DYNAMICS OF A CLASS OF VIRAL INFECTION MODELS WITH DIFFUSION

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Abstract. The aim of this work is to study the dynamics of a class of viral infection models with diffusion and loss of viral particles due to the absorption into uninfected cells. We prove the global stability of equilibria by constructing suitable Lyapunov functionals for two cases: continuous and discrete. Also, some examples are given to illustrate the theoretical results.

Keywords: viral infection; reaction-diffusion equations; basic reproduction number; Lyapunuv functional; global stability.

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1. INTRODUCTION

Reaction-diffusion equations modeling several phenomena in different fields such as physics, biology, economics, etc. These equations describe the variations in concentration or density distributed in space under the influence of two processes: the local interactions between species and the diffusion that causes the propagation of species in space. In population dynamics, the

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terms of diffusion correspond to a random movement of individuals and the terms of reaction describe their reproduction [1].

Recently, reaction-diffusion equations are used to describe the dynamics of viral infections and to obtain information on the mechanisms of these viral infections in vivo. In [2], the authors introduced a mathematical model formulated by partial differential equations (PDEs) to describe the hepatitis B virus (HBV) infection that represents a major global health problem. They assumed that infection rate is bilinear and they ignored the absorption of the virus by the uninfected cells. The importance of our work is to consider both nonlinear incidence rate and absorption of the virus by the uninfected cells. Therefore, we propose a generalized viral infection model governed by the following nonlinear system of PDEs:

(1)
$$\begin{cases} \dfrac{\partial T}{\partial t} = \lambda - dT(x,t) - f(T(x,t),I(x,t),V(x,t))V(x,t), \\ \dfrac{\partial I}{\partial t} = f(T(x,t),I(x,t),V(x,t))V(x,t) - aI(x,t), \\ \dfrac{\partial V}{\partial t} = d_v\Delta V + kI(x,t) - if(T,I,V)V - \mu V, \end{cases}$$

where $T(x,t)$, $I(x,t)$ and $V(x,t)$ are the densities of susceptible cells, infected cells and free virus at position x and time t, respectively. Susceptible cells are produced at rate λ, die at rate dT and become infected at rate $f(T,I,V)V$. Infected cells die at rate aI. Free viruses are produced from infected cells at rate kI and are removed at rate μV. d_v is the diffusion coefficient, Δ is the Laplacien operator, and $i \in \{0,1\}$ denotes the absorption effect.

As in [3, 4], we suppose that the function $f(T,I,V)$ is continuously differentiable in $\mathbf{R}^3_+$ and satisfies the following hypotheses:

(H₁): $f(0,I,V) = 0$, for all $I \geq 0$ and $V \geq 0$;

(H₂): $f(T,I,V)$ is a strictly monotone increasing function with respect to T, for any fixed $I \geq 0$ and $V \geq 0$;

(H₃): $\frac{\partial f(T,I,V)}{\partial I} \leq 0$ and $\frac{\partial f(T,I,V)}{\partial V} \leq 0$, which means that $f(T,I,V)$ is a monotone decreasing function with respect to I and V.

It is very important to note that our model represented by system (1), extends and improves many cases exiting in the literature. For instance, if $f(T,I,V) = \beta T$ and $i = 0$, we get the basic

PDE model proposed in [2]. Further, the more recent model presented by Yang and Zhou in [5] is a special case of system (1).

In this work, we are interested in system (1) according to two purposes. The first is to investigate the dynamics of system (1) with initial values and Neumann boundary conditions

(2)
$$
\begin{cases}
T(x,0) = \phi_1(x) \geq 0, \ I(x,0) = \phi_2(x) \geq 0, \ V(x,0) = \phi_3(x) \geq 0, \ x \in \overline{\Omega}, \\
\dfrac{\partial V}{\partial n} = 0, \ t > 0, \ x \in \partial\Omega,
\end{cases}
$$

where Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$, $\dfrac{\partial}{\partial n}$ is an outward normal vector of $\partial\Omega$. The second purpose is to propose some applications and an numerical method that preserves the qualitative properties of the continuous model (1).

The rest of this paper is organized as follows. In section 2, we analysis the continuous version by showing well-posedness, equilibria and global stability. The discrete version is treated in section 3. In section 4, we give some applications of our analytical results. Finally, we conclude our work in section 5.

2. ANALYSIS OF CONTINUOUS VERSION

We first show the well-posedness of the model by proving the global existence, uniqueness, non-negativity and boundedness of solution of model (1) under (2). After, we determine the basic reproduction number, study steady states of the model (1) and discuss the global stability of the infection-free equilibrium and the chronic infection equilibrium.

Theorem 2.1. *For any given initial* $\phi = (\phi_1, \phi_2, \phi_3) \in C = [C(\overline{\Omega})]^3$ *satisfying the condition (2), there exists a unique solution of problem (1)-(2) defined on* $[0, +\infty)$ *and this solution remains non-negative and bounded for all* $t \geq 0$.

Proof. For any $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T \in C$ and $x \in \overline{\Omega}$, we define $F = (F_1, F_2, F_3) : C \longrightarrow \mathbf{R}^3$ by

$$
\begin{aligned}
F_1(\varphi)(x) &= \lambda - d\varphi_1(x) - f(\varphi_1(x), \varphi_2(x), \varphi_3(x))\varphi_3(x), \\
F_2(\varphi)(x) &= f(\varphi_1(x), \varphi_2(x), \varphi_3(x))\varphi_3(x) - a\varphi_2(x), \\
F_3(\varphi)(x) &= k\varphi_2(x) - if(\varphi_1(x), \varphi_2(x), \varphi_3(x))\varphi_3(x) - \mu\varphi_3(x).
\end{aligned}
$$

Then system (1) can be rewritten as the following abstract functional differential equation

$$\begin{cases} w'(t) = Aw(t) + F(w(t)), \\ w(0) = \phi \in C, \end{cases}$$

where $w = (T, I, V)^T$ and $Aw = (0, 0, d_v \Delta V)^T$. It is clear that F is locally Lipschitz in C. Thus, we conclude that problem (1)-(2) has a unique local solution on $[0, t_{max})$, where $[0, t_{max})$ is the maximal time interval on which the solutions are guaranteed to exist [6]. In addition, all solutions are non-negative since 0 is a lower solution of each equation of model (1).

Now, we show the boundedness of the solutions. For this reason, we put

$$U(x, t) = T(x, t) + I(x, t).$$

So, we can get

$$\begin{aligned} \frac{\partial U(x, t)}{\partial t} &= \lambda - dT(x, t) - aI(x, t), \\ &\leq \lambda - \delta U(x, t), \end{aligned}$$

where $\delta = \min\{a, d\}$. Hence,

$$U(x, t) \leq \max\left\{\frac{\lambda}{\delta}, \max_{x \in \overline{\Omega}}\{\phi_1(x) + \phi_2(x)\}\right\}.$$

This implies that T and I are bounded. From the boundedness of I and system (1), we deduce that V satisfies the following system

(3)
$$\begin{cases} \dfrac{\partial V}{\partial t} - d_v \Delta V \leq k\rho - \mu V, \\ \dfrac{\partial V}{\partial n} = 0, \\ V(x, 0) = \phi_3(x) \geq 0, \end{cases}$$

where $\rho = \max\left\{\dfrac{\lambda}{\delta}, \max_{x \in \overline{\Omega}}\{\phi_1(x) + \phi_2(x)\}\right\}$. Let $\widetilde{V}(t)$ be a solution to the ordinary differential equation

$$\begin{cases} \dfrac{d\widetilde{V}}{dt} = k\rho - \mu\widetilde{V}, \\ \widetilde{V}(0) = \max_{x \in \overline{\Omega}} \phi_3(x). \end{cases}$$

Then $\widetilde{V}(t) \leq \max\left\{\dfrac{k\rho}{\mu}, \underset{x\in\overline{\Omega}}{max}\{\phi_3(x)\}\right\}$, for all $t \in [0, t_{max})$.

By the comparison principale [7], we get $V(x,t) \leq \widetilde{V}(t)$. Hence,

$$V(x,t) \leq \max\left\{\frac{k\rho}{\mu}, \underset{x\in\overline{\Omega}}{max}\{\phi_3(x)\}\right\}, \forall (x,t) \in \overline{\Omega} \times [0, t_{max}).$$

From the above, we have proved that $T(x,t)$, $I(x,t)$ and $V(x,t)$ are bounded on $\overline{\Omega} \times [0, t_{max})$. Therefore, it follows from the standard theory for semi-linear parabolic systems [8] that $t_{max} = +\infty$. This completes the proof.

Next, we study the existence of steady states of model (1). Obviously, model (1) has an infection-free equilibrium $E_0(T_0, 0, 0)$, where $T_0 = \dfrac{\lambda}{d}$. Then the basic reproduction number of (1) is given by

$$(4) \qquad\qquad R_0 = \frac{(k-ia)f(\frac{\lambda}{d},0,0)}{\mu a}.$$

To find the other equilibrium of system (1), we resolve

$$(5) \qquad \begin{aligned} \lambda - dT - f(T,I,V)V &= 0, \\ f(T,I,V)V - aI &= 0, \\ kI - if(T,I,V)V - \mu V &= 0. \end{aligned}$$

Adding the first two equations of (5), we get

$$T = \frac{\lambda - aI}{d} \geq 0. \text{ So, } I \in \left[0, \frac{\lambda}{a}\right].$$

Using the third equation of (5), we have

$$V = \frac{(k-ia)I}{\mu}.$$

Replacing T, V into the second equation of (5), we obtain

$$(k-ia)f(\frac{\lambda - aI}{d}, I, \frac{(k-ia)I}{\mu}) - \mu a = 0.$$

We define a function h on $\left[0, \dfrac{\lambda}{a}\right]$ as follows

$$h(I) = (k-ia)f(\frac{\lambda - aI}{d}, I, \frac{(k-ia)I}{\mu}) - \mu a.$$

It is easy to see that $h(\frac{\lambda}{a}) = -\mu a < 0$ and

$$h(0) = (k - ia) f(\frac{\lambda}{d}, 0, 0) - \mu a = a\mu (R_0 - 1).$$

Clearly, we have a positive equilibrium $E^*(T^*, I^*, V^*)$ when $R_0 > 1$.

For any $I \in \left[0, \frac{\lambda}{a}\right]$, we have

$$h'(I) = (k - ia) \left(\frac{-a}{d} \frac{\partial f}{\partial T} + \frac{\partial f}{\partial I} + \frac{(k - ia)}{\mu} \frac{\partial f}{\partial V} \right).$$

Using hypotheses (H_2) and (H_3), we prove the uniqueness of the chronic infection equilibrium $E^*(T^*, I^*, V^*)$. By this computation, we get the following result.

Theorem 2.2. *For system (1),*

  (1)

  (i) *if $R_0 \leq 1$, then there exists a unique infection-free equilibrium E_0,*

  (ii) *if $R_0 > 1$, then there exists a unique chronic infection equilibrium E^* besides E_0.*

Now, we establish the global stability of the infection-free equilibrium and chronic infection equilibrium for system (1).

Theorem 2.3. If $R_0 \leq 1$, then the infection-free equilibrium E_0 is globally asymptotically stable and it is unstable if $R_0 > 1$.

Proof. Construct a Lyapunov functional for system (1) at E_0 as follows

$$L_0 = \int_{\Omega} \left\{ T(x, t) - T_0 - \int_{T_0}^{T} \frac{f(T_0, 0, 0)}{f(X, 0, 0)} dX + \frac{k}{k - ia} I(x, t) + \frac{a}{k - ia} V(x, t) \right\} dx.$$

The time derivative of L_0 along the solution of system (1) satisfies

$$\begin{aligned}
\frac{dL_0}{dt} &= \int_{\Omega} \left\{ \left(1 - \frac{f(T_0, 0, 0)}{f(T, 0, 0)} \right) \frac{\partial T}{\partial t} + \frac{k}{k - ia} \frac{\partial I}{\partial t} + \frac{a}{k - ia} \frac{\partial V}{\partial t} \right\} dx, \\
&= \int_{\Omega} \left\{ dT_0 \left(1 - \frac{T}{T_0} \right) \left(1 - \frac{f(T_0, 0, 0)}{f(T, 0, 0)} \right) + \frac{\mu a}{k - ia} V (R_0 \frac{f(T, I, V)}{f(T, 0, 0)} - 1) \right\} dx.
\end{aligned}$$

Using the hypothesis (H_3), we get

$$\frac{dL_0}{dt} \leq \int_{\Omega} \left\{ dT_0 \left(1 - \frac{T}{T_0} \right) \left(1 - \frac{f(T_0, 0, 0)}{f(T, 0, 0)} \right) + \frac{\mu a}{k - ia} V (R_0 - 1) \right\} dx.$$

Since $f(T,I,V)$ is strictly monotonically increasing with respect to T, we have

$$\left(1 - \frac{T}{T_0}\right)\left(1 - \frac{f(T_0,0,0)}{f(T,0,0)}\right) \leq 0.$$

Therefore, $\dfrac{dL_0}{dt} \leq 0$ when $R_0 \leq 1$. In addition, $\dfrac{dL_0}{dt} = 0$ if and only if $T = T_0$, $V = 0$ and $I = 0$. From LaSalle's invariance principle [9], we deduce that E_0 is globally asymptotically stable if $R_0 \leq 1$. Similarly to [3], we can easily prove the instability of E_0 when $R_0 > 1$.

For the global stability of the chronic infection equilibrium E^*, we assume that $R_0 > 1$ and f satisfies the following hypothesis

$$(\mathbf{H_4}) \quad \left(1 - \frac{f(T,I,V)}{f(T,I^*,V^*)}\right)\left(\frac{f(T,I^*,V^*)}{f(T,I,V)} - \frac{V}{V^*}\right) \leq 0, \ \forall T,I,V > 0.$$

Theorem 2.4. *If $R_0 > 1$ and the hypthesis (H_4) holds, then the chronic infection equilibrium E^* is globally asymptotically stable when $i = 0$.*

Proof. Let us define the following Lyapunov functional:

$$L_1 = \int_\Omega \left\{ T - T^* - \int_{T^*}^T \frac{f(T^*,I^*,V^*)}{f(X,I^*,V^*)} dX + \frac{k}{k-ia} I^* g\left(\frac{I}{I^*}\right) + \frac{a}{k-ia} V^* g\left(\frac{V}{V^*}\right) \right\} dx,$$

with $g(z) = z - 1 - \ln z$. The function $g(z)$ has its minimum 0 at $z = 1$. So, $g(z) \geq 0$ for all $z > 0$.

The time derivative of L_1 along the solution of system (1) satisfies

$$\begin{aligned}
\frac{dL_1}{dt} &= \int_\Omega \left\{ \left(1 - \frac{f(T^*,I^*,V^*)}{f(T,I^*,V^*)}\right)(\lambda - dT - f(T,I,V)V) \right. \\
&\quad + \frac{k}{k-ia}\left(1 - \frac{I^*}{I}\right)\frac{\partial I}{\partial t} + \frac{a}{k-ia}\left(1 - \frac{V^*}{V}\right)\frac{\partial V}{\partial t} \bigg\} dx \\
&= \int_\Omega \left\{ \left(1 - \frac{f(T^*,I^*,V^*)}{f(T,I^*,V^*)}\right)(\lambda - dT - f(T,I,V)V) \right. \\
&\quad + \frac{k}{k-ia}\left(1 - \frac{I^*}{I}\right)(f(T,I,V)V - aI) \\
&\quad + \frac{a}{k-ia}\left(1 - \frac{V^*}{V}\right)(d_v \Delta V + kI - if(T,I,V)V - \mu V) \bigg\} dx.
\end{aligned}$$

Since

$$\lambda = dT^* + f(T^*,I^*,V^*)V^*,$$

$$f(T^*,I^*,V^*)V^* = aI^*,$$

$$(k - ia)I^* = \mu V^*,$$

we have

$$
\begin{aligned}
\frac{dL_1}{dt} = & \int_{\Omega} \Bigg\{ dT^* \left(1 - \frac{T}{T^*}\right)\left(1 - \frac{f(T^*,I^*,V^*)}{f(T,I^*,V^*)}\right) \\
& - aI^* \left[g\left(\frac{f(T,I^*,V^*)}{f(T,I,V)}\right) + g\left(\frac{f(T^*,I^*,V^*)}{f(T,I^*,V^*)}\right) + g\left(\frac{f(T,I,V)VI^*}{f(T^*,I^*,V^*)V^*I}\right) + g\left(\frac{IV^*}{I^*V}\right) \right] \\
& + aI^* \left[-1 - \frac{V}{V^*} + \frac{f(T,I^*,V^*)}{f(T,I,V)} + \frac{f(T,I,V)V}{f(T,I^*,V^*)V^*} \right] \Bigg\} dx \\
& - \frac{ad_vV^*}{k} \int_{\Omega} \frac{|\nabla V|^2}{V^2} dx.
\end{aligned}
$$

Since $f(T,I,V)$ is strictly monotonically increasing with respect to T, we have

$$
\left(1 - \frac{T}{T^*}\right)\left(1 - \frac{f(T^*,I^*,V^*)}{f(T,I^*,V^*)}\right) \leq 0.
$$

Based on the hypothesis (H_4), we have

$$
-1 - \frac{V}{V^*} + \frac{f(T,I^*,V^*)}{f(T,I,V)} + \frac{V}{V^*}\frac{f(T,I,V)}{f(T,I^*,V^*)} = \left(1 - \frac{f(T,I,V)}{f(T,I^*,V^*)}\right)\left(\frac{f(T,I^*,V^*)}{f(T,I,V)} - \frac{V}{V^*}\right) \leq 0.
$$

Therefore, $\dfrac{dL_1}{dt} \leq 0$ when $R_0 > 1$. Further, $\dfrac{dL_1}{dt} = 0$ if and only if $T = T^*$, $V = V^*$ and $I = I^*$. If follows from LaSalle's invariance principle that E^* is globally asymptotically stable when $R_0 > 1$.

3. ANALYSIS OF DISCRETE VERSION

In this section, we discretize system (1) by using 'mixed' Euler method that is mixture of both forward and backward Euler method [10]. The choice of this numerical method is motivated by the work of Hattaf et al. [11].

Let $\Omega = [p,q]$ with $p,q \in \mathbf{R}$. Denote

$$
t_m = m\triangle t \text{ and } x_n = p + n\triangle x,
$$

where $\triangle t$ and $\triangle x = \dfrac{q-p}{N}$ are time and space step sizes, respectively. Let

$$
T(x_n,t_m) = T_n^m, \ I(x_n,t_m) = I_n^m, \ V(x_n,t_m) = V_n^m.
$$

So, our discrete model is as follows

(6)
$$\begin{cases} \dfrac{T_n^{m+1} - T_n^m}{\triangle t} = \lambda - dT_n^{m+1} - f(T_n^{m+1}, I_n^m, V_n^m)V_n^m, \\[2mm] \dfrac{I_n^{m+1} - I_n^m}{\triangle t} = f(T_n^{m+1}, I_n^m, V_n^m)V_n^m - aI_n^{m+1}, \\[2mm] \dfrac{V_n^{m+1} - V_n^m}{\triangle t} = d_v \dfrac{V_{n+1}^{m+1} - 2V_n^{m+1} + V_{n-1}^{m+1}}{(\Delta x)^2} + kI_n^{m+1} - if(T_n^{m+1}, I_n^m, V_n^m)V_n^m - \mu V_n^{m+1}, \end{cases}$$

where $n \in \{0, 1, ..., N\}$ and $m \in \mathbf{N}$. The discrete initial and boundary conditions are

$$T_n^0 = \phi_1(x_n), \ I_n^0 = \phi_2(x_n), \ V_n^0 = \phi_3(x_n), \ \text{for } n \in \{0, 1, ..., N\},$$

and

$$V_{-1}^m = V_0^m, \ V_N^m = V_{N+1}^m \text{ for } m \in \mathbf{N}.$$

It is clear that discrete system (6) and continuous (1) has the same equilibrium points. First, we establish that the solution of system (6) is nonnegative and bounded.

Theorem 3.1. *For any* $\triangle t > \dfrac{i}{k - ia}$ *and* $\triangle x > 0$*, the solution of system* (6) *is nonnegative and bounded for all* $m \in \mathbf{N}$*.*

Proof. System (6) can be writen as

(7)
$$\begin{cases} T_n^{m+1} = T_n^m + \triangle t \left(\lambda - dT_n^{m+1} - f(T_n^{m+1}, I_n^m, V_n^m)V_n^m \right), \\[2mm] I_n^{m+1} = \dfrac{I_n^m + \triangle t f(T_n^{m+1}, I_n^m, V_n^m)V_n^m}{1 + a\triangle t}, \\[2mm] BV^{m+1} = V^m + k\triangle t I^{m+1} - \triangle t M, \end{cases}$$

where $M = \left(if(T_0^{m+1}, I_0^m, V_0^m)V_0^m, if(T_1^{m+1}, I_1^m, V_1^m)V_1^m, ..., if(T_N^{m+1}, I_N^m, V_N^m)V_N^m \right)^T$. The square matrix B of dimension $(N+1) \times (N+1)$ is given by

$$\begin{pmatrix} \sigma_1 & \sigma_2 & 0 & . & . & . & 0 & 0 & 0 \\ \sigma_2 & \sigma_3 & \sigma_2 & . & . & . & 0 & 0 & 0 \\ 0 & \sigma_2 & \sigma_3 & . & . & . & 0 & 0 & 0 \\ . & . & . & . & & & . & . & . \\ . & . & . & . & & & . & . & . \\ . & . & . & . & & & . & . & . \\ 0 & 0 & 0 & . & . & . & \sigma_3 & \sigma_2 & 0 \\ 0 & 0 & 0 & . & . & . & \sigma_2 & \sigma_3 & \sigma_2 \\ 0 & 0 & 0 & . & . & . & 0 & \sigma_2 & \sigma_1 \end{pmatrix},$$

where

$$
\begin{aligned}
\sigma_1 &= 1 + \frac{d_v \triangle t}{(\triangle x)^2} + \mu \triangle t, \\
\sigma_2 &= -\frac{d_v \triangle t}{(\triangle x)^2}, \\
\sigma_3 &= 1 + \frac{2d_v \triangle t}{(\triangle x)^2} + \mu \triangle t.
\end{aligned}
$$

Thus, for any $j \in \{0, 1, ..., N\}$, we have

$$
\begin{aligned}
V_j^m &+ k\triangle t I_j^{m+1} - \triangle t i f \left(T_j^{m+1}, I_j^m, V_j^m \right) V_j^m \\
&= V_j^m + \frac{k\triangle t I_j^m}{1 + a\triangle t} + \frac{k(\triangle t)^2 f \left(T_j^{m+1}, I_j^m, V_j^m \right) V_j^m}{1 + a\triangle t} - \triangle t i f \left(T_j^{m+1}, I_j^m, V_j^m \right) V_j^m \\
&= V_j^m + \frac{k\triangle t I_j^m}{1 + a\triangle t} + \frac{\triangle t \left((k - ai)\triangle t - i \right)}{1 + a\triangle t} f \left(T_j^{m+1}, I_j^m, V_j^m \right) V_j^m.
\end{aligned}
$$

Since $\triangle t > \dfrac{i}{k - ai}$, we have $(k - ai)\triangle t - i > 0$. This means that

$$
V^m + k\triangle t I^{m+1} - \triangle t M \geq 0.
$$

Note that B is a M-matrix. Thus, from the third equation of (7), we have

$$
V^{m+1} = B^{-1} \left(V^m + k\triangle t I^{m+1} - \triangle t M \right).
$$

Therefore, by the method of induction, the solution remains nonnegative for all $m \in \mathbf{N}$.

Next, we prove the boundedness of the solution. Define a sequence

$$
G^m = T_n^m + I_n^m.
$$

Then

$$
\begin{aligned}
\frac{G_n^{m+1} - G_n^m}{\triangle t} &= \lambda - dT_n^{m+1} - aI_n^{m+1} \\
&\leq \lambda - \delta G_n^{m+1}.
\end{aligned}
$$

By mathematical induction, we have

$$
\begin{aligned}
G_n^{m+1} &\leq \frac{1}{1 + \widetilde{d}\triangle t} G_n^m + \frac{\lambda}{1 + \delta\triangle t} \\
&\leq \left(\frac{1}{1 + \delta\triangle t} \right)^m G_n^0 + \frac{\lambda}{\delta} \left[1 - \left(\frac{\lambda}{1 + \delta\triangle t} \right)^m \right].
\end{aligned}
$$

So,

$$\limsup_{m\to\infty} G_n^m \le \frac{\lambda}{\delta}.$$

This implies that $\{G_n^m\}$ is bounded. Therefore, T_n^m and I_n^m are also bounded.

Let $\widetilde{V}^m = \sum_{j=0}^{N} V_j^m$. By the third equation of (7), we get

$$
\begin{aligned}
\frac{\widetilde{V}^{m+1} - \widetilde{V}^m}{\triangle t} &= k\sum_{j=0}^{N} I_j^{m+1} - \mu \sum_{j=0}^{N} V_j^{m+1} - \sum_{j=0}^{N} i f\left(T_j^{m+1}, I_j^m, V_j^m\right) V_j^m \\
&\le k(N+1)\frac{\lambda}{\delta} - \mu \widetilde{V}^{m+1}.
\end{aligned}
$$

Thus, we have

$$\limsup_{m\to\infty} V^m \le \frac{k\lambda(N+1)}{\delta\mu}.$$

This completes the proof.

Next, we will establish the global stability of the infection-free equilibrium and chronic infection equilibrium for system (6). First, we need the following lemma.

Lemma 3.2 ([12]). *$\overline{T}, \overline{I}, \overline{V}$ and σ be four nonnegative real numbers and $\overline{E}(\overline{T}, \overline{I}, \overline{V})$ be an arbitrary point. The function $\psi_{(\overline{E},\sigma)}$ defined on interval $[0,+\infty)$ by*

$$\psi_{(\overline{E},\sigma)}(T) = T - \sigma - \int_{\sigma}^{T} \frac{f(\overline{T},\overline{I},\overline{V})}{f(X,\overline{I},\overline{V})} dX$$

has the global minimum at $T = \overline{T}$ and satisfies

$$\left(1 - \frac{f(\overline{T},\overline{I},\overline{V})}{f(\sigma,\overline{I},\overline{V})}\right)(T - \sigma) \le \psi_{(\overline{E},\sigma)}(T) \le \left(1 - \frac{f(\overline{T},\overline{I},\overline{V})}{f(T,\overline{I},\overline{V})}\right)(T - \sigma), \text{ for all } T > 0.$$

Theorem 3.3. *For any $\triangle t > 0$ and $\triangle x > 0$, if $R_0 \le 1$, then the infection-free equilibrium E_0 of system (6) is globally asymptotically stable.*

Proof. Consider a Lyapunov functional

$$L^m = \sum_{n=0}^{N} \frac{1}{\triangle t}\left[T_n^m - T_0 - \int_{T_0}^{T_n^m} \frac{f(T_0,0,0)}{f(X,0,0)} dX + \frac{k}{k-ia}I_n^m + \frac{a}{k-ia}(1+\mu\triangle t)V_n^m\right].$$

Then, we get

$$
\begin{aligned}
L^{m+1} - L^m &= \sum_{n=0}^{N} \frac{1}{\triangle t} \left[T_n^{m+1} - T_n^m + \int_{T_n^m}^{T_n^{m+1}} \frac{f(T_0,0,0)}{f(X,0,0)} dX + \frac{k}{k-ia} \left(I_n^{m+1} - I_n^m \right) \right. \\
&\quad \left. + \frac{a}{k-ia} \left(1 + \mu \triangle t \right) \left(V_n^{m+1} - V_n^m \right) \right].
\end{aligned}
$$

Consider

$$
\Psi_{(E_0,T_n^m)} \left(T_n^{m+1} \right) = T_n^{m+1} - T_n^m + \int_{T_n^m}^{T_n^{m+1}} \frac{f(T_0,0,0)}{f(X,0,0)} dX.
$$

Then we have

$$
\begin{aligned}
L^{m+1} - L^m &\leq \sum_{n=0}^{N} \frac{1}{\triangle t} \left[\Psi_{(E_0,T_n^m)} \left(T_n^{m+1} \right) + \left(I_n^{m+1} - I_n^m \right) \right. \\
&\quad \left. + \frac{a}{k} \left(1 + \mu \triangle t \right) \left(V_n^{m+1} - V_n^m \right) \right].
\end{aligned}
$$

Using Lemma 3.2, we obtain

$$
\begin{aligned}
L^{m+1} - L^m &\leq \sum_{n=0}^{N} \frac{1}{\triangle t} \left[\left(1 - \frac{f(T_0,0,0)}{f\left(T_n^{m+1},0,0\right)} \right) \left(T_n^{m+1} - T_n^m \right) + \left(I_n^{m+1} - I_n^m \right) \right. \\
&\quad \left. + \frac{a}{k} \left(1 + \mu \triangle t \right) \left(V_n^{m+1} - V_n^m \right) \right] \\
&= \sum_{n=0}^{N} \left[dT_0 \left(1 - \frac{T_n^{m+1}}{T_0} \right) \left(1 - \frac{f(T_0,0,0)}{f\left(T_n^{m+1},0,0\right)} \right) \right. \\
&\quad + \frac{\mu a}{k} V_n^m \left\{ \frac{f\left(T_n^{m+1},I_n^m,V_n^m\right)}{f\left(T_n^{m+1},0,0\right)} \frac{kf(T_0,0,0)}{\mu a} - 1 \right\} \\
&\quad \left. + \frac{a d_v}{k} \frac{V_{n+1}^{m+1} - 2V_n^{m+1} + V_{n-1}^{m+1}}{(\Delta x)^2} \right].
\end{aligned}
$$

We have $\dfrac{kf(T_0,0,0)}{\mu a} = R_0$ and

$$
\begin{aligned}
\sum_{n=0}^{N} \frac{a d_v}{k} \frac{V_{n+1}^{m+1} - 2V_n^{m+1} + V_{n-1}^{m+1}}{(\Delta x)^2} &= \frac{a d_v}{k(\Delta x)^2} \left[\sum_{n=0}^{N} \left(V_{n+1}^{m+1} - V_n^{m+1} \right) + \sum_{n=0}^{N} \left(V_{n-1}^{m+1} - V_n^m \right) \right] \\
&= \frac{a d_v}{k(\Delta x)^2} \left[V_{N+1}^{m+1} - V_0^{m+1} + V_{-1}^{m+1} - V_N^{m+1} \right] = 0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
L^{m+1} - L^m &\leq \sum_{n=0}^{N} \left[dT_0 \left(1 - \frac{T_n^{m+1}}{T_0}\right) \left(1 - \frac{f(T_0,0,0)}{f(T_n^{m+1},0,0)}\right) + \frac{\mu a}{k} V_n^m \left(\frac{f(T_n^{m+1}, I_n^m, V_n^m)}{f(T_n^{m+1},0,0)} R_0 - 1\right) \right] \\
&\leq \sum_{n=0}^{N} \left[dT_0 \left(1 - \frac{T_n^{m+1}}{T_0}\right) \left(1 - \frac{f(T_0,0,0)}{f(T_n^{m+1},0,0)}\right) + \frac{\mu a}{k} (R_0 - 1) V_n^m \right].
\end{aligned}
$$

If $R_0 \leq 1$, then $L^{m+1} - L^m \leq 0$, for all $m \in \mathbf{N}$ and the equality holds if and only if $\lim_{m \to \infty} T_n^{m+1} = T_0$.

Therefore, $\{L^m\}_{m \in \mathbf{N}}$ is a monotone decreasing sequence. Due to $L^m \geq 0$, there is $\lim_{m \to \infty} L^m \geq 0$.

Thus,

$$
\lim_{m \to \infty} \left(L^{m+1} - L^m\right) = 0.
$$

From $\lim_{m \to \infty} T_n^{m+1} = T_0$ and $\lim_{m \to \infty} ((R_0 - 1) V_n^m) = 0$ for all $n \in \{0, 1, ..., N\}$, we discuss two cases:

- If $R_0 < 1$, then $\lim_{m \to \infty} V_n^m = 0$ for all $n \in \{0, 1, ..., N\}$. From the third equation of system (6), we obtain $\lim_{m \to \infty} I_n^m = 0$ for all $n \in \{0, 1, ..., N\}$.
- If $R_0 = 1$. By $\lim_{m \to \infty} T_n^{m+1} = T_0$ and the first equation of (6), we get $\lim_{m \to \infty} V_n^m = 0$ for all $n \in \{0, 1, ..., N\}$.

We get

$$
\lim_{m \to \infty} T_n^m = T_0, \ \lim_{m \to \infty} I_n^m = 0, \ \lim_{m \to \infty} V_n^m = 0, \ \text{for all } n \in \{0, 1, ..., N\}.
$$

Thus, E_0 is globally asymptotically stable if $R_0 \leq 1$.

Theorem 3.4. *For any $\triangle t > 0$ and $\triangle x > 0$, if $R_0 > 1$ and (H_4) holds, then the infection equilibrium E^* of system (6) is globally asymptotically stable.*

Proof. Consider the following Lyapunov functional:

$$
G^m = \sum_{n=0}^{N} \frac{1}{\triangle t} \left[T_n^m - T^* - \int_{T^*}^{T_n^m} \frac{f(T^*, I^*, V^*)}{f(X, I^*, V^*)} dX + I^* g\left(\frac{I_n^m}{I^*}\right) + \frac{a}{k} (1 + \mu \triangle t) V^* g\left(\frac{V_n^m}{V^*}\right) \right].
$$

According to the Lemma 3.2, we consider

$$
\Psi_{(E^*, T_n^m)} (T_n^{m+1}) = T_n^{m+1} - T_n^m + \int_{T_n^m}^{T_n^{m+1}} \frac{f(T^*, I^*, V^*)}{f(X, I^*, V^*)} dX \leq \left(1 - \frac{f(T^*, I^*, V^*)}{f(T_n^{m+1}, I^*, V^*)}\right) (T_n^{m+1} - T_n^m),
$$

Hence,

$$
\begin{aligned}
G^{m+1} - G^m \ \leq \ & \sum_{n=0}^{N} \left\{ dT^* \left(1 - \frac{T_n^{m+1}}{T^*}\right) \left(1 - \frac{f(T^*, I^*, V^*)}{f\left(T_n^{m+1}, I^*, V^*\right)}\right) \right. \\
& + \frac{ad_v}{k(\Delta x)^2} \left(V_{n+1}^{m+1} - 2V_n^{m+1} + V_{n-1}^{m+1}\right) \left(1 - \frac{V^*}{V_n^{m+1}}\right) \\
& - aI^* \left[g\left(\frac{f(T^*, I^*, V^*)}{f(T_n^{m+1}, I^*, V^*)}\right) + g\left(\frac{f(T_n^{m+1}, I^*, V^*)}{f(T_n^{m+1}, I_n^m, V_n^m)}\right) + g\left(\frac{f(T_n^{m+1}, I_n^m, V_n^m)V_n^m I^*}{f(T^*, I^*, V^*)V^* I_n^m}\right) \right. \\
& \left. + g\left(\frac{I_n^{m+1}V^*}{I^* V_n^{m+1}}\right) \right] + aI^* \left[-1 - \frac{V_n^m}{V^*} + \frac{f(T_n^{m+1}, I^*, V^*)}{f(T_n^{m+1}, I_n^m, V_n^m)} + \frac{f(T_n^{m+1}, I_n^m, V_n^m)V_n^m}{f(T_n^{m+1}, I^*, V^*)V^*} \right] \right\}.
\end{aligned}
$$

We have

$$
\sum_{n=0}^{N} \frac{ad_v}{k(\Delta x)^2} \left(V_{n+1}^{m+1} - 2V_n^{m+1} + V_{n-1}^{m+1}\right) \left(1 - \frac{V^*}{V_n^{m+1}}\right) \ = \ -\frac{ad_v V^*}{k(\Delta x)^2} \left(\sum_{n=0}^{N-1} \frac{\left(V_{n+1}^{m+1} - V_n^{m+1}\right)^2}{V_n^{m+1} V_{n+1}^{m+1}} \right) \leq 0.
$$

According to (H_2) and (H_4), we obtain

$$
\left(1 - \frac{T_n^{m+1}}{T^*}\right) \left(1 - \frac{f(T^*, I^*, V^*)}{f\left(T_n^{m+1}, I^*, V^*\right)}\right) \leq 0,
$$

and

$$
-1 - \frac{V_n^m}{V^*} + \frac{f(T_n^{m+1}, I^*, V^*)}{f(T_n^{m+1}, I_n^m, V_n^m)} + \frac{f(T_n^{m+1}, I_n^m, V_n^m)V_n^m}{f(T_n^{m+1}, I^*, V^*)V^*} \leq 0.
$$

Thus,

$$
G^{m+1} - G^m \leq 0, \text{ for all } m \in \mathbf{N}.
$$

This yields that there exists a constant $\tilde{G}$ such that $\lim\limits_{m \longrightarrow +\infty} G^m = \tilde{G}$. Then

$$
\lim_{m \to \infty} \left(G^{m+1} - G^m\right) = 0.
$$

Hence, $\lim\limits_{m \to \infty} T_n^m = T^*$ and $\lim\limits_{m \to \infty} \dfrac{I_n^{m+1}}{V_n^{m+1}} = \dfrac{I^*}{V^*}$. Combined with system (6), we deduce that

$$
\lim_{m \to \infty} I_n^m = I^* \text{ and } \lim_{m \to \infty} V_n^m = V^*, \text{ for all } n \in \{0, 1, ..., N\}.
$$

Consequently, E^* is globally asymptotically stable when $R_0 > 1$.

4. APPLICATIONS

Here, we give some examples for which we apply our theoretical results.

Example 1: Consider the following system

(8)
$$
\begin{cases}
\dfrac{T_n^{m+1} - T_n^m}{\triangle t} = \lambda - dT_n^{m+1} - T_n^{m+1} F\left(V_n^m\right), \\[2mm]
\dfrac{I_n^{m+1} - I_n^m}{\triangle t} = T_n^{m+1} F\left(V_n^m\right) - aI_n^{m+1}, \\[2mm]
\dfrac{V_n^{m+1} - V_n^m}{\triangle t} = d_v \dfrac{V_{n+1}^{m+1} - 2V_n^{m+1} + V_{n-1}^{m+1}}{(\Delta x)^2} + kI_n^{m+1} - iT_n^{m+1} F\left(V_n^m\right) - \mu V_n^{m+1},
\end{cases}
$$

where $n \in \{0,1,...,N\}$ and $m \in \mathbf{N}$. The function $F(x)$ is a twice differentiable and satisfies

(9)
$$
\begin{cases}
F(x) \geq 0 \text{ with the equality if and only if } x = 0, \\[2mm]
F'(x) \geq 0, \\[2mm]
F''(x) \leq 0.
\end{cases}
$$

The discrete initial and boundary conditions are

$$
T_n^0 = \phi_1(x_n), \ I_n^0 = \phi_2(x_n), \ V_n^0 = \phi_3(x_n), \text{ for } n \in \{0,1,...,N\},
$$

and

$$
V_{-1}^m = V_0^m, \ V_N^m = V_{N+1}^m \text{ for } m \in \mathbf{N}.
$$

This model was studied by Yang and Zhou [5], which is a special case of our model (6), it suffices to take

$$
f(T,I,V) = \begin{cases}
\dfrac{TF(V)}{V}, \ V \neq 0; \\[2mm]
TF'(0), \ V = 0.
\end{cases}
$$

It is easy to show that the function f verified the four assumptions (H_1)-(H_4). Also, the basic reproduction number of system (8) is given by

(10)
$$
R_0 = \frac{\lambda (k - ia) F'(0)}{d\mu a}.
$$

By applying Theorems 3.3 and 3.4, we obtain the following result.

Corollary 4.1. *For any $\triangle t > 0$ and $\triangle x > 0$, we have:*

(1)

(i) *If $R_0 \leq 1$, then the infection-free equilibrium E_0 of system (8) is globally asymptotically stable.*

(ii) *If $R_0 > 1$ and $i = 0$, then the chronic infection equilibrium E^* of system (8) is globally asymptotically stable.*

Example 2: Consider the following system

(11)
$$
\begin{cases}
\dfrac{T_n^{m+1} - T_n^m}{\triangle t} = \lambda - dT_n^{m+1} - \dfrac{\beta T_n^{m+1} V_n^m}{\alpha_0 + \alpha_1 I_n^m + \alpha_2 V_n^m + \alpha_3 I_n^m V_n^m}, \\[2ex]
\dfrac{I_n^{m+1} - I_n^m}{\triangle t} = \dfrac{\beta T_n^{m+1} V_n^m}{\alpha_0 + \alpha_1 I_n^m + \alpha_2 V_n^m + \alpha_3 I_n^m V_n^m} - aI_n^{m+1}, \\[2ex]
\dfrac{V_n^{m+1} - V_n^m}{\triangle t} = d_v \dfrac{V_{n+1}^{m+1} - 2V_n^{m+1} + V_{n-1}^{m+1}}{(\Delta x)^2} + kI_n^{m+1} - \dfrac{i\beta T_n^{m+1} V_n^m}{\alpha_0 + \alpha_1 I_n^m + \alpha_2 V_n^m + \alpha_3 I_n^m V_n^m} - \mu V_n^{m+1}.
\end{cases}
$$

The discrete initial and boundary conditions are

$$
T_n^0 = \phi_1(x_n), \ I_n^0 = \phi_2(x_n), \ V_n^0 = \phi_3(x_n), \text{ for } n \in \{0, 1, ..., N\},
$$

and

$$
V_{-1}^m = V_0^m, \ V_N^m = V_{N+1}^m \text{ for } m \in \mathbf{N}.
$$

System (11) is a particular case of our model (6), it suffices to take

$$
f(T, I, V) = \frac{\beta T}{\alpha_0 + \alpha_1 I + \alpha_2 V + \alpha_3 IV},
$$

where $\beta > 0$ is the infection rate and $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are non-negative constants. This functional response was introduced Hattaf and Yousfi [4] and it covers various types of incidence rate existing in the literature.

Obviously, the function f satisfied the four assumptions (H_1)-(H_4). Therefore, we get the following result.

Corollary 4.2. *For any $\triangle t > 0$ and $\triangle x > 0$, we have:*

(1)

(i) *If $R_0 \leq 1$, then the infection-free equilibrium E_0 of system (11) is globally asymptotically stable.*

(ii) *If $R_0 > 1$ and $i = 0$, then the chronic infection equilibrium E^* of system (11) is globally asymptotically stable.*

5. CONCLUSION

In this paper, we have proposed a class of virus infection model with diffusion and general incidence function. The continuous and discrete versions are rigorously analyzed by established the well-posedness of solutions and the global stability of equilibria. Furthermore, the discrete model and the corresponding results presented in the recent work [5] are improved and generalized.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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