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MATHEMATICAL STUDY OF CORONAVIRUS (MERS-COV)

IMANE EL BERRAI*, KHALID ADNAOUI, JAMAL BOUYAGHROUMNI

Laboratory Analysis, Modeling and Simulation LAMS. Department of Mathematics and Computer Science,

Faculty of Sciences Ben MSik, Hassan II University of Casablanca, Morocco

Abstract. In this paper a mathematical model of Coronavirus (MERS-COV) formulated as a system of parabolic partial differential equations. Immunity is forced through vaccine distribution considered a control variable. Our objective is to prove the existence of solutions to the state system and also the existence of an optimal control.

Keywords: coronavirus; spread of epidemic and compartmental model.

2010 AMS Subject Classification: 92B05.

1. Introduction

In recent years our understanding of infectious disease epidemiology and control has been greatly increased through mathematical modeling with infectious diseases frequently dominating new headlines, public health and pharmaceutical industry professionals, policy makers and infectious disease researchers increasingly need to understand the transmission patters of infectious diseases.

Compartmental models are a technique used to simplify the mathematical modeling of infectious disease. The population is divided into compartments, with the assumption that every

*Corresponding author

E-mail address: im.elberrai@gmail.com

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individual in the same compartment has the same characteristics.
The compartmental model of Kermak-McKendrick [2] is based on relatively simple assumptions on the rate of flow between different classes of members of the population. After Kermak-McKendrick model, different epidemic models have been proposed and studies in the literature(see Hethcote and Tudor [6], Liu [10], Derrick and vanden Driessche [7], Song et al [8]).
We are interested in our study to mathematical model of Coronavirus (MERS-COV). Coronaviruses are a family of viruses that range from the common cold to MERS Coronavirus, which is Middle East Respiratory Syndrome Coronavirus and SARs, Severe acute respiratory syndrome Coronavirus. they are circulating in animals and some of these Coronaviruses have the capability of transmitting between animals and humans. We call that a spillover event.
The remaining parts of this paper are organized as follows: Section 2 is devoted to the mathematical model of Coronavirus and the associated optimal control problem. In Section 3, we prove the existence of a global strong solution for our system. In Section 4, we prove the existence of an optimal solution. Finally, we conclude the paper in Section 5.

2. SPREAD OF CORONAVIRUS

2.1. Mathematical model. A mathematical model of Coronavirus(MERS-COV) transmission is based on the model in Chowell et al[9]. It categorizes each individual into one of six compartment, susceptible(S), exposed(E), symptomatic and infectious(I), infection but asymptomatic(B), hospitalized (F) and recovery (R). It assumed that only infectious and hospitalized individuals can infect others and asymptomatic individuals cannot.
The model takes the following form:

$$
\left\{
\begin{array}{lcl}
\dot{S}(t) & = & \dfrac{-\beta SI - lSF}{N} \\
\dot{E}(t) & = & \dfrac{\beta SI + lSF}{N} - kE \\
\dot{I}(t) & = & k\delta E - (\gamma_b + \gamma_i)I \\
\dot{B}(t) & = & k(1-\delta)E \\
\dot{F}(t) & = & \gamma_b I - \gamma_r F \\
\dot{R}(t) & = & \gamma_i I + \gamma_r F
\end{array}
\right.
\tag{1}
$$

Where $\beta\frac{SI}{N}$ is the total number of infection per unit of time, N is the total population $(N(t) = S(t)+E(t)+I(t)+B(t)+F(t)+R(t) = N(0) = N)$, β is the human-to-human transmission rate per unit time (day) and l quantifies the relative transmissibility of hospitalized patients; k is the rate at which an individual leaves the exposed class by becoming infectious (symptomatic or asymptomatic); δ is the proportion of progression from exposed class E to symptomatic infectious class I, and $(1-\delta)$ is that of progression to asymptomatic class; B; γ_b is the average rate at which symptomatic individuals hospitalize and γ_i is the recovery rate without being hospitalized; γ_r is the recovery rate of hospitalized patients.

We propose another extension of this model, in which we incorporate the spatial behavior of the populations and a term of control representing a vaccination program. The main motivation is to study the effect of a vaccination campaign on the spread of infectious diseases in the context of a more realistic model that takes into account the spatial diffusion. We chose the vaccination as strategy of control because it still remains among the powerful tool that prevent and control the spread of infection. We Assume that the population habitat is a spatially heterogeneous environment, the populations tend to move to regions and their densities will depend on space. The subpopulation in all three compartments are thus tracked not only on time t but also on the spatial location x, leading to the notations $S(t,x), E(t,x), I(t,x), B(t,x), F(t,x)$ and $R(t,x)$ which represent the densities of the three populations at the time t and the spatial position x.

In addition, we assume that the spatial diffusion is through space with λ_1, λ_2, λ_3, λ_4,and λ_6 are the self-diffusion coefficients for each class. With the assumptions explained above in mind, we get the following system of reaction-diffusion equations as a model for the spatial spread of the Coronavirus (MERS-COV):

$$(2) \qquad \begin{cases} \dot{S}(t) &= \lambda_1 \Delta S + \dfrac{-\beta SI - lSF}{N} \\ \dot{E}(t) &= \lambda_2 \Delta E + \dfrac{\beta SI + lSF}{N} - kE \\ \dot{I}(t) &= \lambda_3 \Delta I + k\delta E - (\gamma_b + \gamma_i)I, \qquad (t,x) \in Q = [0,T] \times \Omega \\ \dot{B}(t) &= \lambda_4 \Delta B + k(1-\delta)E \\ \dot{F}(t) &= \gamma_b I - \gamma_r F \\ \dot{R}(t) &= \lambda_6 \Delta R + \gamma_i I + \gamma_r F \end{cases}$$

with the homogeneous Neumann boundary conditions

$$\frac{\partial S}{\partial \eta}=\frac{\partial E}{\partial \eta}=\frac{\partial I}{\partial \eta}=\frac{\partial B}{\partial \eta}=\frac{\partial F}{\partial \eta}=\frac{\partial R}{\partial \eta}=0, \quad (t,x)\in\Sigma=[0,T]\times\partial\Omega$$

$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ represents the usual Laplacian operator, Ω is fixed and bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ is the outward unit normal vector on the boundary, the time t belongs to a finite interval $[0,T]$, while x varies in Ω. Here the homogeneous Neumann boundary condition implies that the above system is self-contained and there is no emigration across the boundary.

The initial distribution of the five populations is supposed to be

$$\begin{array}{lll} S(0,x)=S_0>0, & E(0,x)=E_0>0, & I(0,x)=I_0>0,\\ B(0,x)=B_0>0, & F(0,x)=F_0>0 \quad \text{and} & R(0,x)=R_0>0, \end{array}$$

Strategy of control, we chose a vaccination program, so into the model (2) we include a control u that represents the density of susceptible individuals being vaccinated per time unit and space. We assume that all susceptible vaccinates are transferred directly to the removed class. The dynamics of the controlled system is given by:

$$\text{(3)} \qquad \left\{\begin{array}{lcl} \dot{S}(t) &=& \lambda_1\Delta S+\frac{-\beta SI-lSF}{N}-uS\\ \dot{E}(t) &=& \lambda_2\Delta E+\frac{\beta SI+lSF}{N}-kE\\ \dot{I}(t) &=& \lambda_3\Delta I+k\delta E-(\gamma_b+\gamma_i)I, \qquad (t,x)\in Q\\ \dot{B}(t) &=& \lambda_4\Delta B+k(1-\delta)E\\ \dot{F}(t) &=& \gamma_b I-\gamma_r F\\ \dot{R}(t) &=& \lambda_6\Delta R+\gamma_i I+\gamma_r F+uS \end{array}\right.$$

with the homogeneous Neumann boundary conditions

$$\text{(4)} \qquad \frac{\partial S}{\partial \eta}=\frac{\partial E}{\partial \eta}=\frac{\partial I}{\partial \eta}=\frac{\partial B}{\partial \eta}=\frac{\partial F}{\partial \eta}\frac{\partial R}{\partial \eta}=0, \quad (t,x)\in\Sigma$$

and for $x\in\Omega$

$$S(0,x) = S_0 \quad E(0,x) = E_0 \quad I(0,x) = I_0 \quad B(0,x) = B_0, \quad F(0,x) = F_0 \quad R(0,x) = R_0$$

Our goal is to minimize the density of infected individuals and the cost of vaccination program. Mathematically, it can be interpreted by optimization of the objective functional

$$J(S,E,I,B,F,R,u) = \| I \|^2_{L^2(Q)} + \| I(T,.) \|^2_{L^2(\Omega)} + \alpha \| u \|^2_{L^2(Q)} \tag{5}$$

Where u belongs to the set U_{ad} of admissible controls

$$U_{ad} = \{u \in L^\infty(Q) : \| u \|_{L^\infty(Q)} < 1 \quad and \quad u > 0\} \tag{6}$$

2.2. Existence of global solution.

For $y = (y_1, y_2, y_3, y_4, y_5, y_6)$ and $y^0 = (y_1^0, y_2^0, y_3^0, y_4^0, y_5^0, y_6^0)$ we can put $y = (S,E,I,B,F,R)$, and $y^0 = (S_0, E_0, I_0, B_0, F_0, R_0)$,

$H(\Omega) = (L^2(\Omega))^6$ and A the linear operator defined as follow
$A : D(A) \subset H(\Omega) \to H(\Omega)$

$$\begin{aligned} Ay &= \Big(\lambda_1 \Delta y_1, \lambda_2 \Delta y_2, \lambda_3 \Delta y_3, \lambda_4 \Delta y_4, 0, \lambda_6 \Delta y_6\Big) \in D(A), \\ \forall y &= (y_1, y_2, y_3, y_4, y_5, y_6) \in D(A) \end{aligned} \tag{7}$$

$$\begin{aligned} D(A) &= \Big\{ y = (y_1, y_2, y_3, y_4, y_5, y_6) \in (H^2(\Omega))^6, \frac{\partial y_1}{\partial \eta} = \frac{\partial y_2}{\partial \eta} = \frac{\partial y_3}{\partial \eta} = \frac{\partial y_4}{\partial \eta} \\ &= \frac{\partial y_5}{\partial \eta} = \frac{\partial y_6}{\partial \eta} = 0, a.e \ \ x \in \partial\Omega \Big\} \end{aligned} \tag{8}$$

If we consider the function

$$f(y(t)) = \Big(f_1(y(t)), f_2(y(t)), f_3(y(t)), f_4(y(t)), f_5(y(t)), f_6(y(t))\Big)$$

with

$$
\begin{cases}
f_1(y(t)) &= \dfrac{-\beta y_1 y_3 - l y_1 y_5}{N} - u y_1 \\
f_2(y(t)) &= \dfrac{\beta y_1 y_3 + l y_1 y_5}{N} - k y_2 \\
f_3(y(t)) &= k\delta y_2 - (\gamma_a + \gamma_i) y_3 \\
f_4(y(t)) &= k(1-\delta) y_2 \\
f_5(y(t)) &= \gamma_b y_3 - \gamma_r y_5 \\
f_6(y(t)) &= \gamma_i y_3 + \gamma_r y_5 + u y_1
\end{cases} \tag{9}
$$

Then problem $(3)-(5)$ can be rewritten in the space $H(\Omega)$ under the form

$$
\begin{cases}
\dfrac{\partial y}{\partial t} &= Ay + f(y(t)) \qquad t \in [0,T] \\
y(0) &= y^0
\end{cases} \tag{10}
$$

we denote $L(T,\Omega) = L^2(0,T;H^2(\Omega) \cap L^\infty(0,T;H^1(\Omega))$

Theorem 2.1. *Let* Ω *be a bounded domain from* $\mathbb{R}^2$, *with the boundary of class* $C^{2+\alpha}, \alpha > 0$ *.If* $\beta, l, k, \delta, \gamma_b, \gamma_i, \gamma_r > 0$ $u \in U_{ad}, y \in D(A)$ *and* $y_i^0 \geq 0$ *on* Ω *(for* $i = 1,2,3,4,5,6$*), the problem* $(3)-(5)$ *has a unique (global) strong solution* $y \in W^{1,2}(0,T;H(\Omega))$ *such that*
$y_1, y_2, y_3, y_4, y_5, y_6 \in L(T,\Omega) \cap L^\infty(Q)$
and $y_i \geq 0$ *on* Q *for* $i = 1,2,3,4,5,6$ *İn addition, there exists* $C > 0$ *independent of* u *(and of the corresponding solution* y *) such that for a* $t \in [0,T]$ *For* $i = 1,2,3,4,5,6$*:*

$$
\| \frac{\partial y_i}{\partial t} \|_{L^2(Q)} + \| y_i \|_{L^2(0,T,H^2(\Omega))} + \| y_i \|_{H^1(\Omega)} + \| y_i \|_{L^\infty(Q)} \leq C, \tag{11}
$$

Proof. As $|y_i| \leq N$ for $i = 1,2,3,4,5,6$, thus function $f = \Big(f_1, f_2, f_3, f_4, f_5, f_6\Big)$ becomes Lipschiz continuous in $y = (y_1, y_2, y_3, y_4, y_5, y_6)$ uniformly with respect to $t \in [0,T]$ (See [13, 14, 15]), Eq.(11) admits a unique strong solution
$y = (y_1, y_2, y_3, y_4, y_5, y_6) \in W^{1,2}(0,T;H(\Omega))$ with $y_1, y_2, y_3, y_4, y_5, y_6 \in L(T,\Omega)$
Let's prove the roundedness of y on Q. If we denote:
M= $max\Big\{ \| f_1 \|_{L^\infty(Q)}, \| y_1^0 \|_{L^\infty(\Omega)} \Big\}$

and $\{S(t), t \geq 0\}$ is the C_0-semi-group generated by the operator

$A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$

Where $Ay = \lambda_1 \Delta y_1$

and $D(A) = \left\{ y_1 \in H^2(\Omega), \dfrac{\partial y_1}{\partial \eta} = 0, \;\; a.e\; \partial\Omega \right\}$

The function $Y_1(t,x) = y_1 - Mt - \| y_1^0 \|_{L^\infty(\Omega)}$ it's clearly satisfies the Cauchy problem

$$(12) \qquad \begin{cases} \dfrac{\partial Y_1}{\partial t} &= \lambda_1 \Delta Y_1 + f_1(y(t)) - M \quad t \in [0,T] \\ \\ Y_1(0,x) &= y_1^0 - \| y_1^0 \|_{L^\infty(\Omega)} \end{cases}$$

The corresponding strong solution is
$Y_1(t) = S(t)(y_1^0 - \| y_1^0 \|_{L^\infty(\Omega)}) + \int_0^t S(t-s)(f_1(y(t)) - M)ds$

Since $y_1^0 - \| y_1^0 \|_{L^\infty(\Omega)} \leq 0$ and $f_1(y(t)) - M \leq 0$, it follows that

$Y_1(t,x) \leq 0, \forall (t,x) \in .Q$

And the function:

$W_1(t,x) = y_1 + Mt + \| y_1^0 \|_{L^\infty(\Omega)}$ satisfies the Cauchy problem

$$(13) \qquad \begin{cases} \dfrac{\partial W_1}{\partial t} &= \lambda_1 \Delta Y_1 + f_1(y(t)) + M \quad t \in [0,T] \\ \\ W_1(0,x) &= y_1^0 + \| y_1^0 \|_{L^\infty(\Omega)} \end{cases}$$

The corresponding strong solution is
$W_1(t) = S(t)(y_1^0 + \| y_1^0 \|_{L^\infty(\Omega)} + \int_0^t S(t-s)(f_1(y(t)) + M)ds$

Since $y_1^0 + \parallel y_1^0 \parallel_{L^\infty(\Omega)} \geq 0$ and $f_1(y(t)) + M \geq 0$ it follows that

$W_1(t,x) \geq 0, \forall (t,x) \in Q$ then

$| Y_1(t,x) | \leq Mt + \parallel y_1^0 \parallel_{L^\infty(\Omega)} \forall (t,x) \in Q$

And analogously

$| Y_i(t,x) | \leq Mt + \parallel y_i^0 \parallel_{L^\infty(\Omega)} \forall (t,x) \in Q, \quad for \quad i = 2,3,4,5,6$

So we have proved that $y_i \in L^\infty(Q)(\forall (t,x) \in Q$ for $i = 1,2,3,4,5,6$. By equation (2), we obtain

$$\begin{aligned}(14)\quad &\int_0^t \int_\Omega \left| \frac{\partial y_1}{\partial S} \right| dsdx + \lambda_1^2 \int_0^t \int_\Omega | \Delta y_1^2 | \, dsdx - 2\lambda_1 \int_0^t \int_\Omega \frac{\partial y_1}{\partial S} \Delta y_1 dsdx \\ &= \int_0^t \int_\Omega \left(\frac{-\beta y_1 y_3 - l y_1 y_5}{N} - u y_1 \right)^2 dsdx\end{aligned}$$

Using the regularity of y_1 and the Green's formula, we have

$$(15)\quad 2\int_0^t \int_\Omega \frac{\partial y_1}{\partial S} \Delta y_1 dx = -\int_0^t \frac{\partial}{\partial S} \left(\int_\Omega | \nabla y_1^k |^2 \, dx \right) ds = -\int_\Omega | \nabla y_1 |^2 \, dx + \int_\Omega | \nabla y_1^0 |^2 \, dx$$

Then

$$\begin{aligned}(16)\quad &\int_0^t \int_\Omega \left| \frac{\partial y_1}{\partial S} \right|^2 dsdx + \lambda_1^2 \int_0^t \int_\Omega | \Delta y_1 |^2 \, dsdx + \lambda_1 \int_\Omega | \nabla y_1 |^2 \, dx - \lambda_1 \int_\Omega | \nabla y_1^0 |^2 \, dx \\ &= \int_0^t \int_\Omega \left(\frac{-\beta y_1 y_3 - l y_1 y_5}{N} - u y_1 \right)^2 dsdx\end{aligned}$$

Since $\parallel y_i \parallel_{L^\infty(Q)}$ for $i = 1,2,3,4,5,6$ are bounded independently of u

and $y_1^0 \in H^2(\Omega)$ we deduce that :

$$(17)\quad y_1 \in L^\infty \left([0,T], H^1(\Omega) \right),$$

We make use of (13), and (18) in order to get

$$y_1 \in \pounds(T,\Omega) \cap L^\infty(Q) \tag{18}$$

and conclude that the inequality in (12) holds for $i = 1$ similarly for y_2, y_3, y_4, y_5 and y_6.

In order to show the positiveness of y_i for $i = 1,2,3,4,5,6$ we write system (2) in the form:

$$\begin{cases} \dfrac{\partial y_1}{\partial t} &= \lambda_1 \Delta y_1 + H_1(y_1,y_2,y_3,y_4,y_5,y_6), \\ \dfrac{\partial y_2}{\partial t} &= \lambda_2 \Delta y_2 + H_2(y_1,y_2,y_3,y_4,y_5,y_6), \\ \dfrac{\partial y_3}{\partial t} &= \lambda_3 \Delta y_3 + H_3(y_1,y_2,y_3,y_4,y_5,y_6), \\ \dfrac{\partial y_4}{\partial t} &= \lambda_4 \Delta y_4 + H_4(y_1,y_2,y_3,y_4,y_5,y_6), \\ \dfrac{\partial y_5}{\partial t} &= H_5(y_1,y_2,y_3,y_4,y_5,y_6), \\ \dfrac{\partial y_6}{\partial t} &= \lambda_6 \Delta y_6 + H_6(y_1,y_2,y_3,y_4,y_5,y_6), \\ (t,x) \in Q. \end{cases} \tag{19}$$

It is easy to see that the functions $H_1(y_1,y_2,y_3,y_4,y_5,y_6), H_2(y_1,y_2,y_3,y_4,y_5,y_6)$, $H_3(y_1,y_2,y_3,y_4,y_5,y_6), H_4(y_1,y_2,y_3,y_4,y_5,y_6), H_5(y_1,y_2,y_3,y_4,y_5,y_6)$, and $H_6(y_1,y_2,y_3,y_4,y_5,y_6)$ are continuously differentiable satisfying

$$H_1(0,y_2,y_3,y_4,y_5,y_6) = 0 \geq 0,\ H_2(y_2,0,y_3,y_4,y_5,y_6) = \frac{\beta y_1 y_3 + l y_1 y_5}{N} \geq 0$$

$$H_3(y_1,y_2,0,y_4,y_5,y_6) = k\delta y_2 \geq 0,\ H_4(y_1,y_2,y_3,0,y_5,y_6) = k(1-\delta)y_2 \geq 0$$

$$H_5(y_1,y_2,y_3,y_4,0,y_6) = \gamma_b y_3 \geq 0, H_6(y_1,y_2,y_3,y_4,y_5,0) = \gamma_i y_3 + \gamma_r y_5 + u y_1 \geq 0$$

for all $y_1,y_2,y_3,y_4,y_5,y_6 \geq 0$ (see [1]).This completes the proof.

□

3. EXISTENCE OF THE OPTIMAL SOLUTION

This section is devoted to the existence of an optimal solution. The main result of this section is following

Theorem 3.1. *If* $\beta, l, k, \delta, \gamma_b, \gamma_i, \gamma_r > 0$ *and* $y^0 \in D(A), y_i^0 \geq 0$ *on* Ω *for* $i = 1,2,3,4,5,6)$*, Then the optimal control problem* $(2)-(7)$ *admits an optimal solution* (y^*, u^*)

Proof. Let $J^* = inf\{J(y,u)\}$ Where $u \in U_{ad}$ and y is the corresponding solution of (3)-(5). So $J^*$is finite. Therefore there exist a sequence (y^n, u^n) with $u^n \in U_{ad}$ $y^n = (y_1^n, y_2^n, y_3^n, y_4^n, y_5^n, y_6^n) \in W^{1,2}(0,T,H(\Omega))$ such that

$$(20)\quad \begin{cases} \frac{\partial y_1^n}{\partial t} &= \lambda_1 \Delta y_1^n + \frac{-\beta y_1^n y_3^n - l y_1^n y_5^n}{y_1^n + y_2^n + y_3^n + y_4^n + y_5^n + y_6^n} - u y_1^n \\ \frac{\partial y_2^n}{\partial t} &= \lambda_2 \Delta y_2^n + \frac{\beta y_1^n y_3^n + l y_1^n y_5^n}{y_1^n + y_2^n + y_3^n + y_4^n + y_5^n + y_6^n} - k y_2^n \\ \frac{\partial y_3^n}{\partial t} &= \lambda_3 \Delta y_3^n + k\delta y_2^n - (\gamma_a + \gamma_i) y_3^n \\ \frac{\partial y_4^n}{\partial t} &= \lambda_4 \Delta y_4^n + k(1-\delta) y_2^n \\ \frac{\partial y_5^n}{\partial t} &= \gamma_b y_3^n - \gamma_r y_5^n \\ \frac{\partial y_6^n}{\partial t} &= \lambda_6 \Delta y_6^n + \gamma_i y_3^n + \gamma_r y_5^n + u y_1^n \end{cases}$$

with the homogeneous Neumann boundary conditions

$$(21)\quad \frac{\partial y_1^n}{\partial \eta} = \frac{\partial y_2^n}{\partial \eta} = \frac{\partial y_3^n}{\partial \eta} = \frac{\partial y_4^n}{\partial \eta} = \frac{\partial y_5^n}{\partial \eta} = \frac{\partial y_6^n}{\partial \eta} = 0, \quad (t,x) \in \Sigma$$

$$(22)\quad y_i^n(0,x) = y_i^0, for \;\; i = 1,2,3,4,5,6 with x \in \Omega$$

and

$$(23)\quad J^* \leq J(y^n, u^n) \leq J^* + \frac{1}{n} (\forall n \geq 1)$$

Since $H^1(\Omega)$is compactly embedded in $L^2(\Omega)$, we infer that $y_1^n(t)$is compact in $L^2(\Omega)$ Show that$\{y_1^n(t), n \geq 1\}$is equicontinuous in $C([0,T] : L^2(\Omega)$.

By the first equation from (21)we have

$$\frac{\partial y_1^n}{\partial t}y_1^n = \lambda_1 \Delta y_1^n y_1^n + \frac{-\beta(y_1^n)^2 y_3^n - l(y_1^n)^2 y_5^n}{y_1^n+y_2^n+y_3^n+y_4^n+y_5^n+y_6^n} - u(y_1^n)^2 \tag{24}$$

Then $\forall t \in [0,T]$

$$\int_\Omega (y_1^n)^2(t,x)dx = \int_\Omega (y_1^0)^2(x)dx + 2\int_0^t \int_\Omega \left[\lambda_1 \Delta y_1^n y_1^n + \Gamma - \Lambda - u(y_1^n)^2\right] dxd\xi$$

with $\Gamma = \dfrac{-\beta(y_1^n)^2 y_3^n}{y_1^n+y_2^n+y_3^n+y_4^n+y_5^n+y_6^n}$ and $\Lambda = \dfrac{l(y_1^n)^2 y_5^n}{y_1^n+y_2^n+y_3^n+y_4^n+y_5^n+y_6^n}$

By theorem (1) there exists a constant $C>0$ independent of n such that for all $n \geq 1, t \in [0,T]$

$$\begin{aligned} \left\| \frac{\partial y_i^n}{\partial t} \right\|_{L^2(Q)} &\leq C \parallel y_i^n \parallel_{L^2(0,T,H^2(\Omega)} \\ &\leq C \parallel y_i^n \parallel_{H^1(\Omega)} \\ &\leq C \quad for \;\; i=1,2,3,4,5,5,6 \end{aligned} \tag{25}$$

For all $n \geq 1, t \in [0,T]$, the sequence y_i^n is bounded in $C\Big([0,T]:L^2(\Omega)\Big); \Delta y_i^n, u_1^n$ and $\dfrac{\partial y_i^n}{\partial t}$ are bounded in $L^2(Q)$ for $i=1,2,3,4,5,6$. This implies that for all $s,t \in [0,T]$

$$\left| \int_\Omega (y_i^n)^2(t,x)dx - \int_\Omega (y_i^n)^2(s,x)dx \right| |\leq K \mid t-s \mid \tag{26}$$

The Ascoli-Arzela Theorem (See [11]) implies that y_1^n is compact in $C\Big([0,T]:L^2(\Omega)\Big)$. If necessary, we have $y_1^n \to y_1^*$ in $L^2(\Omega)$, uniformly with respect to t and analogously

$y_i^n \to y_i^*$ in $L^2(\Omega)$, uniformly with respect to t, for $i=2,3,4,5,6$. then $y_2^n(T) \to y_2^*(T)$ in $L^2(\Omega)$ The boundedness of Δy_i^n in $L^2(Q)$, implies its weak convergence, namely $\Delta y_i^n \to \Delta y_i^*$ in $L^2(Q)$ for $i=2,3,4,5,6$. Here and everywhere below the sign $\rightharpoonup$ denotes the weak convergence in the specified space. Estimates (26) lead to

$\frac{\partial y_i^n}{\partial t} \to \frac{\partial y_i^*}{\partial t}$ in $L^2(Q), i = 1,2,3,4,5,6$

$y_i^n \to y_i^*$ in $L^2\Big(0,T,H^2(\Omega)\Big), i = 1,2,3,4,5,6$

$y_i^n \to y_i^*$ in $L^\infty\Big(0,T,H^1(\Omega)\Big), i = 1,2,3,4,5,6$

We put

$$N_1(y) = \frac{\beta}{y_1+y_2+y_3+y_4+y_5+y_6} \quad \text{and} \quad N_2(y) = \frac{l}{y_1+y_2+y_3+y_4+y_5+y_6}$$

we now show that $y_1^n y_3^n \to y_1^* y_3^*, y_1^n y_5^n \to y_1^* y_5^*, N_1(y^n) y_1^n y_3^n \to N_1(y^*) y_1^* y_3^*$ and $N_2(y^n) y_1^n y_5^n \to N_2(y^*) y_1^* y_5^*$ strongly in $L^2(Q)$, and we write

$$y_1^n y_3^n - y_1^* y_3^* = (y_1^n - y_1^*) y_3^n + (y_3^n - y_3^*) y_1^* \quad \text{and} \quad y_1^n y_5^n - y_1^* y_5^* = (y_1^n - y_1^*) y_5^n + (y_5^n - y_5^*) y_1^*$$

So

$$N_1(y^n) = \frac{\beta}{y_1^n+y_2^n+y_3^n+y_4^n+y_5^n+y_6^n} \quad ; \quad N_1(y^*) = \frac{\beta}{y_1^*+y_2^*+y_3^*+y_4^*+y_5^*+y_6^*}$$

And

$$N_2(y^n) = \frac{l}{y_1^n+y_2^n+y_3^n+y_4^n+y_5^n+y_6^n} \quad ; \quad N_2(y^*) = \frac{l}{y_1^*+y_2^*+y_3^*+y_4^*+y_5^*+y_6^*}$$

Also $N_1(y^n) y_1^n y_3^n - N_1(y^*) y_1^* y_3^* = N_1(y^n)(y_1^n y_3^n - y_1^* y_3^*) + y_1^* y_3^* (N_1(y^n) - N_1(y^*))$

$$N_2(y^n) y_1^n y_5^n - N_2(y^*) y_1^* y_5^* = N_2(y^n)(y_1^n y_5^n - y_1^* y_5^*) + y_1^* y_5^* (N_2(y^n) - N_2(y^*))$$

And we make use of the convergences $y_i^n \to y_i^*$ strongly in $L^2(Q), i = 1,3,5$ and of the boundedness of y_1^n, y_3^n and y_5^n in $L(\infty)(Q)$,
and then $y_1^n y_3^n \to y_1^* y_3^*, y_1^n y_5^n \to y_1^* y_5^*, N_1(y^n) y_1^n y_3^n \to N_1(y^*) y_1^* y_3^*$
and $N_2(y^n) y_1^n y_5^n \to N_2(y^*) y_1^* y_5^*$ strongly in $L^2(Q)$.

We also have $u^n \to u^*$ in $L^2(Q)$ on a subsequence denoted again u^n. Since U_{ad} is a closed and convex set in $L^2(Q)$, *is weakly closed, so* $u^* \in U_{ad}$ *and as above* $u^n y_1^n \to u^* y_1^*$

in $L^2(Q)$. Now we may pass to the limit in $L^2(Q)$ as $n \to +\infty$ in $(21-24)$ to deduce that (y^*, u^*) is an optimal solution. The proof is complete. □

CONCLUSION

The work in this paper contributes to a growing literature on modeling the spatial spread of an infectious disease. We present a novel application of optimal control theory to spatiotemporal epidemic models described by a system of partial differential equations. The control variable is the spatial and temporal distribution of vaccine. We have based our mathematical work on the use of semigroup theory and optimal control to show the existence of solutions for our state system, as well as prove the existence of an optimal control.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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