DYNAMICAL ANALYSIS OF POLLUTED PREY-PREDATOR SYSTEM WITH INFECTED PREY

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Abstract. In this paper, a prey-predator model in polluted environment with disease in prey has been proposed and studied. It is assumed that only prey population is prone to disease whereas, both the populations are affected by the pollutant. Boundedness of the solution of the system is discussed. Existence of all possible equilibrium points has been established. Using Routh Hurwitz criterion, local stability of all the possible equilibrium points has been obtained. Also, interior equilibrium point has been proved to be globally asymptotically stable using Lyapunov function. Then time delay has been introduced in the system making the model more realistic. Existence and direction of Hopf bifurcation in the delay model has been established using normal form theory and center manifold theorem. By taking a set of hypothetical and biologically feasible parameters, model has been studied numerically using MATLAB and the effect of pollutant on the system has been deduced.

Keywords: non-linear incidence rate; holling type-II functional response; equilibrium point; stability analysis; hopf bifurcation.

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1. INTRODUCTION

The relation between the predators and their prey is the building block of ecosystems. Due to its widespread existence and importance, this dynamic relation has always been an important topic of study in ecology as well as mathematical ecology. When a number of prey-predator interactions take place in the environment at different trophic levels, food chains and eventually food webs are formed. There are many factors that influence these food webs such as climate, natural disaster, other food chains etc., due to which species evolve and disperse in order to seek resources for their survival and existence in the ecosystem. So, populations continuously move away from one state to the other state.

There has been huge growth in industry, agriculture etc. which has taken the comfort of mankind to the next level. All these developments including urbanization has helped people attaining a better lifestyle. The byproducts of the processes are not just the products and services that we purchase from the market, but also the waste products that are eliminated into the environment. This waste is sometimes treated and is less harmful for the environment or sometimes untreated, consumption of which by the organisms could be lethal. The different forms of waste could be organic, inorganic, radioactive etc. In case of radioactive, organisms may suffer from harmful diseases, birth defects or even gene mutation.

The incubation period is defined as the time period between exposure to an infection and appearance of the first symptom. There are very less mechanisms in this world that are instantaneous. For example, human body already contains cancer genes. The symptoms start occurring only when those genes are exposed to the trigger. So, there is a time lag or delay which is termed as the incubation period, after which the effect of infection could be seen. When we consider delay while defining certain mechanism, it becomes more appropriate according to the real life environment, making any study more reasonable. In recent decades many investigators have proposed and analyzed mathematical models to study the effects of toxicants on biological species.

In [14], it is assumed that the toxicant affects both prey and predator population where the infected prey is more vulnerable to be affected by toxicant and predation as compared to the susceptible prey population. In [15], the effect of only disease and the effect of disease as well

as toxicant on a plant population has been studied. The problem of ratio-dependent predator-prey model has been studied in [10]. In [9], a prey-predator model has been discussed where prey has logistic growth and the model is modified to include parasitic infection in prey where the infected prey becomes more vulnerable to predation. Four modifications of a predator prey model are developed and analyzed in [11] including parasite infection. In [13], authors have shown that the exposure to the pollutants can lead to immunosuppression and increased disease susceptibility in juvenile salmon. [4] determines the direction of the Hopf bifurcation about the equilibrium using center manifold theorem. Also, a settled modelling approach was proposed to the problem of investigating the effects of a pollutant on an ecological system in [5, 6, 7, 8]. Delay differential equations are widely used in epidemiology and problems related to delay have been studied by various authors [2, 3]. A prey-predator model with harvesting and diseased prey, in absence and presence of time delay has been analysed in [16]. [1] talks about the transmission and control of epidemics, where time delay is associated with the infected species. The chaotic dynamics induced by a disease in an eco-epidemiological prey-predator model with diseased prey and weak Allee in predator has been studied in [12].

Keeping in view the above discussion, in this paper, we have proposed a prey-predator system with combined effect of disease and pollutant in section 2. In our model, we have incorporated disease and pollutant and studied its effect on prey-predator dynamics. After formulating the model, dynamical behavior of the system has been studied in section 3, in which existence of all possible equilibrium point has been obtained and stability, local and global, of the system has been analyzed. Dynamical analysis of the system with time delay has been done in section 4. Section 5 deals with the numerical simulations where a set of hypothetical and biologically feasible parameters has been considered and effect of pollutant on the system has been deduced. In section 6, the results obtained theoretically and numerically have been discussed.

2. MATHEMATICAL MODEL

The prey-predator model under the effect of pollutant is considered. The prey population, which is susceptible and infected is denoted by $S(t)$ and $I(t)$ respectively and the predator is denoted by $P(t)$. Also, $C(t)$ is the environmental concentration of the pollutant and $U(t)$ is the concentration of the pollutant in the organisms. The assumptions adopted are:

1. The reproduction in susceptible prey takes place according to constant growth rate. The transmission of disease from infected to susceptible prey takes place by contact. This transmission occurs according to non linear incidence rate of the form $\frac{\lambda SI}{1+I}$, where λSI is the infection force of disease and $\frac{1}{1+I}$ measures the effect of inhibition from the susceptible prey. This inhibition effect occurs due to behavioral change of susceptible population that includes increase in number or crowding effect of infected prey.

2. The predators attack the susceptible and infected individuals with different rates. The consumption of susceptible prey is according to $\frac{\alpha_1 S}{\beta+S+mI}$ and infected prey is according to $\frac{\alpha_2 I}{\beta+S+mI}$, which are known as modified Holling type-II functional response.

3. Food and the environment both are the sources of pollutant uptake by the populations. The loss of pollutant from the organisms takes place due to metabolic processing and other causes. If q is the constant exogenous input rate of the pollutant into the environment, $C(t)$ is the environmental concentration of the pollutant and the natural loss rate of pollutant from environment can be due to biological transformation, hydrolysis, vitalization, microbial degradation, including other processes then the model proposed is as follows:

$$(1) \qquad \frac{dS}{dt} = \Lambda - \frac{\lambda SI}{1+I} - \frac{\alpha_1 SP}{\beta+S+mI} - r_1 US - d_1 S$$

$$(2) \qquad \frac{dI}{dt} = \frac{\lambda SI}{1+I} - \frac{\alpha_2 IP}{\beta+S+mI} - r_2 UI - d_2 I$$

$$(3) \qquad \frac{dP}{dt} = \frac{(e\alpha_1 S + e\alpha_2 I)P}{\beta+S+mI} - r_3 UP - d_3 P$$

$$(4) \qquad \frac{dC}{dt} = -hC + q$$

$$(5) \qquad \frac{dU}{dt} = a_1 C + \frac{dn\phi}{a_1} - (l_1 + l_2)U$$

Since we know from (4) and (5) that $\limsup\limits_{t\to\infty} C(t) \leq C^*$ and $\limsup\limits_{t\to\infty} U(t) \leq U^*$, thus, using the corollary 1 in [7] in the model we get the limiting system as follows:

$$(6) \qquad \frac{dS}{dt} = \Lambda - \frac{\lambda SI}{1+I} - \frac{\alpha_1 SP}{\beta+S+mI} - r_1 U^* S - d_1 S$$

(7)
$$\frac{dI}{dt} = \frac{\lambda SI}{1+I} - \frac{\alpha_2 IP}{\beta + S + mI} - r_2 U^* I - d_2 I$$

(8)
$$\frac{dP}{dt} = \frac{(e\alpha_1 S + e\alpha_2 I)P}{\beta + S + mI} - r_3 U^* P - d_3 P$$

All the parameters in the above defined systems are assumed to have positive values and are described as follows:

Parameter	Description	Parameter	Description
Λ	growth rate constant	n	concentration of pollutant in resource
α_1	predation rate of S	α_2	predation rate of I
β	half saturation constant	ϕ	average rate of food intake per unit mass organism
$e\alpha_1$	conversion rate of S	d_2	rate at which predator population is decreasing due to pollutant
d_1	natural death rate of S	r_3	loss rate of pollutant from the environment
d_3	natural death rate of P	r_1	rate at which susceptible population is decreasing due to pollutant
h	natural death rate of I	r_2	rate at which infected population is decreasing due to pollutant
$e\alpha_2$	conversion rate of I	a_1	environmental pollutant uptake rate per unit mass organism
λ	infected rate	d	uptake rate of pollutant in food per unit mass organism
m	predator's favorite rate	l_1, l_2	organismal net ingestion and depuration rates of pollutant respectively

TABLE 1. Description of the parameters

Since any infection takes time to get incubated into the organism, so we consider the system defined by equations (6)-(8) with time delay, which is more appropriate according to real

environment. Therefore, the system becomes:

$$(9) \qquad \frac{dS}{dt} = \Lambda - \frac{\lambda SI}{1+I} - \frac{\alpha_1 SP}{\beta + S + mI} - r_1 U^* S - d_1 S$$

$$(10) \qquad \frac{dI}{dt} = \frac{\lambda S(t-\tau)I(t-\tau)}{1+I(t-\tau)} - \frac{\alpha_2 IP}{\beta + S + mI} - r_2 U^* I - d_2 I$$

$$(11) \qquad \frac{dP}{dt} = \frac{(e\alpha_1 S + e\alpha_2 I)P}{\beta + S + mI} - r_3 U^* P - d_3 P$$

where, $\tau \geq 0$ is the time interval for the infection to get incubated into the prey species. Also,

$$S(\theta) = \psi_1(\theta), I(\theta) = \psi_2(\theta), P(\theta) = \psi_3(\theta),$$

$$\psi_i(\theta) \geq 0, \psi_i(0) > 0, i = 1, 2, 3, -\tau \leq \theta \leq 0$$

where $\psi(\theta) = (\psi_1(\theta), \psi_2(\theta), \psi_3(\theta)) \in C([-\tau, 0], \mathbb{R}^3_+)$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^3_+$.

2.1. Basic Properties of the Model. The density of population cannot be negative, so the state space of the system is $\mathbb{R}^3_+ = \{(S, I, P) \in \mathbb{R}^3 : S \geq 0, I \geq 0, P \geq 0\}$. To support the positivity and boundedness of the system, we start with lemmas given below:

Lemma 1: All the solutions of the system defined by equations (6)-(8) are positive $\forall\, t \geq 0$.

Proof: Let $(S(t), I(t), P(t))$ be any solution of the system defined by equations (6)-(8). We assume that there exists a solution of the system that is at least not positive. Following cases arise:

Case 1 $\exists\, t^*$ such that

$$S(0) > 0, S(t^*) = 0, S'(t^*) < 0, I(t) > 0, P(t) > 0, 0 \leq t < t^*$$

Case 2 $\exists\, \hat{t}$ such that

$$I(0) > 0, I(\hat{t}) = 0, I'(\hat{t}) < 0, S(t) > 0, P(t) > 0, 0 \leq t < \hat{t}$$

Case 3 $\exists\, \tilde{t}$ such that

$$P(0) > 0, P(\tilde{t}) = 0, P'(\tilde{t}) < 0, S(t) > 0, I(t) > 0, 0 \leq t < \tilde{t}$$

If case 1 holds then $S'(t^*) = \Lambda > 0$, that contradicts with $S'(t^*) < 0$.

If case 2 holds then $I'(\hat{t}) = 0$, that contradicts with $I'(\hat{t}) < 0$.

If case 3 holds then $P'(\tilde{t}) = 0$, that contradicts with $P'(\tilde{t}) < 0$.

Since $(S(t), I(t), P(t))$ was arbitrary, all the solutions of the system are positive $\forall\, t > 0$.

Lemma 2: All the solutions of the system that initiate in the state space $\mathbb{R}^3_+$ are uniformly bounded.

Proof: Let $(S(t), I(t), P(t))$ be any solution of the system with non-negative initial conditions. Consider, $W(t) = S(t) + I(t) + P(t)$, then

$$\frac{dW}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dP}{dt}$$

Since the constant of rate of conversion from prey population to predator population cannot exceed the maximum predation rate constant of predator population to prey population, therefore $e\alpha_i \leq \alpha_i, i = 1, 2$. Also, take $d = min\{d_1, d_2, d_3\}$

$$\implies \limsup_{t \to \infty} W \leq \frac{\Lambda}{d}$$

Thus, all the solutions are bounded. Hence Proved.

3. DYNAMICAL BEHAVIOR OF THE MODEL

3.1. Existence of Equilibrium Points.
There is a possibility of the system to have five equilibrium points: $E_1 = (0, 0, 0)$, $E_2 = (\bar{S}, 0, 0)$, $E_3 = (\tilde{S}, 0, \tilde{P})$, $E_4 = (\hat{S}, \hat{I}, 0)$, $E_5 = (S^*, I^*, P^*)$

(a) The equilibrium point $E_1 = (0, 0, 0)$ obviously exists. Now to check the existence of other equilibrium points.

(b) $E_2 = (\bar{S}, 0, 0)$ exists uniquely where $\bar{S} = \frac{\Lambda}{r_1 U^* + d_1}$.

(c) $E_3 = (\tilde{S}, 0, \tilde{P})$ is a disease free equilibrium and we can see that it always has a unique positive value:

$$\tilde{S} = \frac{d_3 + r_3 U^*}{e\alpha_1 - d_3 - r_3 U^*} \beta.$$

$$\tilde{P} = \left(\frac{\beta + \tilde{S}}{\alpha_1}\right)\left(\frac{\Lambda}{\tilde{S}} - r_1 U^* - d_1\right)$$

which exists provided that,

$$U^* < \frac{1}{r_1}\left(\frac{\Lambda}{\tilde{S}} - d_1\right)$$

$$e\alpha_1 > d_3 + r_3 U^*$$

i.e. disease free equilibrium exists if pollution is under certain level.

(d) $E_4 = (\hat{S}, \hat{I}, 0)$ is a predator free equilibrium and has a unique positive value:

$$\hat{S} = \frac{1+\hat{I}}{\lambda}(d_2 + r_2 U^*)$$

$$\hat{I} = \frac{\lambda\Lambda - (r_1 U^* + d_1)(d_2 + r_2 U^*)}{(\lambda + r_1 U^* + d_1)(d_2 + r_2 U^*)}$$

which exists provided that

$$\lambda\Lambda > (r_1 U^* + d_1)(r_2 U^* + d_2)$$

(e) $E_5 = (S^*, I^*, P^*)$ has a unique existence

$$I^* = \frac{\beta(d_3 + r_3 U^*) + S^*(d_3 + r_3 U^* - e\alpha_1)}{e\alpha_2 - m(d_3 + r_3 U^*)} = h_1(S^*)$$

$$P^* = \frac{1}{\alpha_2}\left(\frac{\lambda S^*}{1 + h_1(S^*)} - d_2 - r_2 U^*\right)\left(\beta + S^* + mh_1(S^*)\right) = h_2(S^*)$$

where $S^* \in \left(0, \frac{\Lambda}{d_1}\right)$ represents a positive root of the equation

$$H(S) = \Lambda - \frac{\lambda S h_1(S)}{1 + h_1(S)} - \frac{\alpha_1 S h_2(S)}{\beta + S + mh_1(S)} - r_1 U^* S - d_1 S$$

We now prove the existence of S^*. It can be easily verified that $h_1(S)$ and $h_2(S)$ are positive for all values of $S \in \left(0, \frac{\Lambda}{d_1}\right)$ under the following conditions:

$$(12) \qquad\qquad\qquad e\alpha_1 < d_3 + r_3 U^*$$

$$(13) \qquad\qquad\qquad e\alpha_2 > m(d_3 + r_3 U^*)$$

$$(14) \qquad\qquad\qquad \frac{\lambda S^*}{1 + h_1(S)} > d_2 + r_2 U^*$$

We have, $H(0) = \Lambda$, which is greater than zero. Also,

$$H\left(\frac{\Lambda}{d_1}\right) = -\left[\frac{\lambda\frac{\Lambda}{d_1}h_1(\frac{\Lambda}{d_1})}{1 + h_1(\frac{\Lambda}{d_1})} + \frac{\frac{\Lambda}{d_1}\alpha_1 h_2(\frac{\Lambda}{d_1})}{\beta + \frac{\Lambda}{d_1} + mh_1(\frac{\Lambda}{d_1})} + r_1 U^*\frac{\Lambda}{d_1}\right]$$

$h_1(\frac{\Lambda}{d_1})$, $h_2(\frac{\Lambda}{d_1})$, r_1, d_1 and U^* are all greater than zero. So, $H\left(\frac{\Lambda}{d_1}\right) < 0$. Moreover,

$$\frac{dH}{dS} = -\lambda \left[\frac{(1+h_1(S))(Sh'_1(S)+h_1(S))-Sh_1(S)h'_1(S)}{(1+h_1(S))^2} \right.$$
$$\left. - \alpha_1 \left[\frac{(\beta+S+mh_1(S))(Sh'_2(S)+h_2(S))-Sh_2(S)(1+mh'_1(S))}{(\beta+S+mh_1(S))^2} \right] - r_1 U^* - d_1 \right.$$

where,

$$h'_1(S) = \frac{d_3+r_3 U^* - e\alpha_1}{e\alpha_2 - m(d_3+r_3 U^*)} > 0$$

$$h'_2(S) = \frac{1}{\alpha_2}\left[(\beta+S+mI)\frac{\lambda\left(1+h_1(S)-Sh'_1(S)\right)}{(1+h_1(S))^2} + \left(\frac{\lambda S}{1+h_1(S)} - d_2 - r_2 U\right)\left(1+mh'_1(S)\right)\right]$$

Now, $h_1(S) > Sh'_1(S)$ holds true.

$\implies \frac{dH}{dS} < 0$ for all values of $S \in \left(0, \frac{\Lambda}{d_1}\right)$. Therefore, by Intermediate Value Theorem, there

exists $S^* \in \left(0, \frac{\Lambda}{d_1}\right)$ which is a unique root of $H(S)$ equation and hence interior equilibrium point

E_5 exists uniquely in $Int\mathbb{R}^3_+$ iff equations (12)-(14) are satisfied.

3.2. Stability Analysis.

In this section, stability analysis of all the five equilibrium points is

carried out using Routh Hurwitz criterion or Lyapunov function. The Jacobian matrix of the

system is given by $V(E) = (a_{ij})_{3X3}$ and $i, j = 1, 2, 3$; where

$a_{11} = \frac{-\lambda I^*}{1+I^*} - \alpha_1 P^* \frac{\beta+mI^*}{(\beta+S^*+mI^*)^2} - r_1 U^* - d_1, a_{12} = \frac{-\lambda S^*}{(1+I^*)^2} + \alpha_1 P^* \frac{\beta+mI^*}{(\beta+S^*+mI^*)^2}$

$a_{13} = \frac{-\alpha_1 S^*}{\beta+S^*+mI^*}, a_{21} = \frac{\lambda I^*}{1+I^*} + \frac{\alpha_2 I^* P^*}{(\beta+S^*+mI^*)^2}$

$a_{22} = \frac{\lambda S^*}{(1+\hat{I})^2} - \frac{\alpha_2 P^*(\beta+S^*)}{(\beta+S^*+mI^*)^2} - d_2 - r_2 U^*, a_{23} = \frac{-\alpha_2 I^*}{\beta+S^*+mI^*}$

$a_{31} = \frac{P^*}{(\beta+S^*+mI^*)^2}[(\beta+mI^*)e\alpha_1 - e\alpha_2 I^*], a_{32} = \frac{P^*}{(\beta+S^*+mI^*)^2}[(\beta+S^*)e\alpha_2 - e\alpha_1 mS^*]$

$a_{33} = \frac{e\alpha_1 S^* + e\alpha_2 I^*}{\beta+S^*+mI^*} - d_3 - r_3 U^*$

Theorem 1: The equilibrium point $E_1 = (0,0,0)$ is locally asymptotically stable.

Proof: From the Jacobian matrix at $E_1 = (0,0,0)$, the eigen values obtained are

$-d_1 - r_1 U^* < 0, -d_2 - r_2 U^* < 0, -d_3 - r_3 U^* < 0$. So, E_1 is asymptotically stable.

Theorem 2: The equilibrium point $E_2 = (\bar{S},0,0)$ of the system is locally asymptotically stable

provided that the following conditions are satisfied:

(15) $$\lambda\bar{S} < d_2 + r_2 U^*$$

(16)
$$\frac{e\alpha_1 \bar{S}}{\beta + \bar{S}} < d_3 + r_3 U^*$$

Proof: The characteristic equation of the Jacobian matrix at E_2 is given by:

$$\left(\lambda \bar{S} - d_2 - r_2 U^* - \gamma\right)\left(\frac{e\alpha_1 \bar{S}}{\beta + \bar{S}} - d_3 - r_3 U^* - \gamma\right)\left(-r_1 U^* - d_1 - \gamma\right) = 0$$

So, $\gamma = \lambda \bar{S} - d_2 - r_2 U^*$, $\gamma = \dfrac{e\alpha_1 \bar{S}}{\beta + \bar{S}} - d_3 - r_3 U^*$ and $\gamma = -r_1 U^* - d_1$, which are all less than zero provided that

$$\lambda \bar{S} < d_2 + r_2 U^*$$

$$\frac{e\alpha_1 \bar{S}}{\beta + \bar{S}} < d_3 + r_3 U^*$$

hold. Since all the eigen values are negative, therefore $E_2 = (\bar{S}, 0, 0)$ is locally asymptotically stable if (15)-(16) hold.

Theorem 3 The equilibrium point $E_3 = (\tilde{S}, 0, \tilde{P})$ of the system is locally asymptotically stable provided that the following conditions are satisfied:

(17)
$$\lambda \tilde{S} < \frac{\alpha_2 \tilde{P}}{\beta + \tilde{S}} + d_2 + r_2 U^*$$

(18)
$$\frac{e\alpha_1 \tilde{S}}{\beta + \tilde{S}} < d_3 + r_3 U^*$$

Proof: The characteristic equation of the Jacobian matrix $V(E_3)$ is given by:

$$\left(\lambda \tilde{S} - \frac{\alpha_2 \tilde{P}}{\beta + S} - d_2 - r_2 U^* - \gamma\right) *$$

$$\left[\left(-\frac{\alpha_1 \beta \tilde{P}}{(\beta + \tilde{S})^2} - d_1 - r_1 U^* - \gamma\right)\left(\frac{e\alpha_1 \tilde{S}}{\beta + \tilde{S}} - d_3 - r_3 U^* - \gamma\right) - \left(\frac{-\alpha_1 \tilde{S}}{\beta + \tilde{S}}\right)\left(\frac{\beta e \alpha_1 \tilde{P}}{(\beta + \tilde{S})^2}\right)\right] = 0$$

So, $\gamma = \lambda \tilde{S} - \frac{\alpha_2 \tilde{P}}{\beta + S} - d_2 - r_2 U^* < 0$ because of equation (17) and other two eigen values are the roots of equation

$$\left[\left(-\frac{\alpha_1 \beta \tilde{P}}{(\beta + \tilde{S})^2} - d_1 - r_1 U^* - \gamma\right)\left(\frac{e\alpha_1 \tilde{S}}{\beta + \tilde{S}} - d_3 - r_3 U^* - \gamma\right) - \left(\frac{-\alpha_1 \tilde{S}}{\beta + \tilde{S}}\right)\left(\frac{\beta e \alpha_1 \tilde{P}}{(\beta + \tilde{S})^2}\right)\right] = 0$$

which is of the form $\gamma^2 + A\gamma + B = 0$, where,

$$A = \frac{\alpha_1 \beta \tilde{P}}{(\beta + \tilde{S})^2} + d_1 + r_1 U^* - \left(\frac{e\alpha_1 \tilde{S}}{\beta + \tilde{S}} - d_3 - r_3 U^* \right)$$

$$B = \left(-\frac{\alpha_1 \beta \tilde{P}}{(\beta + \tilde{S})^2} - d_1 - r_1 U^* \right) \left(\frac{e\alpha_1 \tilde{S}}{\beta + \tilde{S}} - d_3 - r_3 U^* \right) + \left(\frac{\alpha_1 \tilde{S}}{\beta + \tilde{S}} \right) \left(\frac{\beta e\alpha_1 \tilde{P}}{(\beta + \tilde{S})^2} \right)$$

Using the given conditions, we get that $A > 0$ and $B > 0$. Using Routh Hurwitz criteria, there exist two roots of the polynomial $\gamma^2 + A\gamma + B = 0$ i.e. the eigen values of $V(E)$ at E_3 with negative real parts. Since all the eigen values are negative, therefore $E_3 = (\tilde{S}, 0, \tilde{P})$ is locally asymptotically stable.

Theorem 4 Assume that the predator free equilibrium point $E_4 = (\hat{S}, \hat{I}, 0)$ exists. Then it is locally asymptotically stable provided that the following conditions are satisfied:

(19)
$$\frac{\lambda \hat{S}}{(1 + \hat{I})^2} < d_2 + r_2 U^*$$

(20)
$$\frac{e\alpha_1 \hat{S} + e\alpha_2 \hat{I}}{\beta + \hat{S} + m\hat{I}} < d_3 + r_3 U^*$$

Proof: The characteristic equation of the Jacobian matrix $V(E_4)$ is given by:

$$\left(\frac{e\alpha_1 \hat{S} + e\alpha_2 \hat{I}}{\beta + \hat{S} + m\hat{I}} - d_3 - r_3 U^* - \gamma \right)$$

$$\left[\left(\frac{-\lambda \hat{I}}{1 + \hat{I}} - d_1 - r_1 U^* - \gamma \right) \left(\frac{\lambda \hat{S}}{(1 + \hat{I})^2} - d_2 - r_2 U^* - \gamma \right) - \left(\frac{-\lambda \hat{S}}{(1 + \hat{I})^2} \right) \left(\frac{\lambda \hat{I}}{1 + \hat{I}} \right) \right] = 0$$

So, using the given condition,

$$\gamma = \frac{e\alpha_1 \hat{S} + e\alpha_2 \hat{I}}{\beta + \hat{S} + m\hat{I}} - d_3 - r_3 U^* < 0$$

and other two eigen values are the roots of equation

$$\left(\frac{-\lambda \hat{I}}{1 + \hat{I}} - d_1 - r_1 U^* - \gamma \right) \left(\frac{\lambda \hat{S}}{(1 + \hat{I})^2} - d_2 - r_2 U^* - \gamma \right) - \left(\frac{-\lambda \hat{S}}{(1 + \hat{I})^2} \right) \left(\frac{\lambda \hat{I}}{1 + \hat{I}} \right) = 0$$

which is of the form $\gamma^2 + A\gamma + B = 0$

where,

$$A = \frac{\lambda \hat{I}}{1+\hat{I}} + r_1 U^* + d_1 - \frac{\lambda \hat{S}}{(1+\hat{I})^2} + d_2 + r_2 U^*$$

$$B = \left(\frac{\lambda \hat{S}}{(1+\hat{I})^2}\right)\left(\frac{\lambda \hat{I}}{1+\hat{I}}\right) + \left(\frac{\lambda \hat{I}}{1+\hat{I}} + r_1 U^* + d_1\right)\left(\frac{-\lambda \hat{S}}{(1+\hat{I})^2} + d_2 + r_2 U^*\right)$$

Using (19) we get that $A > 0$ and $B > 0$. Using Routh Hurwitz criteria, all the eigen values of $V(E)$ at E_4 with negative real parts. Since all the eigen values are negative, therefore $E_4 = (\hat{S}, \hat{I}, 0)$ is locally asymptotically stable.

Theorem 5 Assume that the interior equilibrium point $E_5 = (S^*, I^*, P^*)$ of the system exists. Let the following conditions are satisfied:

(21)
$$\frac{\lambda I^*}{1+I^*} + \alpha_1 P^* \frac{\beta + mI^*}{(\beta + S^* + mI^*)^2} + r_1 U^* + d_1 > 0$$

(22)
$$\frac{\alpha_2 P^* (\beta + S^*)}{(\beta + S^* + mI^*)^2} + d_2 + r_2 U^* > \frac{\lambda S^*}{(1+I^*)^2}$$

(23)
$$d_3 + r_3 U^* > \frac{e\alpha_1 S^* + e\alpha_2 I^*}{\beta + S^* + mI^*}$$

(24)
$$(\beta + S^*)\alpha_2 > \alpha_1 S^* m$$

(25)
$$\frac{\lambda}{(1+I^*)^2} > \frac{\alpha_1 P^* m}{(\beta + S^* + mI^*)^2}$$

(26)
$$(\beta + mI^*)\alpha_1 > \alpha_2 I^*$$

Then, E_5 is locally asymptotically stable.

Proof: The characteristic equation of the Jacobian matrix $V(E_5)$ is given by $\gamma^3 + A\gamma^2 + B\gamma + C = 0$ where,

$$A = (-a_{11} - a_{22} - a_{33})$$

$$B = R_1 + R_2 + R_3 + R_4 + R_5 + R_6$$

$$C = C_1 + C_2 + C_3 + C_4 + C_5 + C_6$$

where,

$R_1 = a_{22}a_{33}$	$R_2 = -a_{23}a_{32}$	$R_3 = -a_{12}a_{21}$
$R_4 = -a_{13}a_{31}$	$R_5 = a_{11}a_{33}$	$R_6 = a_{11}a_{22}$
$C_1 = -a_{11}a_{22}a_{33}$	$C_2 = a_{11}a_{23}a_{32}$	$C_3 = a_{33}a_{12}a_{21}$
$C_4 = -a_{12}a_{31}a_{23}$	$C_5 = -a_{13}a_{21}a_{32}$	$C_6 = a_{22}a_{13}a_{31}$

From the given conditions we get that $A > 0$, $R_i > 0$ where $i = 1,2,3,4,5,6$ and $C_j > 0$ where $j = 1,2,3,4,5,6$. Therefore, $A, B, C > 0$.

Now applying Routh Hurwitz criterion, $\Delta = AB - C$

$$AB - C = a_{11}^2 a_{33} + a_{11}^2 a_{22} + a_{11}a_{22}a_{33} - a_{11}a_{12}a_{21} - a_{11}a_{13}a_{31} + a_{11}a_{22}^2 + a_{22}^2 a_{33}$$

$$-a_{22}a_{23}a_{32} - a_{12}a_{21}a_{22} + a_{11}a_{33}^2 + a_{11}a_{22}a_{33} + a_{22}a_{33}^2 - a_{23}a_{32}a_{33}$$

$$-a_{12}a_{21}a_{33} - a_{13}a_{31}a_{33} - a_{12}a_{31}a_{23} - a_{13}a_{21}a_{32}$$

Using the given conditions, we get that $\Delta > 0 \implies AB > C$. Therefore, by Routh Hurwitz criteria, all the roots of the polynomial $\gamma^3 + A\gamma^2 + B\gamma + C = 0$ have negative real parts. Since all the eigen values are negative, therefore $E_5 = (S^*, I^*, P^*)$ is locally asymptotically stable.

Theorem 6: Assume that the interior equilibrium point $E_5 = (S^*, I^*, P^*)$ of the system defined by equations (6)-(8) is locally asymptotically stable. Then it is globally asymptotically stable in the sub region Ω of $Int\mathbb{R}_+^3$ that satisfies the following conditions:

$$(27) \qquad P^* < min\left\{ \frac{\Lambda P_2(S,I)}{P_3(S)\alpha_1}, \frac{\lambda S^* P_2(S,I)}{\alpha_2 m P_1(I)} \right\}$$

$$(28) \qquad (q_{12})^2 < 4q_{11}q_{22}$$

Where, $P_1(I) = (1+I)(1+I^*) = AA^*$, $P_2(S,I) = (\beta + S + mI)(\beta + S^* + mI^*) = BB^*$, $P_3(S) = SS^*$ and $q'_{ij}s, i, j = 1,2,3$ are given in the proof.

Proof: Consider the following function:

$$V(S,I,P) = C_1\left[S - S^* - S^* \ln\frac{S}{S^*}\right] + C_2\left[I - I^* - I^* \ln\frac{I}{I^*}\right] + C_3\left[P - P^* - P^* \ln\frac{P}{P^*}\right]$$

where, C_1, C_2, C_3 are constants to be determined. It is easy to see that $V(S,I,P) \in C^1(\mathbb{R}^3, \mathbb{R})$ and $V(S^*, I^*, P^*) = 0$ while $V(S,I,P) > 0$ for all $(S,I,P) \in \mathbb{R}_+^3$ with $(S,I,P) \neq (S^*, I^*, P^*)$ then,

$$\frac{dV}{dt} = -C_1 \left[\frac{\Lambda}{P_3(S)} - \frac{\alpha_1 P^*}{P_2(S,I)}\right](S-S^*)^2 - C_2\left[\frac{\lambda S^*}{P_1(I)} - \frac{\alpha_2 m P^*}{P_2(S,I)}\right](I-I^*)^2$$

$$+\left[C_1\left(\frac{-\lambda}{P_1(I)} + \frac{\alpha_1 m P^*}{P_2(S,I)}\right) + C_2\left(\frac{\lambda A^*}{P_1(I)} + \frac{\alpha_2 P^*}{P_2(S,I)}\right)\right](S-S^*)(I-I^*)$$

$$+\left[C_1\left(\frac{-\alpha_1 B^*}{P_2(S,I)}\right) + C_3\left(\frac{e\alpha_1 B^* - e\alpha_1 S^* - e\alpha_2 I^*}{P_2(S,I)}\right)\right](P-P^*)(S-S^*)$$

$$+\left[C_2\left(\frac{-\alpha_2 B^*}{P_2(S,I)}\right) + C_3\left(\frac{e\alpha_2 B^* - e\alpha_1 S^* - e\alpha_1 m I^*}{P_2(S,I)}\right)\right](P-P^*)(I-I^*)$$

Now, choosing constants as $C_1 = 1$,

$$C_2 = \frac{\alpha_1\left(e\alpha_2\beta + (e\alpha_2 - e\alpha_1 m)S^*\right)}{\alpha_2\left(e\alpha_1\beta + (e\alpha_1 m - e\alpha_2)I^*\right)}, C_3 = \frac{\alpha_1(\beta + S^* + mI^*)}{e\alpha_1\beta + (e\alpha_1 m - e\alpha_2)I^*}$$

which are all positive due to the local stability condition from theorem 5. Then applying the

Sylvester's criterion we get that:

$$\frac{dV}{dt} = -q_{11}(S-S^*)^2 + q_{12}(S-S^*)(I-I^*) - q_{22}(I-I^*)^2 + q_{13}(S-S^*)(P-P^*) + q_{23}(I-I^*)(P-P^*)$$

$$= -q_{11}(S-S^*)^2 + q_{12}(S-S^*)(I-I^*) - q_{22}(I-I^*)^2$$

where,

$$q_{11} = \frac{\Lambda}{P_3(S)} - \frac{\alpha_1 P^*}{P_2(S,I)}, \quad q_{22} = \left(\frac{\alpha_1\left(e\alpha_2\beta + (e\alpha_2 - e\alpha_1 m)S^*\right)}{\alpha_2\left(e\alpha_1\beta + (e\alpha_1 m - e\alpha_2)I^*\right)}\right)\left(\frac{\lambda S^*}{P_1(I)} - \frac{\alpha_2 m P^*}{P_2(S,I)}\right),$$

$$q_{12} = \left(\frac{-\lambda}{P_1(I)} + \frac{\alpha_1 m P^*}{P_2(S,I)}\right) + \left(\frac{\alpha_1\left(e\alpha_2\beta + (e\alpha_2 - e\alpha_1 m)S^*\right)}{\alpha_2\left(e\alpha_1\beta + (e\alpha_1 m - e\alpha_2)I^*\right)}\right)\left(\frac{\lambda A^*}{P_1(I)} + \frac{\alpha_2 P^*}{P_2(S,I)}\right), q_{13} = q_{23} = 0$$

From the condition given in equation (27) we get that $q_{11} > 0$ and $q_{22} > 0$. Then from (28), we obtain that $\frac{dV}{dt} < 0$ is negative definite and hence V is a Lyapunov function with respect to E_5. So, E_5 is globally asymptotically stable in $\Omega \in Int\mathbb{R}^3_+$ that satisfies the given conditions.

4. DYNAMICAL ANALYSIS OF SYSTEM WITH DELAY

4.1. Existence of Hopf Bifurcation. In order to study the stability of the system with delay at interior equilibrium point, we begin with linearizing the system defined by equations (9)-(11) at E_5 and obtain the following system:

$$\frac{dS}{dt} = a_{11}S(t) + a_{12}I(t) + a_{13}P(t);$$

$$\frac{dI}{dt} = a_{12}S(t) + a_{22}I(t) + a_{13}P(t) + b_{21}S(t-\tau) + b_{22}I(t-\tau);$$

$$\frac{dP}{dt} = a_{31}S(t) + a_{32}I(t) + a_{33}P(t)$$

where,

$$a_{11} = \frac{-\lambda I^*}{1+I^*} - \alpha_1 P^* \frac{\beta+mI^*}{(\beta+S^*+mI^*)^2} - r_1 U^* - d_1, a_{12} = \frac{-\lambda S^*}{(1+I^*)^2} + \alpha_1 P^* \frac{\beta+mI^*}{(\beta+S^*+mI^*)^2}$$

$$a_{13} = \frac{-\alpha_1 S^*}{\beta+S^*+mI^*}, a_{21} = \frac{\alpha_2 I^* P^*}{(\beta+S^*+mI^*)^2}, a_{22} = -\frac{\alpha_2 P^*(\beta+S^*)}{(\beta+S^*+mI^*)^2} - d_2 - r_2 U^*$$

$$a_{23} = \frac{-\alpha_2 I^*}{\beta+S^*+mI^*}, a_{31} = \frac{P^*}{(\beta+S^*+mI^*)^2}[(\beta+mI^*)e\alpha_1 - e\alpha_2 I^*]$$

$$a_{32} = \frac{P^*}{(\beta+S^*+mI^*)^2}[(\beta+S^*)e\alpha_2 - e\alpha_1 mS^*], a_{33} = \frac{e\alpha_1 S^* + e\alpha_2 I^*}{\beta+S^*+mI^*} - d_3 - r_3 U^*$$

$$b_{21} = \frac{\lambda I^*}{1+I^*}, b_{22} = \frac{\lambda S^*}{(1+I^*)^2}$$

The characteristic equation of the system at the interior equilibrium point $E_5 = (S^*, I^*, P^*)$ is given by:

$$(29) \qquad f(\rho, \tau) = \rho^3 + k_2\rho^2 + k_1\rho + k_0 + (m_2\rho^2 + m_1\rho + m_0)e^{-\rho\tau}$$

where,

$$k_0 = -a_{11}a_{22}a_{33} - a_{12}a_{23}a_{31} + a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} - a_{13}a_{21}a_{32}$$

$$k_1 = a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{13}a_{31} - a_{23}a_{32} - a_{12}a_{21}$$

$$k_2 = -a_{11} - a_{22} - a_{33}$$

$$m_0 = -a_{11}a_{33}b_{22} + a_{13}a_{31}b_{22} - a_{13}a_{32}b_{21} + a_{12}a_{33}b_{21}$$

$$m_1 = a_{11}b_{22} + a_{33}b_{22} - a_{12}b_{21}$$

$$m_2 = -b_{22}$$

Now, two cases arise:

Case 1: $\tau = 0$, then equation (29) becomes:

$$(30) \qquad \rho^3 + (k_2+m_2)\rho^2 + (k_1+m_1)\rho + k_0 + m_0 = 0$$

If (C3) $k_0+m_0 > 0, k_1+m_1 > 0, k_2+m_2 > 0$ and $(k_2+m_2)(k_1+m_1) > (k_0+m_0)$, then all roots of (30) must have negative real parts (Using Routh-Hurwitz criteria).

Thus, if C3 holds then the interior equilibrium point E_5 is locally asymptotically stable at $\tau = 0$.

Case 2: $\tau \neq 0$

Let $\rho = i\sigma$ $(\sigma > 0)$ be the root or solution of (29).

$$\implies i\sigma^3 - k_2\sigma^2 + ik_1\sigma + k_0 + (-m_2\sigma^2 + im_1\sigma + m_0)(cos\sigma\tau - isin\sigma\tau) = 0$$

Now, we separate the imaginary and real parts and get

$$-\sigma^3 + k_1\sigma = -m_2\sigma^2 sin\sigma\tau - m_1\sigma cos\sigma\tau + m_0 sin\sigma\tau,$$

$$-k_2\sigma^2 + k_0 = m_2\sigma^2 ccos\sigma\tau - m_1\sigma sin\sigma\tau - m_0 cos\sigma\tau$$

$\implies \sigma$ satisfies the following equation:

$$(31) \qquad \sigma^6 + (k_2^2 - 2k_1 - m_2^2)\sigma^4 + (k_1^2 - 2k_0k_2 - m_1^2 + 2m_0m_2)\sigma^2 + (k_0^2 + m_0^2) = 0$$

Let $\omega = \sigma^2$, (31) becomes

$$(32) \qquad\qquad\qquad\qquad \omega^3 + s_2\omega^2 + s_1\omega + s_0 = 0$$

where

$$s_2 = k_2^2 - 2m_1 - m_2^2, \; s_1 = k_1^2 - 2k_0k_2 - m_1^2 + 2m_0m_2, \; s_0 = k_0^2 + m_0^2$$

Define a function f as:

$$f(\omega) = \omega^3 + s_2\omega^2 + s_1\omega + s_0$$

Thus, equation (32) has at least one positive root if $s_0 < 0$; it has no positive roots if $s_0 \geq 0$ and $\Delta = s_2^2 - 3s_1 \leq 0$; and if $s_0 \geq 0$ and $\Delta = s_2^2 - 3s_1 > 0$, then it has positive roots $\iff$ $\omega^* = \dfrac{-s_2 + \sqrt{\Delta}}{3}$ and $f(\omega^*) \leq 0$. In addition, it is assumed that the coefficients in $f(\omega)$ satisfy the condition (C4): $s_0 < 0$ or $s_0 \geq 0$ and $f(\omega^*) \leq 0$. If this condition holds then (32) has at least one positive root. WLOG we have assumed that (32) has three positive roots, namely ω_1, ω_2 and ω_3. Consequently, (31) has three positive roots $\sigma_k = \sqrt{\omega_k}, k = 1, 2, 3$.

So, we get

$$\tau_k^{(j)} = \frac{1}{\sigma_k}\left\{arccos\frac{(m_1 - m_2k_2)\sigma_k^4 + (m_0k_2 + m_2k_0 - m_1k_1)\sigma_k^2 - m_0k_0}{m_2^2\sigma_k^4 + (m_1^2 - 2m_0m_2)\sigma_k^2 + m_0^2} + 2j\Pi\right\},$$

$k = 1, 2, 3; \; j = 0, 1, 2...$, then pair of imaginary roots of (31) are $\pm i\sigma$ when $\tau = \tau_k^j$.

Let $\tau' = min\{\tau_k\}$, $\sigma' = \sigma_k|_{\tau=\tau'}$, (k=1,2,3). Let the root of equation (31) near $\tau = \tau'$ be $\rho(\tau) = \gamma(\tau) + i\sigma(\tau)$ that satisfies $\gamma(\tau') = 0, \sigma(\tau') = \sigma'$. To establish Hopf Bifurcation, we will show

that if $f'(\omega) > 0$, then $\dfrac{dRe(\rho)}{d\tau}|_{\tau=\tau'}$ and $f'(\omega)$ have same sign and $\dfrac{dRe(\rho)}{d\tau}|_{\tau=\tau'} > 0$.

Consider equation (31) then derivative of ρ w.r.t. τ, we get:

$$\left(\frac{d\rho}{d\tau}\right)^{-1} = \frac{(3\rho^2 + 2k_2\rho + k_1)e^{\rho\tau} + 2m_2\rho + m_1}{\rho(m_2\rho^2 + m_1\rho + m_0)} - \frac{\tau}{\rho}$$

Put $\rho = i\sigma'$ in the above equation, we get

$$
\begin{aligned}
Re\left[\frac{d\rho}{d\tau}\right]^{-1}_{\tau=\tau'} &= Re\left[\frac{(-3\sigma'^2 + i2k_2\sigma' + k_1)(\cos\sigma'\tau' + i\sin\sigma'\tau')}{-m_1\sigma'^2 + i(-m_2\sigma'^3 + m_0\sigma')}\right] + Re\left[\frac{m_1 + i2m_2\sigma'}{-m_1\sigma'^2 + i(-m_2\sigma'^3 + m_0\sigma')}\right] \\
&= -\frac{1}{\Pi}\Big\{\left[(k_1 - 3\sigma'^2)\cos\sigma'\tau' - 2k_2\sigma'\sin\sigma'\tau'\right]m_1\sigma'^2 \\
&\quad + \left[(k_1 - 3\sigma'^2)\sin\sigma'\tau' + 2m_2\sigma'\cos\sigma'\tau'\right](m_2\sigma'^3 - m_0\sigma') + M\Big\} \\
&= \frac{\sigma'^2}{\Pi}\Big\{3\sigma_0^4 + 2(k_2^2 - 2k_1 - m_2^2)\sigma'^2 + (k_1^2 - 2k_0k_2 - m_1^2 + 2m_0m_2)\Big\} \\
&= \frac{f'(\sigma'^2)}{(m_1\sigma')^2 + (m_2\sigma'^2 - m_0)^2}
\end{aligned}
$$

where, $\Pi = \left[(m_1\sigma')^2 + (m_2\sigma'^2 - m_0)^2\right]\sigma'^2$ and $M = m_1^2\sigma'^2 + 2m_2\sigma'(m_2\sigma'^3 - m_0\sigma')$.

Therefore,

$$sign\left\{\frac{dRe(\rho)}{d\tau}\Big|_{\tau=\tau'}\right\} = sign\left\{Re\left[\frac{d\rho}{d\tau}\right]^{-1}_{\tau=\tau'}\right\} = sign\{f'(\sigma_0^2)\}$$

Since $f'(\sigma'^2) \neq 0$, therefore $\dfrac{dRe(\rho)}{d\tau}|_{\tau=\tau'} \neq 0$. Let us assume that $\dfrac{dRe(\rho)}{d\tau}|_{\tau=\tau'} < 0$. So, when $\tau < \tau'$, the characteristic equation will have roots with positive real parts. This is a contradiction to the local stability of the interior equilibrium point E_5. Hence, $\dfrac{dRe(\rho)}{d\tau}|_{\tau=\tau'} > 0$.

Thus, based on this analysis we obtain, if (C3)-(C4) hold, then the system defined by equations (9)-(11) at the interior equilibrium point E_5 is locally asymptotically stable when $\tau \in [0, \tau']$ and unstable when $\tau > \tau'$ and the system undergoes a Hopf bifrcation at the interior equilibrium point E_5 when $\tau = \tau'$.

4.2. Direction of Hopf Bifurcation.
In the previous section, we obtained certain conditions under which the given system of equations undergoes Hopf bifurcation, with time delay $\tau = \tau'$ being the critical parameter. In this section, by taking into account the normal form theory and

the center manifold theorem which were introduced by [12], we will be presenting the formula determining the direction of Hopf bifurcation and will be obtaining conditions for the stability of bifurcating periodic solutions, as well. Since Hopf bifurcation occurs at the critical value τ' of τ, there exists a pair of pure imaginary roots $\pm\iota\sigma(\tau')$ of the characteristic equation (29). Next let, $x_1 = S - S^*$, $x_2 = I - I^*$, $x_3 = P - P^*$. We also let $t \to \tau t$, and $\tau = \tau' + \mu$. Then, the system finally takes the form of an FDE in $C = C([-1,0],\mathbb{R}^3$ as :

$$(33) \qquad\qquad \dot{x}(t) = L_\mu(x_t) + F(\mu, x_t)$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^T \in \mathbb{R}^3$ and $L_\mu : C \to \mathbb{R}^3$, $F : C \times \mathbb{R} \to \mathbb{R}^3$ are given respectively by :

$L_\mu(\psi) = (\tau' + \mu)L_1\psi(0) + (\tau' + \mu)L_2\psi(-1)$ and $F(\mu, \psi) = (\tau' + \mu)F_1$

where,

$$L_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

$$L_2 = \begin{bmatrix} 0 & 0 & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$F_1 = \begin{bmatrix} -\lambda\,\psi_1(0)\,\psi_2(0) - \alpha_1\,\psi_1(0)\,\psi_3(0) \\ -\alpha_2\,\psi_2(0)\,\psi_3(0) + \lambda\,\psi_1(-1)\,\psi_2(-1) \\ e\alpha_1\,\psi_1(0)\,\psi_3(0) - e\alpha_2\,\psi_2(0)\,\psi_3(0) \end{bmatrix}$$

We also have that, $\psi = (\psi_1, \psi_2, \psi_3)^T \in C$, and $x_t(\theta) = x(t + \theta)$ for $\theta \in [-1,0]$.

By the Riesz Representation theorem, there exists a function $\sigma(\theta, \mu)$ of bounded variation for $\theta \in [-1,0]$, such that

$$(34) \qquad\qquad L_\mu(\psi) = \int_{-1}^{0} d\sigma(\theta, \mu)\psi(\theta)$$

for $\psi \in C$.

Infact, we can take

$$(35) \qquad\qquad \sigma(\theta, \mu) = (\tau' + \mu)L_1\delta(\theta) + (\tau' + \mu)L_2\delta(\theta + 1)$$

where, L_1, L_2 have already been given above, and $\delta(\theta)$ is Dirac delta function.

Next, for $\psi \in C^1([-1,0], \mathbb{R}^3)$, we define the following :

$$A(\mu)\psi = \begin{cases} \frac{d\psi(\theta)}{d\theta}, & \theta \in [-1,0) \\ \int_{-1}^{0} d\sigma(s,\mu)\psi(s), & \theta = 0 \end{cases}$$

and,

$$R(\mu)\psi = \begin{cases} 0, & \theta \in [-1,0) \\ F(\mu,\psi), & \theta = 0 \end{cases}$$

Then, the system (34) is equivalent to,

(36)
$$\dot{x}_t = A(\mu)x_t + R(\mu)x_t$$

where, $x_t(\theta) = x(t+\theta)$ for $\theta \in [-1,0]$.

Next, for $\varphi \in C^1([0,1], \mathbb{R}^3)$, the adjoint operator A^* of A can be defined as,

$$A^*\varphi(s) = \begin{cases} \frac{-d\varphi(s)}{ds}, & s \in (0,1] \\ \int_{-1}^{0} d\sigma^T(t,0)\varphi(-t), & s = 0 \end{cases}$$

and hence for $\psi \in ([-1,0], \mathbb{R}^3)$, $\varphi \in ([0,1], \mathbb{R}^3)$ a bilinear inner product , in order to normalize the eigenvalues of A and A^* can be defined as follows:

(37)
$$\langle \varphi(s), \psi(\theta) \rangle = \bar{\varphi}(0)\psi(0) - \int_{-1}^{0} \int_{\gamma=0}^{\theta} \bar{\varphi}(\gamma-\theta) d\sigma(\theta)\psi(\gamma)d\gamma$$

where $\sigma(\theta) = \sigma(\theta, 0)$, and $\bar{\varphi}$ is the complex conjugate of φ. It can be verified that the operators A and A^* are adjoint operators with respect to this bilinear form. Thus, since $\pm\iota\sigma'\tau'$ are eigenvalues of $A(0)$, they are the eigenvalues of A^* as well.

We need to compute the eigenvectors of $A(0)$ and A^* corresponding to the eigenvalues $\iota\sigma'\tau'$ and $-\iota\sigma'\tau'$, respectively.

Let us suppose that $q(\theta) = (1, \alpha', \beta')^T e^{\iota\sigma'\tau'\theta}$ is the eigenvector of A(0) corresponding to $\iota\sigma'\tau'$. Then, $A(0)q(\theta) = \lambda q(\theta)$, that is,

$A(0)q(\theta) = \iota\sigma'\tau'q(\theta)$ or $[\lambda I - A(0)]q(0) = 0$ which gives the following :

$$\tau'\begin{bmatrix} \iota\sigma' - a_{11} & -a_{12} & -a_{13} \\ -a_{21} - b_{21}e^{-\iota\sigma'\tau'} & \iota\sigma' - a_{22} - b_{22}e^{-\iota\tau'\sigma'} & -a_{23} \\ -a_{31} & -a_{32} & \iota\sigma' - a_{33} \end{bmatrix}\begin{bmatrix} 1 \\ \alpha' \\ \beta' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} \iota\sigma' - a_{11} & -a_{12} & -a_{13} \\ -a_{21} - b_{21}e^{-\iota\sigma'\tau'} & \iota\sigma' - a_{22} - b_{22}e^{-\iota\tau'\sigma'} & -a_{23} \\ -a_{31} & -a_{32} & \iota\sigma' - a_{33} \end{bmatrix}\begin{bmatrix} 1 \\ \alpha' \\ \beta' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(since $\tau' \neq 0$)

And, on solving this we get, $q(0) = (1, \alpha', \beta')^T$, where,

$\alpha' = \dfrac{a_{23}(\iota\sigma' - a_{11}) - a_{13}a_{12}(a_{21} + b_{21}e^{-\iota\sigma'\tau'})}{a_{12}a_{23} - a_{12}a_{13}(\iota\sigma' - a_{22} - b_{22}e^{-\iota\sigma'\tau'})}$, and $\beta' = \dfrac{a_{31} + a_{32}\alpha'}{\iota\sigma' - a_{33}}$

Next, let us suppose that $q^*(\theta) = D(1, (\alpha')^*, (\beta')^*)e^{\iota\sigma'\tau'\theta}$ be the eigenvector of A^* corresponding to the eigenvalue $-\iota\sigma'\tau'$, and hence in a similar manner we can obtain

$(\alpha')^* = -\dfrac{(\iota\sigma' + a_{11})a_{32} + a_{31}a_{12}a_{21} + a_{31}a_{12}b_{21}e^{\iota\sigma'\tau'}}{a_{21}a_{32} + a_{21}a_{31}(\iota\sigma' + a_{22}) + (b_{21}a_{32} + b_{22}a_{21}a_{31})e^{\iota\sigma'\tau'}}$ and

$(\beta')^* = -\dfrac{a_{13} + a_{23}(\bar{\alpha'})^*}{\iota\sigma' + a_{33}}$

From (37) we get,

$$\begin{aligned} \langle q^*(s), q(\theta) \rangle &= \bar{D}(1, (\bar{\alpha'})^*, (\bar{\beta'})^*)(1, \alpha', \beta')^T \\ &\quad - \int_{-1}^{0}\int_{\gamma=0}^{\theta} \bar{D}(1, (\bar{\alpha'})^*, (\bar{\beta'})^*)e^{-\iota\sigma'\tau'(\gamma-\theta)}d\eta(\theta)(1, \alpha', \beta')^T e^{\iota\sigma'\tau'\gamma}d\gamma \\ &= \bar{D}[1 + (\alpha')^*\alpha' + (\beta')^*\beta' - (1, (\bar{\alpha'})^*, (\bar{\beta'})^*)\int_{-1}^{0}\theta e^{\iota\sigma'\tau'\theta}d\eta(\theta)(1, \alpha', \beta')^T \end{aligned}$$

Now let $\psi(\theta) = \theta e^{\iota\sigma'\tau'\theta}$

$\Rightarrow \psi(0) = 0$, and $\psi(-1) = -e^{-\iota\sigma'\tau'}$

Thus, from (34) and the definition of ψ as taken above we finally get that,

$\langle q^*(s), q(\theta) \rangle = \bar{D}[1 + (\alpha')^*\alpha' + (\beta')^*\beta' - \tau'(I^* + \alpha'S^*)(\bar{\alpha'})^*\beta e^{-b\tau'}e^{-\iota\sigma'\tau'}]$

Hence,

$$\bar{D} = \frac{1}{[1 + (\alpha')^* \alpha' + (\beta')^* \beta' - \tau'(I^* + \alpha' S^*)(\bar{\alpha'})^* \beta e^{-b\tau'} e^{-\iota \sigma' \tau'}]}$$

such that $\langle q^*(s), q(\theta) \rangle = 1$, and $\langle q^*(s), \bar{q}(\theta) \rangle = 0$.

In the remaining part of this section, using the same ideas as in [12], we now compute the coordinates in order to describe the center manifold C_0 at $\mu = 0$. Let x_t be the solution of (36) when $\mu = 0$.

Next, define

(38) $$\tilde{z}(t) = \langle q^*, x_t \rangle, W(t, \theta) = x_t - 2Re[\tilde{z}(t) q(\theta)]$$

Now, on the center manifold C_0, we have

(39) $$W(t, \theta) = W(\tilde{z}(t), \bar{\tilde{z}}(t), \theta) = W_{20}(\theta)\frac{\tilde{z}^2}{2} + W_{11}(\theta)\tilde{z}\bar{\tilde{z}} + W_{02}(\theta)\frac{\bar{\tilde{z}}^2}{2} + ...$$

where $\tilde{z}$ and $\bar{\tilde{z}}$ are local coordinates for the center manifold C_0 in the direction of q^* and $\bar{q}^*$. We note that W is real if x_t is real and we will be considering the real solutions only.

From (38) we have,

$\dot{\tilde{z}}(t) = \langle q^*, \dot{x}_t \rangle$

$= \langle q^*, A(\mu)x_t + R(\mu)x_t \rangle$ (from (36))

$= \langle A^*(\mu)q^*, x_t \rangle + \langle q^*, R(\mu)x_t \rangle$

$= \iota \sigma' \tau' \tilde{z}(t) + \langle q^*, R(\mu)x_t \rangle$ (since $A^* q^* = \bar{\lambda} q^*$)

$= \iota \sigma' \tau' \tilde{z}(t) + \bar{q}^*(0)F(0, x_t)$ (from the definition of bilinear product, i.e (37) and taking $\theta = 0$)

$= \iota \sigma' \tau' \tilde{z}(t) + \bar{q}^*(0)F(0, W(\tilde{z}, \bar{\tilde{z}}, 0) + 2Re[\tilde{z}q(0)])$ (from (38))

$= \iota \sigma' \tau' \tilde{z}(t) + \bar{q}^*(0)F_0(\tilde{z}, \bar{\tilde{z}})$

$= \iota \sigma' \tau' \tilde{z}(t) + g(\tilde{z}, \bar{\tilde{z}})$

where,

(40) $$g(\tilde{z}, \bar{\tilde{z}}) = \bar{q}^*(0)F_0(\tilde{z}, \bar{\tilde{z}}) = g_{20}\frac{\tilde{z}^2}{2} + g_{11}\tilde{z}\bar{\tilde{z}} + g_{02}\frac{\bar{\tilde{z}}^2}{2} + g_{21}\frac{\tilde{z}^2\bar{\tilde{z}}}{2} + ...$$

From (38) we have,

$x_t(\theta) = (x_{1t}(\theta), x_{2t}(\theta), x_{3t}(\theta)) = W(t, \theta) + \tilde{z}q(\theta) + \bar{\tilde{z}}\bar{q}(\theta)$ and $q(\theta) = (1, \alpha', \beta')^T e^{\iota \theta \sigma' \tau'}$

And thus we can obtain that,

$$x_{1t}(0) = W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \bar{z} + \bar{\bar{z}} + O(|\bar{z},\bar{\bar{z}}|^3),$$

$$x_{2t}(0) = W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + \alpha'\bar{z} + \bar{\alpha}'\bar{\bar{z}} + O(|\bar{z},\bar{\bar{z}}|^3),$$

$$x_{3t}(0) = W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0)z\bar{z} + W_{02}^{(3)}(0)\frac{\bar{z}^2}{2} + \beta'\bar{z} + \bar{\beta}'\bar{\bar{z}} + O(|\bar{z},\bar{\bar{z}}|^3),$$

$$x_{1t}(-1) = W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + \bar{z} + \bar{\bar{z}} + O(|\bar{z},\bar{\bar{z}}|^3),$$

$$x_{2t}(-1) = W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z} + W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2} + \alpha'\bar{z}e^{-\iota\sigma'\tau'} + \bar{\alpha}'\bar{\bar{z}}e^{\iota\sigma'\tau'} + O(|\bar{z},\bar{\bar{z}}|^3),$$

$$x_{3t}(-1) = W_{20}^{(3)}(-1)\frac{z^2}{2} + W_{11}^{(3)}(-1)z\bar{z} + W_{02}^{(3)}(-1)\frac{\bar{z}^2}{2} + \beta'\bar{z}e^{-\iota\sigma'\tau'} + \bar{\beta}'\bar{\bar{z}}e^{\iota\sigma'\tau'} + O(|\bar{z},\bar{\bar{z}}|^3),$$

From the definition of $F(\mu, x_t)$ we get,

$$g(\bar{z},\bar{\bar{z}}) = \tau'\bar{D}(1,(\bar{\alpha}')^*,(\bar{\beta}')^*)\begin{bmatrix} -\lambda x_{1t}(0)x_{2t}(0) - \alpha_1 x_{1t}(0)x_{3t}(0) \\ -\alpha_2 x_{2t}(0)x_{3t}(0) + \lambda x_{1t}(-1)x_{2t}(-1) \\ e\alpha_1 x_{1t}(0)x_{3t}(0) - e\alpha_2 x_{2t}(0)x_{3t}(0) \end{bmatrix}$$

Simplifying and comparing the coefficients with (40), we get:

$$g_{20} = 2\tau'\bar{D}\Big[-\lambda\alpha' + \lambda\alpha'(\bar{\alpha}')^*e^{-\iota\sigma'\tau'} - \alpha_1\beta' + e\alpha_1\beta'(\bar{\beta}')^* - \alpha_2\alpha'(\bar{\alpha}')^*\beta' + e\alpha_2\alpha'\beta'(\bar{\beta}')^*\Big]$$

$$g_{11} = 2\tau'\bar{D}\Big[-\lambda Re\{\alpha'\} + (\bar{\alpha}')^*\lambda Re\{\alpha'e^{-\iota\sigma'\tau'}\} + (-\alpha_1 + (\bar{\beta}')^*e\alpha_1)Re\{\beta'\}$$
$$+ (-\alpha_2(\bar{\alpha}')^* + (\bar{\beta}')^*e\alpha_2)Re\{\alpha'\bar{\beta}'\}\Big]$$

$$g_{02} = 2\tau'\bar{D}\Big[-\lambda\bar{\alpha}' + \lambda(\bar{\alpha}')^*\bar{\alpha}'e^{\iota\sigma'\tau'} + (-\alpha_1 + (\bar{\beta}')^*e\alpha_1)\bar{\beta}' + (-\alpha_2(\bar{\alpha}')^* + (\bar{\beta}')^*e\alpha_2)\bar{\alpha}'\bar{\beta}'\Big]$$

$$g_{21} = \tau'\bar{D}\Big[-\lambda(\bar{\alpha}'W_{20}^{(1)}(0) + W_{20}^{(2)}(0)) + \lambda(\bar{\alpha}')^*(\bar{\alpha}'e^{\iota\sigma'\tau'}W_{20}^{(1)}(-1) + W_{20}^{(2)}(-1))$$
$$+ (-\alpha_1 + (\bar{\beta}')^*e\alpha_1)(\bar{\beta}'W_{20}^{(1)}(0) + W_{30}^{(3)}(0))$$
$$+ (-\alpha_2(\bar{\alpha}')^* + (\bar{\beta}')^*e\alpha_2)(\bar{\beta}'W_{20}^{(2)}(0) + W_{20}^{(3)}(0))\Big]$$

We can clearly see that in order to determine g_{21}, we will have to compute $W_{20}(\theta)$ and $W_{11}(\theta)$.

From (36) and (38) we have:

$$\dot{W} = \dot{x}_t - 2Re[\dot{\bar{z}}(t)q(\theta)]$$

$$= A(\mu)x_t + R(\mu)x_t - 2Re[(\iota\sigma'\tau'\bar{z}(t) + \bar{q}^*(0)F_0(\bar{z},\bar{\bar{z}}))q(\theta)]$$

$$= A(\mu)x_t + R(\mu)x_t - 2Re[\iota\sigma'\tau'\bar{z}(t)q(\theta)] - 2Re[\bar{q}^*(0)F_0(\bar{z},\bar{\bar{z}})q(\theta)]$$

Therefore,

$$\dot{W} = \begin{cases} AW - 2Re[\bar{q}^*(0)F_0(\tilde{z},\bar{\tilde{z}})q(\theta)], & \theta \in [-1,0) \\ AW - 2Re[\bar{q}^*(0)F_0(\tilde{z},\bar{\tilde{z}})q(\theta)] + F_0, & \theta = 0 \end{cases}$$

(using the definition of AW and $R(\mu)x_t$)

Therefore, let

(41)
$$\dot{W} = AW + \tilde{H}(\tilde{z},\bar{\tilde{z}},\theta)$$

where,

(42)
$$\tilde{H}(\tilde{z},\bar{\tilde{z}},\theta) = \tilde{H}_{20}(\theta)\frac{\tilde{z}^2}{2} + \tilde{H}_{11}(\theta)\tilde{z}\bar{\tilde{z}} + \tilde{H}_{02}(\theta)\frac{\bar{\tilde{z}}^2}{2} + ...$$

On the other hand, on the center manifold C_0 near the origin, $\dot{W} = W_{\tilde{z}}\dot{\tilde{z}} + W_{\bar{\tilde{z}}}\dot{\bar{\tilde{z}}}$

Using (41) to compare the coefficients, we finally arrive at the following,

(43)
$$(A - 2\iota\sigma'\tau')W_{20}(\theta) = -\tilde{H}_{20}(\theta), AW_{11}(\theta) = -\tilde{H}_{11}(\theta)$$

From (41) we also have that $\tilde{H}(\tilde{z},\bar{\tilde{z}},\theta) = -2Re[\bar{q}^*(0)F_0(\tilde{z},\bar{\tilde{z}})q(\theta)]$, for $\theta \in [-1,0)$. That is,

$$\begin{aligned}\tilde{H}(\tilde{z},\bar{\tilde{z}},\theta) &= -\bar{q}^*(0)F_0(\tilde{z},\bar{\tilde{z}})q(\theta) - q^*(0)\bar{F}_0(\tilde{z},\bar{\tilde{z}})\bar{q}(\theta) \\ &= -(g_{20}\frac{\tilde{z}^2}{2} + g_{11}\tilde{z}\bar{\tilde{z}} + g_{02}\frac{\bar{\tilde{z}}^2}{2} + g_{21}\frac{\tilde{z}^2\bar{\tilde{z}}}{2} + ...)q(\theta) - (\bar{g}_{20}\frac{\bar{\tilde{z}}^2}{2} + \bar{g}_{11}\tilde{z}\bar{\tilde{z}} + \bar{g}_{02}\frac{\tilde{z}^2}{2} + \bar{g}_{21}\frac{\bar{\tilde{z}}^2\tilde{z}}{2})q(\theta)\end{aligned}$$

Now equating this with (42), and comparing the coefficients, we have,

(44)
$$\tilde{H}_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \tilde{H}_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta)$$

From (43), (44) and the definition of A for $\theta \in [-1,0)$, we get,

(45)
$$\dot{W}_{20}(\theta) = 2\iota\sigma'\tau'W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta)$$

Note that, $q(\theta) = q(0)e^{\iota\sigma'\tau'\theta}$. Hence, putting this value in (45), and solving it((45) being a linear differential equation), we get:

(46)
$$W_{20}(\theta) = \frac{\iota g_{20}}{\sigma'\tau'}q(0)e^{\iota\sigma'\tau'\theta} + \frac{\iota\bar{g}_{02}}{3\sigma'\tau'}\bar{q}(0)e^{-\iota\sigma'\tau'\theta} + \tilde{E}_1e^{2\iota\sigma'\tau'\theta}$$

where $\tilde{E}_1 = (\tilde{E}_1^{(1)}, \tilde{E}_1^{(2)}, \tilde{E}_1^{(3)}) \in \mathbb{R}^3$ is a constant vector. Similarly we can get,

(47)
$$W_{11}(\theta) = -\frac{\iota g_{11}}{\sigma'\tau'}q(0)e^{\iota\sigma'\tau'\theta} + \frac{\iota\bar{g}_{11}}{\sigma'\tau'}\bar{q}(0)e^{-\iota\sigma'\tau'\theta} + \tilde{E}_2$$

where $\tilde{E}_2 = (\tilde{E}_2^{(1)}, \tilde{E}_2^{(2)}, \tilde{E}_2^{(3)}) \in \mathbb{R}^3$ is a constant vector.

Further, we will be finding $\tilde{E}_1$ and $\tilde{E}_2$.

From the definition of A at $\theta = 0$, and (43), we have,

$$(48) \qquad \int_{-1}^{0} d\sigma(\theta) W_{20}(\theta) = 2\iota\sigma'\tau' W_{20}(0) - \tilde{H}_{20}(0)$$

$$(49) \qquad \int_{-1}^{0} d\sigma(\theta) W_{11}(\theta) = -\tilde{H}_{11}(0)$$

where, $\sigma(\theta) = \sigma(0, \theta)$ (since $\mu = 0$)

Also, from(41) we have that for $\theta = 0$, $\tilde{H}(\tilde{z}, \bar{\tilde{z}}, \theta) = -2Re[\bar{q}^*(0)F_0(\tilde{z}, \bar{\tilde{z}})q\{\theta\}] + F_0$. That is,

$$
\begin{aligned}
\tilde{H}(\tilde{z}, \bar{\tilde{z}}, \theta) &= -\bar{q}^*(0)F_0(\tilde{z}, \bar{\tilde{z}})q(\theta) - q^*(0)\bar{F}_0(\tilde{z}, \bar{\tilde{z}})\bar{q}(\theta) + F_0 \\
&= -(g_{20}\frac{\tilde{z}^2}{2} + g_{11}\tilde{z}\bar{\tilde{z}} + g_{02}\frac{\bar{\tilde{z}}^2}{2} + g_{21}\frac{\tilde{z}^2\bar{\tilde{z}}}{2} + ...)q(\theta) - (\bar{g}_{20}\frac{\tilde{z}^2}{2} + \bar{g}_{11}\tilde{z}\bar{\tilde{z}} + \bar{g}_{02}\frac{\bar{\tilde{z}}^2}{2} + \bar{g}_{21}\frac{\bar{\tilde{z}}^2\tilde{z}}{2})q(\theta) \\
&\quad + F_0
\end{aligned}
$$

where

$$
\begin{aligned}
F_0 &= \tau' \begin{bmatrix} -\lambda x_{1t}(0)x_{2t}(0) - \alpha_1 x_{1t}(0)x_{3t}(0) \\ -\alpha_2 x_{2t}(0)x_{3t}(0) + \lambda x_{1t}(-1)x_{2t}(-1) \\ e\alpha_1 x_{1t}(0)x_{3t}(0) - e\alpha_2 x_{2t}(0)x_{3t}(0) \end{bmatrix} \\
&= \tau' \begin{bmatrix} -\lambda\alpha' - \alpha_1\beta' \\ -\alpha_2\alpha'\beta' + \lambda\alpha'e^{-\iota\sigma'\tau'} \\ e\alpha_1\beta' + e\alpha_2\alpha'\beta' \end{bmatrix} \tilde{z}^2 + \begin{bmatrix} -2\lambda Re\{\alpha'\} - 2\alpha_1 Re\{\beta'\} \\ -2\alpha_2 Re\{\alpha'\bar{\beta}'\} + 2\lambda Re\{\alpha'e^{-\iota\sigma'\tau'}\} \\ 2e\alpha_1 Re\{\beta'\} + 2e\alpha_2 Re\{\alpha'\bar{\beta}'\} \end{bmatrix} \tilde{z}\bar{\tilde{z}} + ...
\end{aligned}
$$

And thus, after comparing the coefficients, we get,

$$(50) \qquad \tilde{H}_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\tau' \begin{bmatrix} -\lambda\alpha' - \alpha_1\beta' \\ -\alpha_2\alpha'\beta' + \lambda\alpha'e^{-\iota\sigma'\tau'} \\ e\alpha_1\beta' + e\alpha_2\alpha'\beta' \end{bmatrix}$$

and,

$$(51) \qquad \tilde{H}_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + 2\tau' \begin{bmatrix} -2\lambda Re\{\alpha'\} - 2\alpha_1 Re\{\beta'\} \\ -2\alpha_2 Re\{\alpha'\bar{\beta}'\} + 2\lambda Re\{\alpha'e^{-\iota\sigma'\tau'}\} \\ 2e\alpha_1 Re\{\beta'\} + 2e\alpha_2 Re\{\alpha'\bar{\beta}'\} \end{bmatrix}$$

Substituting (50) and (46) in (48), and noticing that, $(\iota\sigma'\tau'I - \int_{-1}^{0} d\eta(\theta)e^{\iota\sigma'\tau'\theta})q(0) = 0$, and $(-\iota\sigma'\tau'I - \int_{-1}^{0} d\eta(\theta)e^{-\iota\sigma'\tau'\theta})\bar{q}(0) = 0$ (since $\iota\sigma'\tau'$ is the eigenvalue of $A(0)$ and $q(0)$ is the corresponding eigenvector), we obtain,

$$\tilde{E}_1(2\iota\sigma'\tau'I - \int_{-1}^{0} d\eta(\theta)e^{2\iota\sigma'\tau'}) = 2\tau' \begin{bmatrix} -\lambda\alpha' - \alpha_1\beta' \\ -\alpha_2\alpha'\beta' + \lambda\alpha'e^{-\iota\sigma'\tau'} \\ e\alpha_1\beta' + e\alpha_2\alpha'\beta' \end{bmatrix}$$

which leads to,

$$\tilde{E}_1 \begin{bmatrix} 2\iota\sigma' - a_{11} & -a_{12} & -a_{13} \\ -a_{21} - b_{21}e^{-\iota\sigma'\tau'} & 2\iota\sigma' - a_{22} - b_{22}e^{-\iota\tau'\sigma'} & -a_{23} \\ -a_{31} & -a_{32} & 2\iota\sigma' - a_{33} \end{bmatrix} = 2 \begin{bmatrix} -\lambda\alpha' - \alpha_1\beta' \\ -\alpha_2\alpha'\beta' + \lambda\alpha'e^{-\iota\sigma'\tau'} \\ e\alpha_1\beta' + e\alpha_2\alpha'\beta' \end{bmatrix}$$

And, from Cramer's rule for solving system of linear equations, we get,

$$\tilde{E}_1^{(1)} = \frac{2}{\tilde{M}_1} \begin{vmatrix} -\lambda\alpha' - \alpha_1\beta' & -a_{12} & -a_{13} \\ -\alpha_2\alpha'\beta' + \lambda\alpha'e^{-\iota\sigma'\tau'} & 2\iota\sigma' - a_{22} - b_{22}e^{-\iota\tau'\sigma'} & -a_{23} \\ e\alpha_1\beta' + e\alpha_2\alpha'\beta' & -a_{32} & 2\iota\sigma' - a_{33} \end{vmatrix}$$

$$\tilde{E}_1^{(2)} = \frac{2}{\tilde{M}_1} \begin{vmatrix} 2\iota\sigma' - a_{11} & -\lambda\alpha' - \alpha_1\beta' & -a_{13} \\ -a_{21} - b_{21}e^{-\iota\sigma'\tau'} & -\alpha_2\alpha'\beta' + \lambda\alpha'e^{-\iota\sigma'\tau'} & -a_{23} \\ -a_{31} & e\alpha_1\beta' + e\alpha_2\alpha'\beta' & 2\iota\sigma' - a_{33} \end{vmatrix}$$

$$\tilde{E}_1^{(3)} = \frac{2}{\tilde{M}_1} \begin{bmatrix} 2\iota\sigma' - a_{11} & -a_{12} & -\lambda\alpha' - \alpha_1\beta' \\ -a_{21} - b_{21}e^{-\iota\sigma'\tau'} & 2\iota\sigma' - a_{22} - b_{22}e^{-\iota\tau'\sigma'} & -\alpha_2\alpha'\beta' + \lambda\alpha'e^{-\iota\sigma'\tau'} \\ -a_{31} & -a_{32} & e\alpha_1\beta' + e\alpha_2\alpha'\beta' \end{bmatrix}$$

$$\text{where, } \tilde{M}_1 = \begin{bmatrix} 2\iota\sigma' - a_{11} & -a_{12} & -a_{13} \\ -a_{21} - b_{21}e^{-\iota\sigma'\tau'} & 2\iota\sigma' - a_{22} - b_{22}e^{-\iota\tau'\sigma'} & -a_{23} \\ -a_{31} & -a_{32} & 2\iota\sigma' - a_{33} \end{bmatrix}$$

Next, substituting (51) and (47) in (49), and working in a similar pattern as above, we finally get,

$$\tilde{E}_2^{(1)} = \frac{2}{\tilde{M}_2} \begin{vmatrix} -\lambda Re\{\alpha'\} - \alpha_1 Re\{\beta'\} & a_{12} & a_{13} \\ -\alpha_2 Re\{\alpha'\bar{\beta}'\} + \lambda Re\{\alpha'e^{-\iota\sigma'\tau'}\} + b_{21} & a_{22} + b_{22} & a_{23} \\ e\alpha_1 Re\{\beta'\} + e\alpha_2 Re\{\alpha'\bar{\beta}'\} & a_{32} & a_{33} \end{vmatrix}$$

$$\tilde{E}_2^{(2)} = \frac{2}{\tilde{M}_2} \begin{vmatrix} a_{11} & -\lambda Re\{\alpha'\} - \alpha_1 Re\{\beta'\} & a_{13} \\ a_{21} + b_{21} & -\alpha_2 Re\{\alpha'\bar{\beta}'\} + \lambda Re\{\alpha'e^{-\iota\sigma'\tau'}\} & a_{23} \\ a_{31} & e\alpha_1 Re\{\beta'\} + e\alpha_2 Re\{\alpha'\bar{\beta}'\} & a_{33} \end{vmatrix}$$

$$\tilde{E}_2^{(3)} = \frac{2}{\tilde{M}_2} \begin{vmatrix} a_{11} & a_{12} & -\lambda Re\{\alpha'\} - \alpha_1 Re\{\beta'\} \\ a_{21} + b_{21} & a_{22} + b_{22} & -\alpha_2 Re\{\alpha'\bar{\beta}'\} + \lambda Re\{\alpha'e^{-\iota\sigma'\tau'}\} \\ a_{31} & a_{32} & e\alpha_1 Re\{\beta'\} + e\alpha_2 Re\{\alpha'\bar{\beta}'\} \end{vmatrix}$$

$$\text{where, } \tilde{M}_2 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Thus, we can determine $W_{20}(\theta)$ and $W_{11}(\theta)$ from (46) and (47), and hence, we can compute g_{21}.

Therefore, the behaviour of bifurcating periodic solutions in the center manifold at the critical value $\tau = \tau'$ is computed by the following values:

$$\tilde{C}_1(0) = \frac{\iota}{2\sigma'\tau'}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}) + \frac{g_{21}}{2},$$

$$\tilde{\mu}_2 = -\frac{Re\{\tilde{C}_1(0)\}}{Re\{\frac{d\lambda(\tau')}{d\tau}\}},$$

$$\tilde{\beta}_2 = 2Re\{\tilde{C}_1(0)\},$$

$$\tilde{T}_2 = -\frac{Im\{\tilde{C}_1(0)\} + \tilde{\mu}_2 Im\{\frac{d\lambda(\tau')}{d\tau}\}}{\sigma'\tau'}$$

where,

- $\tilde{\mu}_2$ determines the direction of Hopf bifurcation, for if $\tilde{\mu}_2 > 0$, the Hopf bifurcation will be supercritical, and if $\tilde{\mu}_2 < 0$, the Hopf bifurcation will be subcritical, and the bifurcating periodic solutions exist for $\tau > \tau'$ or $\tau < \tau'$.

- $\tilde{\beta}_2$ determines the stability of the bifurcating periodic solutions, for if $\tilde{\beta}_2 < 0$, the bifurcating periodic solutions will be stable, and if $\tilde{\beta}_2 > 0$, the bifurcating periodic solutions will be unstable.

- $\tilde{T}_2$ determines the period of the bifurcating periodic solutions, for if $\tilde{T}_2 > 0$, the period increases, and if $\tilde{T}_2 < 0$, the period decreases.

5. NUMERICAL EXAMPLES

We investigate the dynamics of the system numerically. We consider a hypothetical and biologically feasible set of parameters illustrated below:

$\Lambda = 50; \lambda = 0.05; \alpha_1 = 0.1; \alpha_2 = 0.1; \beta = 20; m = 1; e\alpha_1 = 0.1; e\alpha_2 = 0.1;$

$d_1 = 0.075; d_2 = 0.05; d_3 = 0.09; r_1 = 0.0001; r_2 = 0.001; r_3 = 0.0002; h = 0.1;$

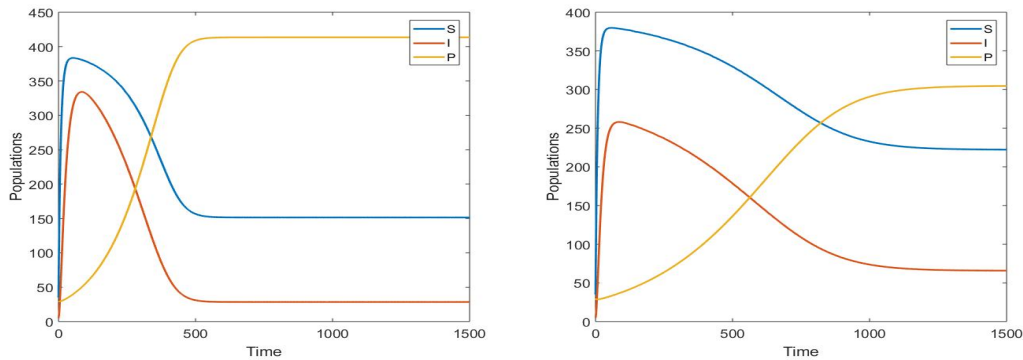$a = 0.07; n = 0.05; \phi = 0.3; l_1 = 0.1; l_2 = 0.1; q = 5; d = 0.02$

FIGURE 1. (a),(b) Trajectories in absence and presence of pollutant

Figure 1 (a) shows the solution of the system with above set of parameters in absence of pollutant and the equilibrium point so obtained is $(S^*, I^*, P^*) = (151.4877, 28.5114, 413.4734)$. (b)
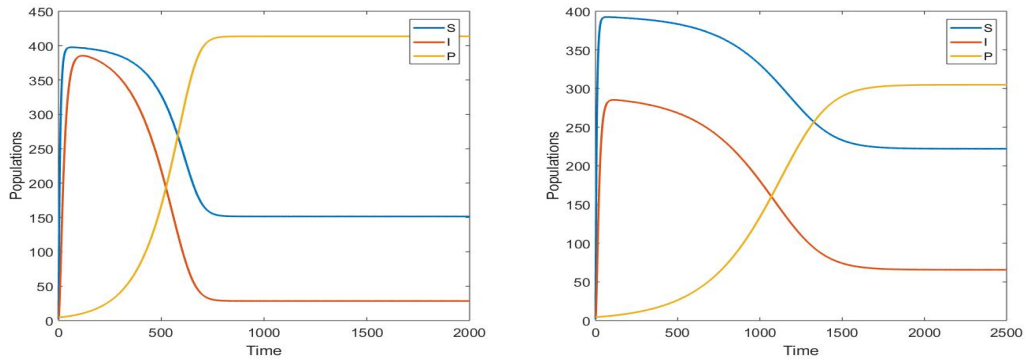
FIGURE 2. (c),(d) Trajectories in absence and presence of pollutant

shows the solution of the system with above set of parameters in presence of pollutant and the

equilibrium point comes out to be $(S^*, I^*, P^*) = (222.3478, 65.8217, 304.6972)$. We observe

that for the above set of parameters, due to presence of pollutant in the system, prey population

increases whereas predator population decreases. So, pollutant does not always have negative

effect on the populations as in this case prey population not only survives but also increases as

predator population decreases due to pollutant. Also, we can see from figure 1(a) and figure 2(c)

that the system in absence of pollutant approaches same interior equilibrium point starting from

different initial conditions and from figure 1(b) and figure 2(d) that the system in presence of

pollutant approaches same interior equilibrium point starting from different initial conditions.

So, for the above set of data, the system has a globally asymptotically stable interior equilibrium
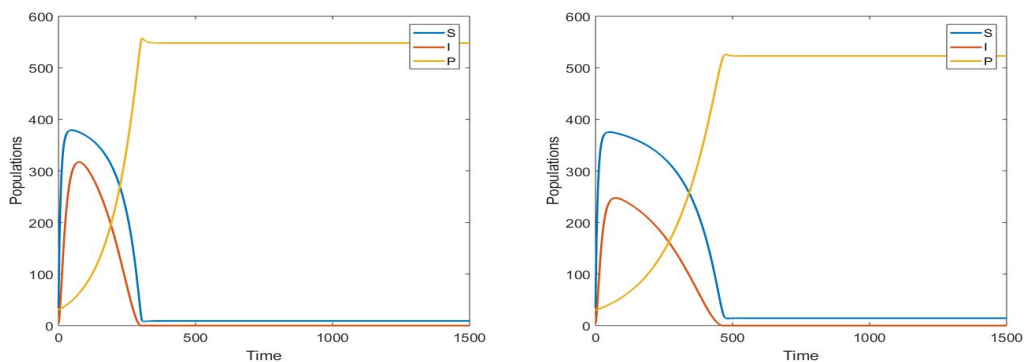
point.

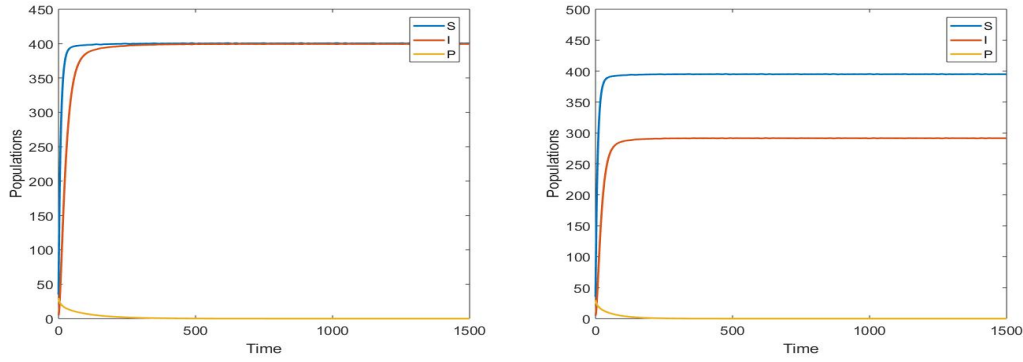FIGURE 3. (e),(f) Trajectories in absence and presence of pollutant when $\beta = 1$

FIGURE 4. (g),(h) Trajectories in absence and presence of pollutant when $\beta = 190$

Further the effect of varying half saturation constant β on the dynamics of the system is investigated. Figure 3 (e) shows the solution of the system when $\beta = 1$ in absence of pollutant and the equilibrium point so obtained is $(S^*, I^*, P^*) = (9.0000, 0.0000, 548.0556)$. (f) shows the solution of the system with $\beta = 1$ in presence of pollutant and the equilibrium point comes out to be $(S^*, I^*, P^*) = (14.3948, 0.0000, 522.9190)$. Thus, keeping β under some threshold, level system becomes disease free. Also, due to pollutant, predator population decreases and prey population increases. Now taking $\beta = 190$, we obtain figure 4. Figure 4 (g) Shows the solution of the system in absence of pollutant that approaches asymptotically to the point $(S^*, I^*, P^*) = (400.3761, 399.4159, 0.0000)$ and (h) shows the solution of the system in presence of pollutant and the equilibrium point comes out to be $(S^*, I^*, P^*) = (394.9940, 291.5103, 0.0000)$. Thus, on increasing the value of β, system becomes predator free. Thus, when we take the half saturation constant $\beta \leq 1$, the solution of the system approaches equilibrium point E_3, which is locally asymptotically stable and due to pollutant, the susceptible population increases and predator population decreases. When $1 < \beta < 190$, the system approaches interior equilibrium point as shown in Figure 1. When we take $\beta \geq 190$, it leads to extinction of the predator population and due to presence of pollutant, prey population decreases.

Now take $e\alpha_1 = 0.16, e\alpha_2 = 0.16$. Figure 5 (i) shows the solution of the system in absence of pollutant that approaches asymptotically to the point $(S^*, I^*, P^*) = (25.7254, 0.0000, 854.5850)$ and (j) shows the solution of the system in presence of pollutant and the equilibrium point comes out to be $(S^*, I^*, P^*) = (28.1243, 0.0000, 818.6385)$. So, on increasing the conversion
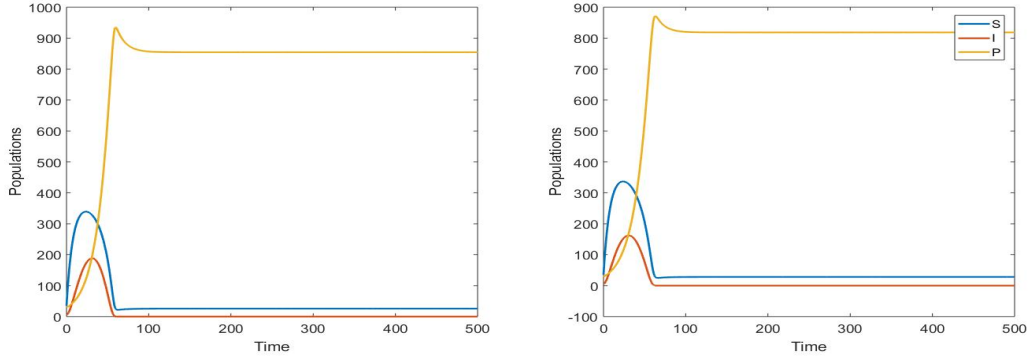
FIGURE 5. (i),(j) Trajectories in absence and presence of pollutant when $e\alpha_1 = 0.16, e\alpha_2 = 0.16$
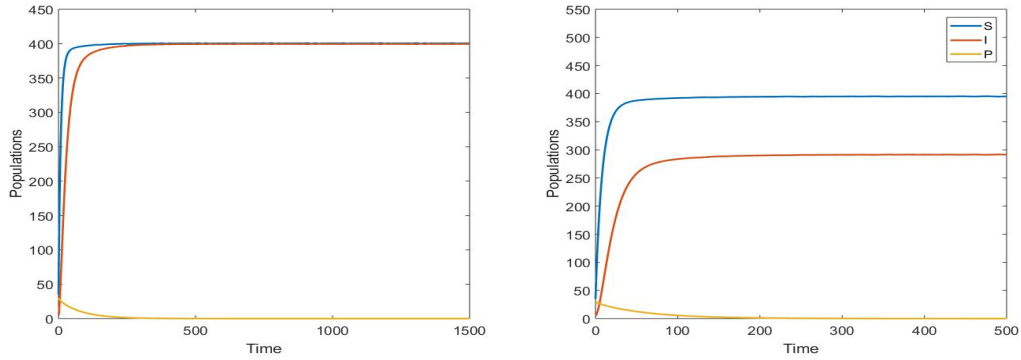

FIGURE 6. (k),(l) Trajectories in absence and presence of pollutant when $e\alpha_1 = 0.08, e\alpha_2 = 0.08$

rates, infection is getting eliminated from the system. Take $e\alpha_1 = 0.08, e\alpha_2 = 0.08$. Figure 6 (k) shows the solution of the system in absence of pollutant that approaches asymptotically to the point $(S^*, I^*, P^*) = (400.3659, 399.4227, 0.0000)$ and (l) shows the solution of the system in presence of pollutant and the equilibrium point comes out to be $(S^*, I^*, P^*) = (394.9921, 291.5119, 0.0000)$. Here, predator is getting extinct on decreasing conversion rates simultaneously. We observe that when the conversion rates $e\alpha_1, e\alpha_2 \leq 0.08$, system approaches predator free equilibrium E_4, which is locally asymptotically stable and pollutant leads to decrease in prey population. When $0.08 < e\alpha_1, e\alpha_2 < 0.16$, system approaches interior equilibrium point as shown in Figure 1. When $e\alpha_1, e\alpha_2 \geq 0.16$ the system approaches infective free equilibrium point E_3 which is locally asymptotically stable and due to presence of pollutant susceptible prey increases as predator population decreases.
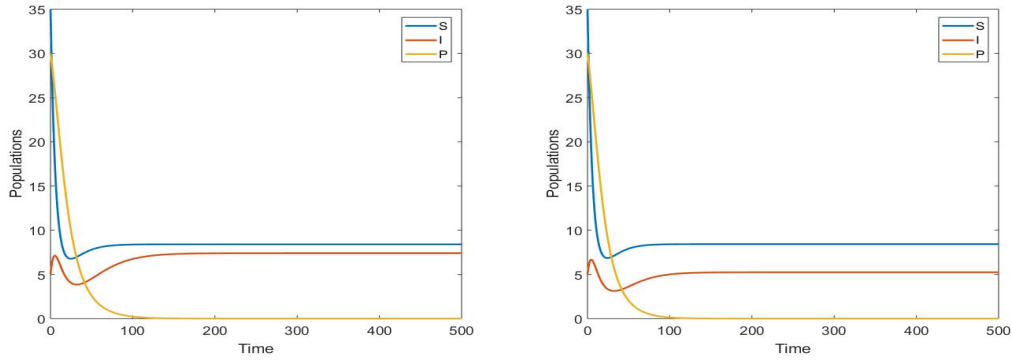
FIGURE 7. (m),(n) Trajectories in absence and presence of pollutant when $\Lambda = 1$

When we take $\Lambda = 1$, we obtain figure 7, where (m) shows the solution of the system in absence of pollutant that approaches asymptotically to the point $(S^*, I^*, P^*) = (8.4001, 7.3999, 0.0000)$ and (n) show the solution of the system in presence of pollutant that approaches asymptotically to the point $(S^*, I^*, P^*) = (8.4221, 5.2366, 0.0000)$. We observe that taking $\Lambda \leq 1$ leads to extinction of predator population. Also, due to presence of pollutant susceptible prey population increases and infective prey population decreases.
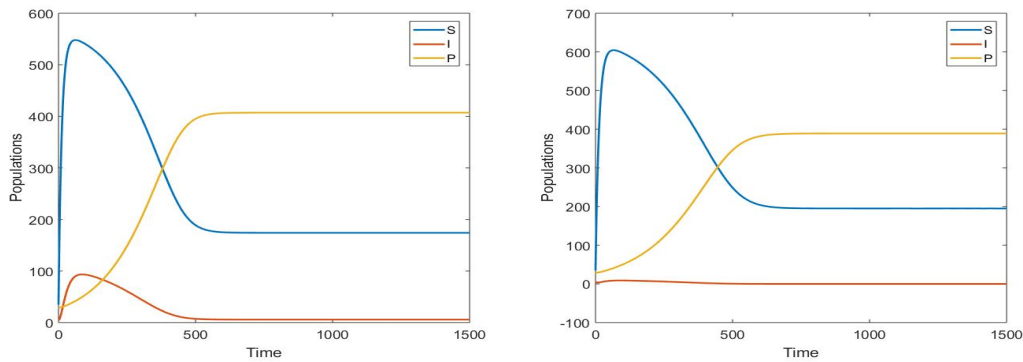

FIGURE 8. (o),(p) Trajectories in absence and presence of pollutant when $\lambda = 0.001$ and $q = 1$

When we take $\lambda = 0.001$ and $q = 1$, we obtain figure 8, where (o) shows the solution of the system in absence of pollutant and approaches asymptotically to the point $(S^*, I^*, P^*) = (174.1301, 5.8694, 407.1852)$ and (p) shows the solution of the system in presence of pollutant that approaches asymptotically to the point $(S^*, I^*, P^*) = (195.1533, 0.0000, 389.1198)$. We observe that keeping infection and pollutant under some limit not only eliminates infection from

the system but also increases susceptible prey population and decreases the predator population, which is in sync with real life scenario.
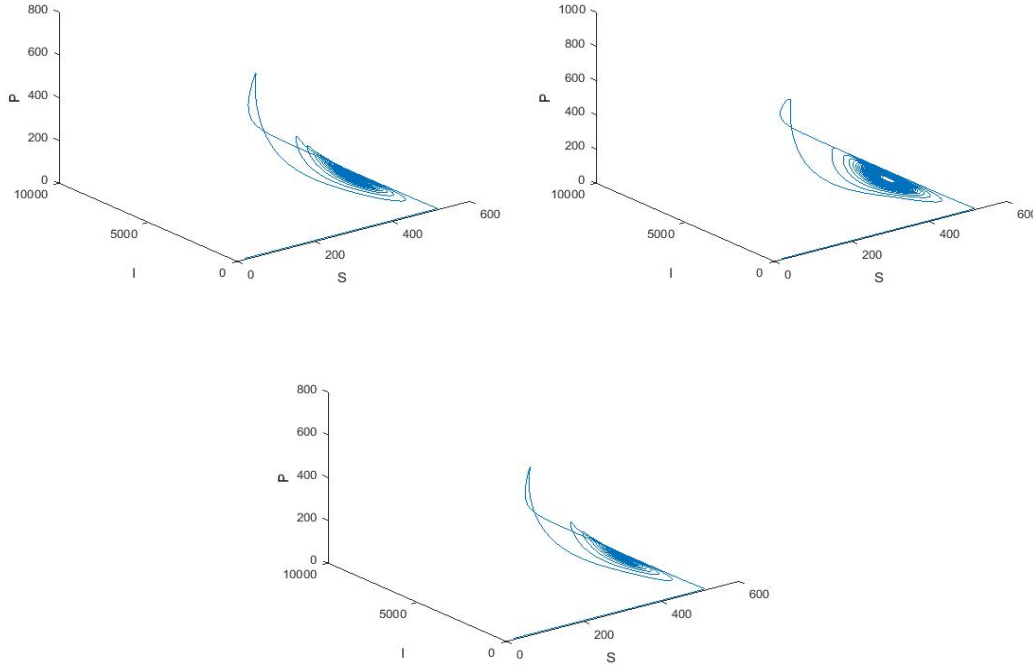

FIGURE 9. (q):$\tau = 5.5, U^* = 0.0001$, (r):$\tau = 5.9, U^* = 0.0001$, (s):$\tau = 5.9, U^* = 0.001$

When we consider, $\Lambda = 500, \lambda = 0.9103, \alpha_1 = 0.959241, \alpha_2 = 0.56585, \beta = 30, m = 0.005, e = 0.12, d_1 = 0.0534, d_2 = 00.0010, d_3 = 0.50259, r_1 = 0.937, r_2 = 0.91, r_3 = 0.090001$ and $U^* = 0.0001$; we obtain that the periodic solution is locally asymptotically stable when $\tau < \tau_0 = 5.9$(Figure 9(q)) and at $\tau = \tau_0 = 5.9$, Hopf Bifurcation occurs(Figure 9(r)). It can be seen from figure 9(s) that if pollution is increased to $U^* = 0.001$ then the periodic solution is stable. Thus, we can say that increase in pollution upto a certain level has stabilizing effect on the system.

6. CONCLUSION

In this paper, a polluted prey-predator model with disease in prey has been proposed and studied. It is assumed that the pollutant affects both the populations while only prey population is vulnerable to disease. First thing discussed was positivity and boundedness of the solutions of the system. Then we performed stability analysis i.e. local and global stability of the solutions

were analyzed. Then we introduced delay in the model to make it more realistic and studied the stability of the delayed system at interior equilibrium point. The existence and direction of Hopf bifurcation was established i.e. Hopf bifurcation occurs at the interior equilibrium point after the delay crosses certain value τ'. Further, numerical simulations are carried out in order to investigate that which set of parameters control the dynamic behavior of the system. The parameters chosen were hypothetical and biologically feasible. For the set of data chosen, it is observed that pollutant may not always have negative effect on existence of species, rather it could help the species to survive. On varying the half saturation parameter, if we take half saturation constant below a specific value, disease gets eliminated from the system. The solution of the system in absence of pollutant and in presence of pollutant approaches infective free equilibrium point which is locally asymptotically stable and due to presence of pollutant, the susceptible prey population increases and predator population decreases. If we take half saturation constant above a specific value it leads to extinction of predator population and due to presence of pollutant, the prey population decreases. Varying the conversion rates simultaneously, we get that increasing conversion rates above a specific value, system becomes disease free. The solution of the system in absence of pollutant and in presence of pollutant approaches an infective free equilibrium which is locally asymptotically stable. Also, in this case, due to presence of pollutant, prey population increases and predator population decreases. Decreasing conversion rates below a specific value, predator population extincts and prey population decreases. On decreasing the growth rate constant of the susceptible prey below a specific value, the system approaches predator free equilibrium i.e. it leads to the extinction of predator population. Above that specific value of growth rate constant, the system approaches to interior equilibrium point as shown in figure 1. If we take the infection rate λ and exogenous input rate q of pollutant into the environment below a specific value simultaneously, the pollutant helps in eliminating disease from the prey population and thus increasing the number of healthy prey in the system. Whereas, predator population decreases due to presence of pollutant, which is in sync with real life scenario.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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