Approximate Analytical Solutions of Linear Stochastic Differential Models Based on Karhunen-Loéve Expansion With Finite Series Terms

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Abstract: Stochastic Differential Equations (SDEs) as particular forms of Differential Equations (DEs) play immense roles in modeling of various phenomena with applications in physical sciences, and finance—such as stock option practices due to thermal and random fluctuations. The solutions of these SDEs, if they exist, are difficult to obtain, unlike those of the Differential Equations. In this paper, the white noise terms of the linear SDEs in Stratonovich forms are considered on the basis of Karhunen-Loéve Expansion finite series while Daftardar-Jafari Integral Method is proposed for approximate analytical solution of the linear Stratonovich Stochastic Differential Equations. Three numerical examples are considered to test the accuracy and effectiveness of this proposed method. The results obtained show clearly that the approximate solutions converge faster to the exact solutions even with fewer terms; though, higher terms increase the accuracy. The method is direct in terms of application. Thus, it is recommended for nonlinear financial models such as Ito Stochastic Differential Equations.

Keywords: Stratonovich SDEs; population dynamics; Karhunen-Loéve expansion; approximate solution; differential models; option pricing; Daftardar-Jafari method.

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1. LINEAR INTRODUCTION

Stochastic Differential Equations (SDEs) play a critical role in the field of Mathematical Finance, Biological Sciences (Population Dynamics Modelling), Physics, Engineering, and other areas. They are used to model physical phenomena, biological systems, and real-life situations [1]. Probability theory stands as a link connecting SDEs to Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs). Several authors have established that most well-known SDEs are not solvable using the analytical approaches, hence the development of numerical schemes to solve such problems [2, 3, 14].

Considering the Stratonovich SDE of this form:

\[
dY = a(Y, t)dt + \sum_{i=1}^{d} b(Y, t) \circ dB^i(t),
Y(0) = Y_0,
\]

with, \( Y \in \mathbb{R}^n \), \( a_0 = a_0(Y, t) \), \( b_0 = b_0(Y, t) \) are the drift and diffusion coefficients respectively, and \( B(t) \) is the standard Brownian motion (Wiener process), and \( d \) represents the dimension of the Brownian Motion,

\[
B = \{ B(t), 0 \leq t \leq T \} \text{ is a 1-dimensional Brownian motion [4].}
\]

The Itô form of the SDE in (1.1) is:

\[
dY = a(t, Y_t)dt + \sum_{i=1}^{d} b(t, y = Y_t) dB^i(t)
Y(0) = Y_0,
\]

In integral form, (1.1) is written as:

\[
Y(t) = Y_0 + \int_0^t a(s, Y(s))ds + \int_0^t b(s, Y(s)) \circ dB(s).
\]

The first integral in (1.3) is known as the Riemann-Stiltjes and the second is a stochastic integral attached to the Wiener process, \( B(t) \). Many researchers have used different numerical methods to solve SDEs [5-14]. In the process, many of these methods generate complex situations that
cannot be accommodated effectively, most notably when dealing with the nonlinear SDEs, which makes most researchers to streamline their researches to linear SDE, ODE and PDE type. But recently, precisely in 2006, two researchers Daftardar-Gejji and Jafari [15] proposed a new iterative scheme that is capable of solving linear and nonlinear ODEs, PDEs. The method was named after them as Daftardar-Jafari Method (DJM). This is also referred to as New Iterative Method (NIM). This method has been widely accepted and used by many scholars for the solution of linear and nonlinear ODEs and PDEs in integer and fractional orders [16-22]. Azodi [23] used the method to obtain the solution of Itô-SDEs, and it was seen to produce better results than those obtained by ADM, HPM, and VIM [23].

The aim of this research is to apply the DJM to solve Stochastic Differential Equations (SDEs) of the Sratonovich form. This form of SDE plays a dominant role in mathematical finance, physics, engineering, biology, and other related fields via the transformation of the “white noise” into an ODE form using the Karhunen-Loéve expansion.

The remaining part of the work is divided into five sections. Section 2 is devoted to a brief discussion of the linear SDEs. The proposed method is discussed in section 3; numerical examples are considered in section 4, and lastly, conclusion is made in section 5.

### 2. Linear Stochastic Differential Equations (LSDEs)

Consider a linear Stochastic Differential Equation (LSDE) that follows a Geometric Brownian Motion (such as stock price) given price:

\[
dY(t) = eY(t)dt + \sigma Y(t)dB(t),
\]

where \( e \) is the stock drift, \( \sigma > 0 \) is the stock volatility or diffusion, \( dB(t) \) is the “white noise” and \( Y(t) \) is a Stochastic Process. The process \( (B(t))_{t \geq 0} \) is the Brownian Motion defined in \( \mathbb{R} \) on some probability space \( (\Omega, \mathcal{F}, P) \). We refer the readers to [6] for the existence and uniqueness of the solution.
3. THE METHOD OF SOLUTION

This section presents the concepts of New Iterative Method (NIM) and Karhunen-Loeve Expansion (K-L Expansion).

3.1 DAFTAR-GEJII-JAFARI METHOD (DJM)

Consider the general functional equation defined as follows

\[ y = b + L(y) + N[y], \]  \hspace{1cm} (3.1)

where \( b \) is a known function, \( L[\cdot] \) and \( N[\cdot] \) are the linear and nonlinear operators respectively. Suppose we define \( \tilde{N}[y] \) as:

\[ \tilde{N}[y] = L[y] + N[y], \]  \hspace{1cm} (3.2)

then (3.1) becomes:

\[ y = b + \tilde{N}[y]. \]  \hspace{1cm} (3.3)

Now considering a solution, \( y \) of (3.2) having the infinite series form:

\[
\begin{align*}
    y &= \sum_{i=0}^{\infty} y_i, \\
    \tilde{N}[y] &= \tilde{N}\left[ \sum_{i=0}^{\infty} y_i \right].
\end{align*}
\]  \hspace{1cm} (3.4)

The nonlinear operator \( \tilde{N} \) can now be decomposed as

\[ \tilde{N}\left( \sum_{i=0}^{\infty} y_i \right) = \tilde{N}\left[ y_0 \right] + \sum_{i=1}^{\infty} \left[ \tilde{N}\left( \sum_{i=0}^{m} y_i \right) - \tilde{N}\left( \sum_{i=0}^{m-1} y_i \right) \right], \quad m = 1, \ 2, \ldots \]  \hspace{1cm} (3.5)

Therefore, putting (3.4) and (3.5) into (3.3), we obtain

\[ \sum_{i=0}^{\infty} y_i = b + \tilde{N}\left[ y_0 \right] + \sum_{i=1}^{\infty} \left[ \tilde{N}\left( \sum_{i=0}^{m} y_i \right) - \tilde{N}\left( \sum_{i=0}^{m-1} y_i \right) \right], \quad m = 1, \ 2, \ldots \]  \hspace{1cm} (3.6)

Hence, the recurrence relation is:
such that:
\[ y = b + \sum_{i=1}^{\infty} y_i = \sum_{i=0}^{\infty} y_i. \]  
\[ (3.8) \]

The convergence of this method has been discussed by many researchers. We refer the readers to [17, 24-26].

In another dimension of applications, numerical or exact solution methods for linear and or nonlinear differential models including SDEs are linked to the following references and the link therein [27-40].

### 3.2 Karhunen-Loéve Expansion (K-L E) of Brownian Motion

A Brownian Motion (also called Wiener Process) is a Stochastic Process \( \{B(t)\}_{t\geq0} \) indexed by nonnegative real numbers, \( t \) with the following properties listed below:

- **i.** \( B(0) = 0, \)
- **ii.** the probability that a randomly generated Brownian path be continuous is one,
- **iii.** the process \( \{B(t)\}_{t\geq0} \) has stationary, independent increment,
- **iv.** the increments: \( B(t + v) - B(t) \) are normally distributed (Gaussian Process) with zero \( (0) \) and \( v \) as the expected value and variance respectively.

The Karhunen-Loéve Expansion (K-LE) of Brownian motion was used by [41] to characterize the Wiener Process. Let \( Y(t) \) denote the trajectory of a random process, \( Y(t, \omega) \) for a given \( \omega \). The Wiener Process \( B(t) \) has trajectories belonging to \( L^2([0,T]) \) for almost all \( \omega \)'s, and this space K-L expansion takes the form:
\[ B(t) = B(t, \omega) = \sum_{j=0}^{\infty} z_j(w) \Phi_j(t), \quad 0 \leq t \leq T, \]
\[ \Phi_j(t) = \frac{2\sqrt{2T}}{(2j+1)} \sin \left( \frac{(2j+1)\pi t}{2T} \right), \]
where \( z_j \) is a sequence of identically, independent Gaussian random variables, and \( \Phi_j(t) \) form a basis of orthogonal function [41, 42].

Therefore, simplifying (3.9) with \( T = 1 \) gives;
\[ B(t) = \sqrt{2} \sum_{j=0}^{\infty} \left[ \frac{\sin\left( (j+0.5)\pi t \right)}{(j+0.5)\pi} \right] z_j \]
\[ dB(t) = \sqrt{2} \sum_{j=0}^{\infty} \left[ \cos\left( (j+0.5)\pi t \right) \right] z_j . \]

Replacing the finite terms of K-L Expansion in the SDE (1.2), we obtain:
\[ \begin{cases} dY(t) = a(Y,t)dt + \sigma(Y,t)d \left( \sum_{j=0}^{L} z_j \Phi_j(t) \right), \\ Y(0) = Y_0. \end{cases} \]

For the purpose of this work, we choose \( L = 5 \) (which represents the number of functions of K-L Expansion) and \( z_j \) is generated in MATLAB software using normarnd.

\section*{4. Numerical Examples}

In what follows, two kinds of approximate solutions viz: \( \Phi_3 \) and \( \Phi_4 \) will be considered.

Hence,
\[ \begin{cases} \Phi_3 = \sum_{i=0}^{3} Y_i, \\ \Phi_4 = \sum_{i=0}^{4} Y_i, \end{cases} \]

such that \( Y(t) \) represents the exact solution.
Example 4.1 Consider the Stratonovich Stochastic Differential Equation [4]

\[
\begin{cases}
    dY(t) = \beta Y(t) \circ dW(t), \\
    Y(0) = 1.
\end{cases}
\]  
(4.2)

The exact solution of (4.2) is

\[
Y(t) = Y(0) \exp(\beta W(t)), \quad t \in [0, T].
\]

In integral form, (4.1) becomes:

\[
Y(t) = 1 + \beta \int_0^t (Y(s) dW(s)) dt 
= b \cancel{+ N}[Y].
\]  
(4.3)

Thus,

\[
\begin{cases}
    b = 1, \\
    \bar{N}[Y] = \beta \int_0^t (Y(s) dW(s)) dt.
\end{cases}
\]

By applying the DJM to (4.3), we get the following:

\[
y_0 = 1,
\]

\[
y_1 = \bar{N}(y_0) = \beta \int_0^t y_0 dW(s) 
= \beta \int_0^t \left[ \frac{\sqrt{2}}{2} \sum_{j=0}^5 \cos\left( (j + 0.5) \pi t \right) z_j \right] dt,
\]

\[
\bar{N}[y] = \beta \int_0^t \left[ \frac{\sqrt{2}}{2} \sum_{j=0}^5 \cos\left( (j + 0.5) \pi t \right) z_j \right] dt, \quad \beta = 1
\]

\[
y_2 = \bar{N}(1 + y_1) - \bar{N}(y_0)
= \int_0^t (1 + y_1) dW(s) dt - \int_0^t y_0 dW(s) dt
= \int_0^t \left[ (1 + y_1) \left[ \frac{\sqrt{2}}{2} \sum_{j=0}^5 \cos\left( (j + 0.5) \pi t \right) z_j \right] - \{y_1\} \right] dt,
\]

\[
y_3 = \int_0^t \left[ (1 + y_1 + y_2) \left[ \frac{\sqrt{2}}{2} \sum_{j=0}^5 \cos\left( (j + 0.5) \pi t \right) z_j \right] \right] dt
- \int_0^t \left[ (1 + y_1) \left[ \frac{\sqrt{2}}{2} \sum_{j=0}^5 \cos\left( (j + 0.5) \pi t \right) z_j \right] \right] dt,
\]
The computed results and errors analysis are shown in the table 1 and two graphs are plotted for $Y(t)$ against $\Phi_3$ and $Y(t)$ against $\Phi_4$.

Table 1: Solutions and absolute errors via DJM: $Y(t)$, $\Phi_3(t)$, $\Phi_4(t)$ for Example 4.1

| $t$  | $Y(t)$  | $\Phi_3(t)$ | $\Phi_4(t)$ | $|Y(t) - \Phi_3(t)|$ | $|Y(t) - \Phi_4(t)|$ | $|\Phi_4(t) - \Phi_3(t)|$ |
|------|---------|-------------|-------------|----------------------|----------------------|----------------------|
| 0.00 | 1.0000  | 1.0000      | 1.0000      | 0.0000               | 0.0000               | 0.0000               |
| 0.05 | 1.2716  | 1.2714      | 1.2716      | 0.0002               | 0.0000               | 0.0002               |
| 0.10 | 1.5981  | 1.5959      | 1.5979      | 0.0022               | 0.0002               | 0.0020               |
| 0.15 | 1.9399  | 1.9307      | 1.9387      | 0.0092               | 0.0012               | 0.0080               |
| 0.20 | 2.2045  | 2.1853      | 2.2015      | 0.0192               | 0.0030               | 0.0162               |
| 0.25 | 2.3007  | 2.2768      | 2.2968      | 0.0239               | 0.0039               | 0.0200               |
| 0.30 | 2.2390  | 2.2181      | 2.2357      | 0.0209               | 0.0033               | 0.0176               |
| 0.35 | 2.1463  | 2.1296      | 2.1438      | 0.0167               | 0.0025               | 0.0142               |
| 0.40 | 2.1791  | 2.1610      | 2.1763      | 0.0181               | 0.0028               | 0.0153               |
| 0.45 | 2.4635  | 2.4302      | 2.4577      | 0.0333               | 0.0058               | 0.0275               |
| 0.50 | 3.0838  | 2.9983      | 3.0653      | 0.0855               | 0.0185               | 0.0670               |
Figure 1: $Y(t)$ and $\Phi_3(t)$ via DJM for Example 4.1 ($N = 100, L = 5$)

Figure 2: $Y(t)$ and $\Phi_4(t)$ via DJM for Example 4.1 ($N = 100, L = 5$)


**Example 4.2:** Consider the Stochastic Differential Equation [23]:

\[
\begin{align*}
\frac{dY(t)}{dt} &= \frac{1}{2} Y(t) + dY(t) \cdot dB(t), \\
Y(0) &= 1.
\end{align*}
\]  

(4.4)

The Stratonovich exact solution of (4.4) is given as

\[
Y(t) = \exp \left( \frac{t}{2} + B(t) \right).
\]  

(4.5)

In integral form, we have:

\[
Y(t) = 1 + \frac{1}{2} \int_0^t Y(s) \, ds + \int_0^t Y(s) \, dB(s).
\]  

(4.6)

Therefore, with the following definitions:

\[
\tilde{N}(Y(t)) = \frac{1}{2} \int_0^t Y(s) \, ds + \int_0^t Y(s) \, dB(s),
\]  

(4.7)

\[
dB(s) = \sqrt{2} \sum_{j=0}^5 \cos((j + 0.5) \pi t) z_j,
\]  

(4.8)

and

\[
\tilde{N}(Y(t)) = \frac{1}{2} \int_0^t Y(s) \, ds + \int_0^t Y(s) \left( \sqrt{2} \sum_{j=0}^5 \cos((j + 0.5) \pi t) z_j \right) ds,
\]  

(4.9)

we obtain the iterative scheme via DJM as follows:

\[
Y_0 = 1,
\]

\[
Y_1 = \tilde{N}(Y_0)
\]

\[
= \frac{1}{2} \int_0^t Y_0 \, ds + \int_0^t Y_0 \, dB(s) ds
\]

\[
= \frac{1}{2} \int_0^t Y_0 \, ds + \int_0^t \left[ \sqrt{2} \sum_{j=0}^5 \cos((j + 0.5) \pi t) z_j \right] ds,
\]

\[
Y_2 = \tilde{N}(Y_0 + Y_1) - \tilde{N}(Y_0)
\]

\[
= \left\{ \frac{1}{2} \int_0^t (Y_0 + Y_1) ds + \int_0^t ((Y_0 + Y_1) dB(s) ds \right\} - \left\{ \frac{1}{2} \int_0^t (Y_0) ds + \int_0^t (Y_0 dB(s) ds \right\}
\]

\[
= \left\{ \frac{1}{2} \int_0^t ((Y_0 + Y_1) ds + \int_0^t (Y_0 + Y_1) \left( \sqrt{2} \sum_{j=0}^5 \cos((j + 0.5) \pi t) z_j \right) ds \right\} - \{Y_1\},
\]
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\[ Y_3 = \bar{N}(Y_0 + Y_1 + Y_2) - \bar{N}(Y_0 + Y_1) \]
\[ = \left\{ \frac{1}{2} \int_0^t (Y_0 + Y_1 + Y_2) ds + \int_0^t ((Y_0 + Y_1 + Y_2) dB(s)) ds \right\} \]
\[ - \left\{ \frac{1}{2} \int_0^t (Y_0 + Y_1) ds + \int_0^t ((Y_0 + Y_1) dB(s)) ds \right\} \]
\[ = \left\{ \frac{1}{2} \int_0^t (Y_0 + Y_1 + Y_2) ds + \int_0^t \left( (Y_0 + Y_1 + Y_2) \left( \sqrt{2} \sum_{j=0}^{5} \cos \left( (j + 0.5) \pi t \right) z_j \right) \right) ds \right\} \]
\[ - \left\{ \frac{1}{2} \int_0^t (Y_0 + Y_1) ds + \int_0^t \left( (Y_0 + Y_1) \left( \sqrt{2} \sum_{j=0}^{5} \cos \left( (j + 0.5) \pi t \right) z_j \right) \right) ds \right\}. \]

\[ Y_4 = \bar{N}(Y_0 + Y_1 + Y_2 + Y_3) - \bar{N}(Y_0 + Y_1 + Y_2) \]
\[ = \left\{ \frac{1}{2} \int_0^t (Y_0 + Y_1 + Y_2 + Y_3) ds + \int_0^t ((Y_0 + Y_1 + Y_2 + Y_3) dB(s)) ds \right\} \]
\[ - \left\{ \frac{1}{2} \int_0^t (Y_0 + Y_1 + Y_2) ds + \int_0^t ((Y_0 + Y_1 + Y_2) dB(s)) ds \right\} \]
\[ = \left\{ \frac{1}{2} \int_0^t (Y_0 + Y_1 + Y_2 + Y_3) ds + \int_0^t \left( (Y_0 + Y_1 + Y_2 + Y_3) \left( \sqrt{2} \sum_{j=0}^{5} \cos \left( (j + 0.5) \pi t \right) z_j \right) \right) ds \right\} \]
\[ - \left\{ \frac{1}{2} \int_0^t (Y_0 + Y_1 + Y_2) ds + \int_0^t \left( (Y_0 + Y_1 + Y_2) \left( \sqrt{2} \sum_{j=0}^{5} \cos \left( (j + 0.5) \pi t \right) z_j \right) \right) ds \right\}. \]

\[ \vdots \]

The computed results and errors analysis are shown in the table 2; two graphs are plotted for \( Y(t) \) against \( \Phi_3 \) and \( Y(t) \) against \( \Phi_4 \).

Table 2: Solutions and absolute errors via DJM: \( Y(t) \), \( \Phi_3(t) \), \( \Phi_4(t) \) for Example 4.2

| \( t \) | \( Y(t) \)     | \( \Phi_3(t) \) | \( \Phi_4(t) \) | \( |Y(t) - \Phi_3(t)| \) | \( |Y(t) - \Phi_4(t)| \) | \( |\Phi_4(t) - \Phi_3(t)| \) |
|-------|-------------|-------------|-------------|----------------|----------------|----------------|
| 0.00  | 1.0000      | 1.0000      | 1.0000      | 0.0000         | 0.0000         | 0.0000         |
| 0.05  | 1.3038      | 1.3036      | 1.3038      | 0.0002         | 0.0000         | 0.0002         |
| 0.10  | 1.6801      | 1.6767      | 1.6797      | 0.0034         | 0.0004         | 0.0030         |
| 0.15  | 2.0910      | 2.0766      | 2.0889      | 0.0144         | 0.0021         | 0.0123         |
| 0.20  | 2.4363      | 2.4047      | 2.4309      | 0.0316         | 0.0054         | 0.0262         |
| 0.25  | 2.6071      | 2.5640      | 2.5991      | 0.0431         | 0.0080         | 0.0351         |
| 0.30  | 2.6013      | 2.5586      | 2.5934      | 0.0427         | 0.0079         | 0.0348         |
| 0.35  | 2.5567      | 2.5172      | 2.5496      | 0.0395         | 0.0071         | 0.0324         |
| 0.40  | 2.6615      | 2.6143      | 2.6526      | 0.0472         | 0.0089         | 0.0383         |
| 0.45  | 3.0851      | 2.9995      | 3.0666      | 0.0856         | 0.0185         | 0.0671         |
| 0.50  | 3.9596      | 3.7574      | 3.9068      | 0.2022         | 0.0528         | 0.1494         |
Figure 3: $Y(t)$ and $\Phi_3(t)$ via DJM for Example 4.2 ($N = 100, L = 5$)

Figure 4: $Y(t)$ and $\Phi_4(t)$ via DJM for Example 4.2 ($N = 100, L = 5$)
Example 4.3: Consider the Stochastic Differential Equation [43]:

\[
\begin{align*}
  dY(t) &= \frac{1}{2} Y(t) dt + Y(t) \circ dB(t), \\
  Y(0) &= e^{-1}.
\end{align*}
\]  

(4.10)

The Stratonovich exact solution of (4.10) is given as:

\[
Y(t) = \exp\left(\frac{t}{2} + B(t) - 1\right).
\]  

(4.11)

In integral form, we have:

\[
Y(t) = e^{-1} + \frac{1}{2} \int_0^t Y(s) ds + \int_0^t Y(s) dB(s).
\]  

(4.12)

Therefore, with the following definitions:

\[
\tilde{N}(Y(t)) = \frac{1}{2} \int_0^t Y(s) ds + \int_0^t Y(s) dB(s),
\]  

(4.13)

\[
\begin{align*}
  dB(s) &= \sqrt{2} \sum_{j=0}^{5} \left( \cos\left[ \frac{(j+0.5)\pi t}{5} \right] \right) \epsilon_j, \\
  \tilde{N}(Y(t)) &= \frac{1}{2} \int_0^t Y(s) ds + \int_0^t Y(s) dB(s).
\end{align*}
\]  

(4.14)

and

\[
\tilde{N}(Y(t)) = \frac{1}{2} \int_0^t Y(s) ds + \int_0^t Y(s) \left( \sqrt{2} \sum_{j=0}^{5} \left( \cos\left[ \frac{(j+0.5)\pi t}{5} \right] \right) \epsilon_j \right),
\]  

(4.15)

we obtain the iterative scheme via DJM as follows:

\[
Y_0 = e^{-1},
\]

\[
Y_1 = \tilde{N}(Y_0)
\]

\[
= \frac{1}{2} \int_0^t Y_0 ds + \int_0^t Y_0 dB(s) ds
\]

\[
= \frac{1}{2} \int_0^t \left( Y_0 \right) ds + \int_0^t \left( \sqrt{2} \sum_{j=0}^{5} \left( \cos\left[ \frac{(j+0.5)\pi t}{5} \right] \right) \epsilon_j \right) ds,
\]

\[
Y_2 = \tilde{N}(Y_0 + Y_1) - \tilde{N}(Y_0)
\]

\[
= \left\{ \frac{1}{2} \int_0^t (Y_0 + Y_1) ds + \int_0^t \left( (Y_0 + Y_1) dB(s) \right) ds \right\} - \left\{ \frac{1}{2} \int_0^t (Y_0) ds + \int_0^t (Y_0 dB(s)) ds \right\}
\]

\[
= \left\{ \frac{1}{2} \int_0^t (Y_0 + Y_1) ds + \int_0^t \left( Y_0 + Y_1 \right) \left( \sqrt{2} \sum_{j=0}^{5} \left( \cos\left[ \frac{(j+0.5)\pi t}{5} \right] \epsilon_j \right) \right) ds \right\} - \{Y_1\},
\]
\[ Y_3 = N(Y_0 + Y_1 + Y_2) - N(Y_0 + Y_1) \]
\[ = \left\{ \frac{1}{2} \int_0^1 (Y_0 + Y_1 + Y_2) ds + \int_0^1 (Y_0 + Y_1 + Y_2) dB(s) ds \right\} \]
\[ - \left\{ \frac{1}{2} \int_0^1 (Y_0 + Y_1) ds + \int_0^1 (Y_0 + Y_1) dB(s) ds \right\} \]
\[ = \left\{ \frac{1}{2} \int_0^1 (Y_0 + Y_1 + Y_2) ds + \int_0^1 (Y_0 + Y_1 + Y_2) \left( \sqrt{2} \sum_{j=0}^5 \cos((j + 0.5) \pi t) z_j \right) ds \right\} \]
\[ - \left\{ \frac{1}{2} \int_0^1 (Y_0 + Y_1) ds + \int_0^1 (Y_0 + Y_1) \left( \sqrt{2} \sum_{j=0}^5 \cos((j + 0.5) \pi t) z_j \right) ds \right\}, \]

\[ Y_4 = N(Y_0 + Y_1 + Y_2 + Y_3) - N(Y_0 + Y_1 + Y_2) \]
\[ = \left\{ \frac{1}{2} \int_0^1 (Y_0 + Y_1 + Y_2 + Y_3) ds + \int_0^1 (Y_0 + Y_1 + Y_2 + Y_3) dB(s) ds \right\} \]
\[ - \left\{ \frac{1}{2} \int_0^1 (Y_0 + Y_1 + Y_2) ds + \int_0^1 (Y_0 + Y_1 + Y_2) dB(s) ds \right\} \]
\[ = \left\{ \frac{1}{2} \int_0^1 (Y_0 + Y_1 + Y_2 + Y_3) ds + \int_0^1 (Y_0 + Y_1 + Y_2 + Y_3) \left( \sqrt{2} \sum_{j=0}^5 \cos((j + 0.5) \pi t) z_j \right) ds \right\} \]
\[ - \left\{ \frac{1}{2} \int_0^1 (Y_0 + Y_1 + Y_2) ds + \int_0^1 (Y_0 + Y_1 + Y_2) \left( \sqrt{2} \sum_{j=0}^5 \cos((j + 0.5) \pi t) z_j \right) ds \right\}, \]

The computed results and errors analysis are shown in the table 3 and two graphs are plotted for

\( Y(t) \) against \( \Phi_3 \) and \( Y(t) \) against \( \Phi_4 \).

**Table 3:** Solutions and absolute errors via DJM: \( Y(t), \Phi_3(t), \Phi_4(t) \) for Example 4.3

| \( t \) | \( Y(t) \) | \( \Phi_3(t) \) | \( \Phi_4(t) \) | \( |Y(t) - \Phi_3(t)| \) | \( |Y(t) - \Phi_4(t)| \) | \( |\Phi_4(t) - \Phi_3(t)| \) |
|---|---|---|---|---|---|---|
| 0.00 | 0.3679 | 0.3679 | 0.3679 | 0.0000 | 0.0000 | 0.0000 |
| 0.05 | 0.4796 | 0.4796 | 0.4798 | 0.0000 | 0.0002 | 0.0002 |
| 0.10 | 0.6181 | 0.6168 | 0.6198 | 0.0013 | 0.0017 | 0.0030 |
| 0.15 | 0.7692 | 0.7639 | 0.7763 | 0.0053 | 0.0071 | 0.0124 |
| 0.20 | 0.8963 | 0.8846 | 0.9108 | 0.0117 | 0.0145 | 0.0262 |
| 0.25 | 0.9591 | 0.9432 | 0.9784 | 0.0159 | 0.0193 | 0.0352 |
| 0.30 | 0.9570 | 0.9413 | 0.9761 | 0.0157 | 0.0191 | 0.0348 |
| 0.35 | 0.9406 | 0.9260 | 0.9584 | 0.0146 | 0.0178 | 0.0324 |
| 0.40 | 0.9791 | 0.9618 | 1.0000 | 0.0173 | 0.0209 | 0.0382 |
| 0.45 | 1.1350 | 1.1035 | 1.1706 | 0.0315 | 0.0356 | 0.0671 |
| 0.50 | 1.4567 | 1.3823 | 1.5317 | 0.0744 | 0.0750 | 0.1494 |
Figure 5: \( Y(t) \) and \( \Phi_3(t) \) via DJM for Example 4.3 \((N=100, L=5)\)

Figure 6: \( Y(t) \) and \( \Phi_4(t) \) via DJM for Example 4.3 \((N=100, L=5)\)
4. CONCLUDING REMARKS
This paper considered the application of a new iterative integral method referred to as Daftardar-Jafari Integral Method (DJM) for approximate analytical solution of linear Stratonovich Stochastic Differential Equations. The concerned white noise was handled using Karhunen-Loève Expansion in a finite series form, unlike other approaches whose solutions were given in terms of Brownian Motion. Some of the primary advantages of the DJM include its ease of application, direct nature, and less computational activities. Illustrative test cases were considered for testing the effectiveness and efficiency of the method, and the results converged faster to the exact solutions when compared. Based on these, higher order and nonlinear models of Ito or Stratonovich Stochastic Differential Equations (SDEs) can be considered via this approach, and the finite series expansion definition.

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CONFLICT OF INTERESTS
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REFERENCES


