geq \alpha^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t) - a_1 x(t)\bar{v}_1 - E x_m(t),\ t \geq T_2 + \tau.$$

Consider the following system

$$\dot{u}(t) = \alpha^{-\gamma\tau} u(t-\tau) - \beta u^2(t) - a_1 u(t)\bar{v}_1 - E u(t), t \geq T_2 + \tau,$$

$$u(t) = \varphi(t),\ T_2 \leq t \leq T_2 + \tau.$$

Let

$$\begin{aligned} P_3 &= \alpha^{-\gamma\tau} - E - a_1\bar{v}_1 \\ &= \frac{(b\beta - a_1 a_2)(\alpha^{-\gamma\tau} - E) + a_1\beta r}{\beta b} - \frac{a_1(a_2 + b)\varepsilon}{b}. \end{aligned}$$

For the above ε, we obtain $P_3 > 0$. It follows from Lemma 3.1 that $\lim\limits_{t\to\infty} u(t) = \dfrac{\alpha e^{-\gamma\tau} - a_1\bar{v}_1 - E}{\beta}$. By the comparison principle, we have $x_m(t) \geq u(t), t \geq T_2 + \tau$. there exists a $T_3 > T_2 + \tau > 0$ such that

$$x_m(t) > \frac{\alpha e^{-\gamma\tau} - a_1\bar{v}_1 - E}{\beta} - \varepsilon = \underline{u}_1. \tag{3.18}$$

From (3.16) and (3.18), we obtain

$$\underline{u}_1 < \bar{u}_1. \tag{3.19}$$

From (3.18) and the second equation of system (1.5), we obtain

$$\dot{y}(t) \geq \frac{y^2(t)}{A+y(t)}\big(-r+a_2\underline{u}_1-by(t)\big),\ t \geq T_3.$$

Let

$$\begin{array}{rcl} P_4 & = & -r+a_2\underline{u}_1 \\ & = & \dfrac{(b\beta-a_1a_2)\Big(a_2(\alpha e^{-\gamma\tau}-E)-r\beta\Big)}{b\beta^2}-\dfrac{a_2\varepsilon(a_1a_2+a_1b+b\beta)}{b\beta}. \end{array}$$

For the above ε, we obtain $P_4 > 0$. By comparison principle and Lemma 3.3, there exists a $T_4 > T_3$ such that

$$y(t) > \frac{a_2\underline{u}_1-r}{b}-\varepsilon = \underline{v}_1,\ t \geq T_4. \tag{3.20}$$

From (3.17), (3.19) and (3.20), we obtain

$$\underline{v}_1 < \overline{v}_1. \tag{3.21}$$

From (3.20) and the first equation of system (1.5), we obtain

$$\dot{x}_m(t) \leq \alpha^{-\gamma\tau}x_m(t-\tau)-\beta x_m^2(t)-a_1x(t)\underline{v}_1-Ex_m(t),\ t \geq T_4+\tau.$$

From (3.21), we obtain $0 < P_3 = \alpha e^{-\gamma\tau}-a_1\overline{v}_1-E < \alpha e^{-\gamma\tau}-a_1\underline{v}_1-E$. Hence, by the similar arguments as above, for the above $\varepsilon > 0$, there exists a $T_5 > T_4+\tau > 0$ such that

$$x_m(t) < \frac{\alpha e^{-\gamma\tau}-E-a_1\underline{v}_1}{\beta}+\frac{\varepsilon}{2} = \overline{u}_2. \tag{3.22}$$

From (3.16) and (3.22), we can have

$$\overline{u}_1 > \overline{u}_2. \tag{3.23}$$

From (3.22) and the second equation of system (1.5), we obtain

$$\dot{y}(t) \leq \frac{y^2(t)}{A+y(t)}\big(-r+a_2\overline{u}_2-by(t)\big),\ t \geq T_5.$$

From (3.18), (3.21) and (3.22), we obtain

$$\overline{u}_2 > \underline{u}_1 \tag{3.24}$$

From (3.24), we obtain $a_2\overline{u}_2 - r > a_2\underline{u}_1 - r = P_4 > 0$. Hence, by comparison theorem and Lemma 3.3, there exists a $T_6 > T_5$ such that

$$y(t) < \frac{a_2\overline{u}_2 - r}{b} + \frac{\varepsilon}{2} = \overline{v}_2,\ t \geq T_6. \tag{3.25}$$

From (3.18), (3.23) and (3.25), we obtain

$$\overline{v}_2 < \overline{v}_1. \tag{3.26}$$

From (3.25) and the first equation of system (1.5), we obtain

$$\dot{x}_m(t) \geq \alpha e^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t) - a_1 x_m(t)\overline{v}_2 - E x_m(t),\ t > T_6 + \tau.$$

From (3.26), we obtain $\alpha e^{-\gamma\tau} - a_1\overline{v}_2 - E > \alpha e^{-\gamma\tau} - a_1\overline{v}_1 - E = P_3 > 0$. Hence, by the similar arguments as above, for the above ε, there exists a $T_7 > T_6 + \tau > 0$ such that

$$x_m(t) > \frac{\alpha e^{-\gamma\tau} - a_1\overline{v}_2 - E}{\beta} - \frac{\varepsilon}{2} = \underline{u}_2. \tag{3.27}$$

According to (3.21), (3.26) and (3.27), we obtain

$$\underline{u}_2 > \underline{u}_1. \tag{3.28}$$

From (3.27) and the second equation of system (1.5), we obtain

$$\dot{y}(t) \geq \frac{y^2(t)}{A + y(t)}\big(-r + a_2\underline{u}_2 - by(t)\big),\ t \geq T_7.$$

From (3.28), we obtain $-r + a_2\underline{u}_2 > -r + a_2\underline{u}_1 = P_4 > 0$. Hence, by comparison theorem and Lemma 3.3, there exists a $T_8 > T_7$ such that

$$y(t) > \frac{a_2\underline{u}_2 - r}{b} - \frac{\varepsilon}{2} = \underline{v}_2,\ t \geq T_8. \tag{3.29}$$

From (3.20), (3.28) and (3.29) we obtain

$$\underline{v}_2 > \underline{v}_1. \tag{3.30}$$

Repeating the above steps, we can obtain four sequences $\{\overline{u}_n\}_{n=1}^{\infty}$, $\{\underline{u}_n\}_{n=1}^{\infty}$, $\{\underline{v}_n\}_{n=1}^{\infty}$, $\{\overline{v}_n\}_{n=1}^{\infty}$, and $t \geq T_{4n} > 0$, such that

$$\overline{u}_n = \frac{\alpha e^{-\gamma\tau} - a_1\underline{v}_{n-1} - E}{\beta} + \frac{\varepsilon}{n},\ \overline{v}_n = \frac{a_2\overline{u}_n - r}{b} + \frac{\varepsilon}{n}, \tag{3.31}$$

$$\underline{u}_n = \frac{\alpha e^{-\gamma\tau} - a_1\overline{v}_n - E}{\beta} - \frac{\varepsilon}{n},\ \underline{v}_n = \frac{a_2\underline{u}_n - r}{b} - \frac{\varepsilon}{n}. \tag{3.32}$$

We can obtain that

$$0 < \underline{u}_1 < \underline{u}_2 < \cdots < \underline{u}_n < x_m(t) < \overline{u}_n < \cdots < \overline{u}_2 < \overline{u}_1,$$

$$0 < \underline{v}_1 < \underline{v}_2 < \cdots < \underline{v}_n < y(t) < \overline{v}_n < \cdots < \overline{v}_2 < \overline{v}_1.$$

Hence, the limits of $\{\overline{u}_n\}_{n=1}^{\infty}$, $\{\underline{u}_n\}_{n=1}^{\infty}$, $\{\underline{v}_n\}_{n=1}^{\infty}$, $\{\overline{v}_n\}_{n=1}^{\infty}$ exist. Set

$$\overline{u} = \lim_{n\to\infty}\overline{u}_n,\ \overline{v} = \lim_{n\to\infty}\overline{v}_n,\ \underline{u} = \lim_{n\to\infty}\underline{u}_n,\ \underline{v} = \lim_{n\to\infty}\underline{v}_n.$$

It follow from (3.31) and (3.32) that

$$\overline{u} = \frac{\alpha e^{-\gamma\tau} - a_1\underline{v} - E}{\beta},\ \overline{v} = \frac{a_2\overline{u} - r}{b}, \tag{3.33}$$

$$\underline{u} = \frac{\alpha e^{-\gamma\tau} - a_1\overline{v} - E}{\beta},\ \underline{v} = \frac{a_2\underline{u} - r}{b}. \tag{3.34}$$

From (3.33) and (3.34), we obtain

$$b\alpha e^{-\gamma\tau} - bE - a_1a_2\underline{u} + a_1r = \beta b\overline{u},$$

$$b\alpha e^{-\gamma\tau} - bE - a_1a_2\overline{u} + a_1r = \beta b\underline{u},$$

which is equivalent to

$$(a_1a_2 - b\beta)(\overline{u} - \underline{u}) = 0.$$

Since $a_1a_2 < b\beta$, it immediately follows that $\overline{u} = \underline{u}$. Consequently $\overline{v} = \underline{v}$. That is

$$\lim_{t\to\infty} x_m(t) = x_m^*,\ \lim_{t\to\infty} y(t) = y^*.$$

Hence, $E_2(x_m^*, y^*)$ is global asymptotic stability.

4. Examples

Now let us consider the following examples.

The following examples show the feasibility of main results.

Example 4.1 Consider the following system

$$\begin{aligned} \dot{x}_m(t) &= 3e^{-1}x_m(t-5) - x_m^2(t) - x_m(t)y(t) - 2x_m(t), \\ \dot{y}(t) &= \frac{y^2(t)}{0.5+y(t)}\big(-3+0.2x_m(t)-0.5y(t)\big), \end{aligned} \tag{4.1}$$

where corresponding to system (1.5), we take $\alpha = 3$, $\gamma = 0.2$, $\tau = 5$, $\beta = 1$, $a_1 = 1$, $E = 2$, $A = 0.5$, $r = 3$, $a_2 = 0.2$, $b = 0.5$. Clearly, $1.103 \approx \alpha e^{-\gamma\tau} < E = 2$. Hence, it follows from Theorem 3.6 that $E_0(0,0)$ is globally attractive. Numeric simulation (Figure 1) also supports this assertion.

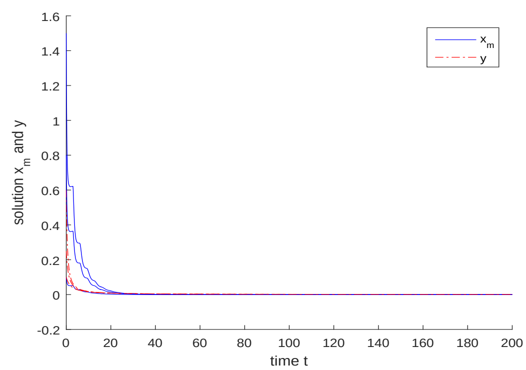

FIGURE 1. Dynamic behaviors of system (4.1) with the initial conditions $\big(x_m(\theta), y(\theta)\big) = (1.5, 0.1), (0.8, 0.25)$ and $(0.1, 0.6)$ for $-5 \le \theta < 0$.

Example 4.2 Consider the following system

$$\begin{aligned} \dot{x}_m(t) &= 3e^{-1}x_m(t-5) - x_m^2(t) - x_m(t)y(t) - 0.5x_m(t), \\ \dot{y}(t) &= \frac{y^2(t)}{0.5+y(t)}\big(-3+0.2x_m(t)-0.5y(t)\big), \end{aligned} \tag{4.2}$$

where corresponding to system (1.5), we take $\alpha = 3$, $\gamma = 0.2$, $\tau = 5$, $\beta = 1$, $a_1 = 1$, $E = 0.5$, $A = 0.5$, $r = 3$, $a_2 = 0.2$, $b = 0.5$, Hence $-13.897 \approx \dfrac{a_2\alpha e^{-\gamma\tau} - \beta r}{a_2} < E = 0.5 < \alpha e^{-\gamma\tau} \approx 1.103$. Hence, it follows from Theorem 3.7 that $E_1(\dfrac{\alpha e^{-\gamma\tau} - E}{\beta}, 0)$ is globally attractive. Numeric simulation (Figure 2) also supports this assertion.

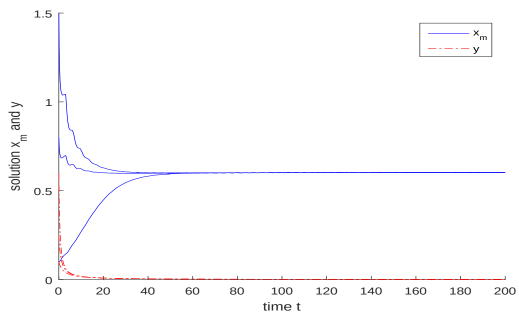

FIGURE 2. Dynamic behaviors of system (4.2) with the initial conditions $\big(x_m(\theta), y(\theta)\big) = (0.1, 0.5), (0.6, 0.25)$ and $(0.3, 0.6)$ for $-5 \le \theta < 0$.

Example 4.3 Consider the following system

$$\begin{aligned} \dot{x}_m(t) &= 10e^{-3}x_m(t-15) - 0.3x_m^2(t) - 0.1x_m(t)y(t) - 0.1x_m(t), \\ \dot{y}(t) &= \frac{y^2(t)}{A+y(t)}\big(-0.4 + 0.5x_m(t) - y(t)\big), \end{aligned} \tag{4.3}$$

where corresponding to system (1.5), we take $\alpha = 10$, $\gamma = 0.2$, $\tau = 15$, $\beta = 0.3$, $a_1 = 0.1$, $E = 0.1$, $r = 0.4$, $a_2 = 0.5$, $b = 1$. One could easily verify that

$$0.02 = a_1a_2r < \beta br = 0.12 < a_2b(\alpha e^{-\gamma\tau} - E) \approx 0.199.$$

Hence, it follows from Theorem 3.8 that $E_2(x_m^*, y^*)$ is globally attractive. Numeric simulation (Figure 3) also supports this assertion. Furthermore, we define the different values of Allee effect $A = 0, A = 2, A = 4$. Numerical simulations (Figure 4, 5) show that the system takes a longer time to reach its stable steady-state solution, as the Allee effect increases and the equilibrium densities of both species at the stable steady state do not change.

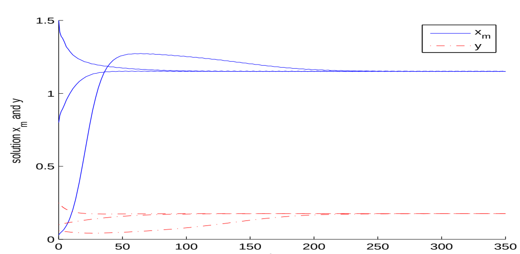

FIGURE 3. Dynamic behaviors of system (4.3) with the initial conditions $\big(x_m(\theta), y(\theta)\big) = (0.1, 0.5), (0.6, 0.25)$ and $(0.3, 0.6)$ for $-15 \le \theta < 0$.

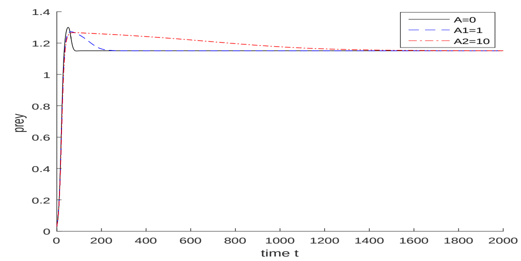

FIGURE 4. Time-population density graphs of the prey population with and without Allee effects, denoted by $A_1 = 2$ and $A_2 = 4$. The initial conditions in both graphs are $x_m(\theta) = 0.3$ and $y(\theta) = 0.6$, for $-15 \leq \theta < 0$.

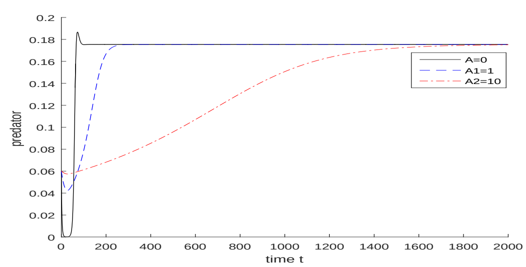

FIGURE 5. Time-population density graphs of the predator population with and without Allee effects, denoted by $A_1 = 2$ and $A_2 = 4$. The initial conditions in both graphs are $x_m(\theta) = 0.3$ and $y(\theta) = 0.6$, for $-15 \leq \theta < 0$.

5. Discussion

In this paper, we consider the dynamic behaviors of the stage-structure predator-prey model with Allee effect on predator species and harvesting on prey species. We discuss the global attractivity of the system. Our results show the harvesting and stage structure play an important role in the dynamic behaviors of the system. If $\alpha e^{-\gamma\tau} < E$, then E_0 is globally attractive. That is, both two species of system goes extinct. If $\alpha e^{-\gamma\tau} > E > \dfrac{a_2\alpha e^{-\gamma\tau} - \beta r}{a_2}$, then E_1 is globally attractive, which implies the predator goes extinct and the prey exists. If $a_1a_2r < \beta br$ and $E < \dfrac{a_2\alpha e^{-\gamma\tau} - \beta r}{a_2}$ then E_2 is global attractive, which implies the two species of system could be coexistence in a stable state. Numeric simulations also support our findings.

In[6], Merdan showed that the Allee effect plays an important role in the local stability and reduces the population densities of both species at the stable steady-state solutions. In [26], Celik showed that the positive equilibrium moves from instability to stability under the Allee effect on prey population. However, our results differ from their results. In our paper, one interesting finding is that the Allee effect has no effect on the local stability of the system and does not change both species at the stable steady-state solutions(see Fig.4 and Fig.5). Furthermore, from the Figure 4 and 5, we see that the system spends a much longer time to achieve its stable steady-state solution as increases in the Allee effect.

Conflict of Interests

The authors declare that there is no conflict of interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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